Supersymmetry and complexified spectrum on Euclidean AdS_2

Imtak Jeon (APCTP)
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Outline

- Motivation
- Problem with SUSY and standard basis
- Construction of supersymmetric Hilbert space
- 1-loop in SUSY Hilbert space
- Conclusion

Motivation: SUSY on $EAdS_2$

- All extremal black holes universally have an **AdS_2** factor in their near horizon geometry.

- **Euclidean path integral** approach provides thermodynamic properties of the black holes [Gibbons, Hawking ‘97]

- **Supersymmetr**y provides us with a powerful tool for quantum study of the Euclidean path integral.

  e.g. SUSY localization method [Nekrasov ’02, Pestun '07]

Motivation: SUSY on $EAdS_2$

- **Quantum entropy function** [Sen '08]: Quantum formula of macroscopic entropy for extremal black holes is defined as the Euclidean partition function with the boundary condition dictated by AdS_2

$$S_{\mathrm{BH}} = k_B \log Z^{\mathrm{sugra}}_{AdS_2} = k_B \log \left\langle \mathrm{e}^{-iq \oint A} \right\rangle^{\mathrm{finite}}_{AdS_2}$$

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- Defined in micro-canonical ensemble

- Generalization of Bekenstein-Hawking entropy to include all the quantum corrections

Motivation: SUSY on EAdS$_2$

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$$S_{\rm BH} = k_B \log Z^{\rm sugra}_{AdS_2} = \frac{k_B A_H(q,p)}{4G_N \hbar / c^3} + c \log A_H(q,p) + \cdots$$

- For various BPS black holes, perturbative 1-loop successfully matches with corresponding microscopic result.
[Sen, Banerjee, Gupta, Mandal, Bhattacharyya, Panda, Lal, Thakur '10~ ; Keeler, Larsen, Lisão '14'15; …]

Motivation: SUSY on $EAdS_2$

- Application of supersymmetric localization:
  Computation of quantum entropy function for 1/8 BPS black hole in type II supergravity reproduces microstate degeneracy as an integer !

$$Z^{\mathrm{sugra}}_{AdS_2} = \left\langle \mathrm{e}^{-iq\oint A} \right\rangle^{\mathrm{finite}}_{AdS_2} = \mathrm{integer}$$

[Banerjee, Banerjee, Gupta, Mandal, Sen ‘09; Dabholkar, Gomes, Murthy ’10,’11’13; Gupta, Murthy ’12; Gupta, Ito, **IJ** ’15; Murthy, Reys, de Wit Murthy, Reys ’18; **IJ**, Murthy ’18; Iliesiu, Murthy, Turiaci, ‘22]

- This is a quantum completion of [Strominger-Vafa ’96] providing exact test of AdS_2 /CFT_1 .

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- Despite those extensive results, there is a problem concerning the asymptotic supersymmetric boundary condition.

The problem

- Imposing **boundary condition** is important to define a theory. If we want supersymmetric theory, then boundary condition should be supersymmetric. However…

- **Supersymmetric** boundary condition and **normalizable** condition are **not** always **compatible**. [David, Gava, Gupta, Narain `18, `19]

- Standard eigenbasis for normalizable fluctuation [Camporesi, Higuchi '94] of boson and fermion are not mapped to each other by supersymmetry [Sen '23].

- Supersymmetry demands 'non-normalizable' modes?
  → path integral ill-defined?!
  → well-defined theories on AdS_2 does not have SUSY?!

The problem

- If there is no SUSY, what does it mean by ‘super’gravity on AdS, given that there is the dual ‘supersymmetric’ field theory?

- How can the 1-loop test using the standard normalizable non-supersymmetric basis agree with results from supersymmetric microscopic theory?

- How is the localization method valid and capable of giving the correct exact result?

- We resolve this problem by showing that EAdS2 requires **complexified spectrum** and constructing the supersymmetric Hilbert space for scalar and spinor fields.

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Problem of SUSY and standard basis

- SUSY relation between boson and fermion is generically given by

$$Q\Phi = \varepsilon\Psi\,.$$

- On the AdS_2 geometry,

$$ds^2 = L^2(d\eta^2 + \sinh^2\eta\, d\theta^2) \qquad 0 \le \eta < \infty\,,\ 0 \le \theta < 2\pi$$

  the Killing spinor equation is given by

$$D_\mu \varepsilon^s = s\frac{1}{2L}\gamma_\mu \varepsilon^s\,, \qquad s = \pm 1\,,$$

  whose solutions are

$$\varepsilon_+^s = \sqrt{L}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\begin{pmatrix}\cosh\frac{\eta}{2}\\ s\sinh\frac{\eta}{2}\end{pmatrix},\ \varepsilon_-^s = \sqrt{L}\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}\begin{pmatrix}s\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}$$

- They have exponential asymptotic growth $\exp(\eta/2)$ for large η .

Problem of SUSY and standard basis

- Eigenbasis of $-\nabla^2$ for scalar:

$$\phi_{\lambda,k}(\eta,\theta) \sim e^{ik\theta}\sinh^{|k|}\eta\, F\Big(\alpha_s, \beta_s; |k|+1; -\sinh^2\frac{\eta}{2}\Big) \qquad \alpha_s = \frac{1}{2}+|k|+i\lambda,\ \beta_s = \frac{1}{2}+|k|-i\lambda$$

with $k \in \mathbb{Z}\,, \quad \lambda \in \mathbb{R}_{>0}$, where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, which have eigenvalue, $L^{-2}(\lambda^2 + 1/4)$.

- Eigenbasis of $i\gamma^\mu D_\mu$ spinor field:

$$\psi^+_{\lambda,k} \sim e^{i(k+\frac{1}{2})\theta}\begin{pmatrix} \cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}\,F(\alpha_f,\beta_f;k+1;-\sinh^2\frac{\eta}{2}) \\ -i\frac{\lambda}{k+1}\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}\,F(\alpha_f,\beta_f;k+2;-\sinh^2\frac{\eta}{2}) \end{pmatrix}$$

$$\psi^-_{\lambda,k} \sim e^{-i(k+\frac{1}{2})\theta}\begin{pmatrix} i\frac{\lambda}{k+1}\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}\,F(\alpha_f,\beta_f;k+2;-\sinh^2\frac{\eta}{2}) \\ -\cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}\,F(\alpha_f,\beta_f;k+1;-\sinh^2\frac{\eta}{2}) \end{pmatrix}$$

$$\alpha_f = k+1+i\lambda,\ \beta_f = k+1-i\lambda$$

with $\lambda \in \mathbb{R}\,, \quad k \in \mathbb{Z}_{\geq 0}$, which have the eigenvalue, $L^{-1}\lambda$.

- They form a Dirac delta-function orthonormal basis : $\langle \lambda, k|\lambda', k'\rangle = \delta_{kk'}\delta(\lambda - \lambda')$

Problem of SUSY and standard basis

- Eigenbasis of scalar and spinor fields grow as

$$\phi_{\lambda,k}(\eta,\theta) \sim \mathrm{e}^{-\frac{1}{2}\eta+\mathrm{i}k\theta}\left(\alpha_{\lambda,k}\mathrm{e}^{\mathrm{i}\lambda\eta}+\alpha_{-\lambda,k}\mathrm{e}^{-\mathrm{i}\lambda\eta}\right)$$

$$\psi^{\pm}_{\lambda,k} \sim \mathrm{e}^{-\frac{\eta}{2}\pm\mathrm{i}(k+\frac{1}{2})\theta}\left(\mathrm{e}^{\mathrm{i}\lambda\eta}\beta_{\lambda,k}\upsilon_{(-)}\pm\mathrm{e}^{-\mathrm{i}\lambda\eta}\beta_{-\lambda,k}\upsilon_{(+)}\right)$$

  having degree of growth -1/2.

- Since the bispinor

$$\varepsilon^{s}_{\pm}\psi^{\pm}_{\lambda,k} \sim \mathrm{e}^{s\mathrm{i}\lambda\eta\pm\mathrm{i}(k+1)\theta}\,, \quad \varepsilon^{s}_{\mp}\psi^{\pm}_{\lambda,k} \sim \mathrm{e}^{s\mathrm{i}\lambda\eta\pm\mathrm{i}k\theta}\,.$$

  has the degree of growth 0 ,

- the left and right hand side of the supersymmetry relation $Q\Phi=\varepsilon\Psi$ do not match when expressed in terms of standard basis.

Problem with SUSY on AdS_2

- Try to find supersymmetry of mode expansion coefficient

$$\Phi = \sum a_m \phi_m \,, \Psi = \sum b_n \psi_n \,,$$

$$Q a_m = \langle \phi_m | \varepsilon \Psi \rangle = \sum_n b_n \langle \phi_m | \varepsilon \psi_n \rangle$$

- The inner product is ill-defined as the integration diverges

$$\left\langle \phi_{\lambda',k'} \left| \varepsilon \psi_{\lambda,k} \right\rangle \sim \int_0^\infty \mathrm{d}\eta \sqrt{g}\, \mathrm{e}^{-\frac{1}{2}\eta} \to \infty$$

- Nevertheless, we can obtain the inner product using analytic continuation: Introduce $\epsilon > 1/2$ such that the integration converges and take $\epsilon \to 0$ at the end.

$$\left\langle \phi_{\lambda',k} \middle| \varepsilon^s \psi_{\lambda,k} \right\rangle \propto \delta\big(\lambda' + \big(\lambda - s\tfrac{\mathrm{i}}{2}\big)\big) + \delta\big(\lambda' - \big(\lambda - s\tfrac{\mathrm{i}}{2}\big)\big)$$

- Since the parameter λ and λ' are real, the inner product is zero. → No SUSY.

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Supersymmetric Hilbert space

- The above result suggest to consider $\lambda \to \lambda + s\,\mathrm{i}/2$.

- The bi-spinor with shifted λ by $\mathrm{i}/2$ is exactly proportional to the eigenfunctions for scalar as

$$-\nabla^2(\varepsilon^s \psi^{\pm}_{\lambda,k}) = \frac{1}{L^2}\left(\left(\lambda - s\frac{\mathrm{i}}{2}\right)^2 + \frac{1}{4}\right)(\varepsilon^s \psi^{\pm}_{\lambda,k})$$

$$\varepsilon^s_{\pm}\psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k} \propto \phi_{\lambda,\pm(|k|+1)}\,, \quad \varepsilon^s_{\mp}\psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k} \propto \phi_{\lambda,\pm|k|}\,.$$

  Therefore, the mapping between boson and fermion becomes manifest.

- We propose a SUSY Hilbert space

$$\begin{array}{ll} \text{Scalar:} & \left\{\phi_{\lambda,k}(\eta,\theta)\,\middle|\,\lambda \in \mathbb{R}_{>0}\,,\quad k \in \mathbb{Z}\right\} \\ \text{Spinor:} & \left\{\psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k}(\eta,\theta)\,\middle|\,\lambda \in \mathbb{R}\,,\quad k \in \mathbb{Z}_{\geq 0}\right\}. \end{array}$$

Supersymmetric Hilbert space

- Complex eigenvalue:

$$\mathrm{i}\gamma^\mu D_\mu \psi^\pm_{\lambda+s\frac{\mathrm{i}}{2},k} = L^{-1}(\lambda + s\frac{\mathrm{i}}{2})\psi^\pm_{\lambda+s\frac{\mathrm{i}}{2},k}$$

  - It is natural: for a space with boundary, the Dirac operator is not necessarily hermitian.
    c.f. non-hermiticity in a box [Bonneau, Faraut, Valent '01]

- Unlike the standard basis, there is no fermionic zero mode:
  If mass is ±1/2, the kinetic operator $\mathrm{i}(\gamma^\mu D_\mu - s/2)$ vanishes at $\lambda = 0$. But there is no spectrum at this point as spectral density of the spinor basis is zero.

$$\mu_{\psi^\pm}(\lambda + s\tfrac{\mathrm{i}}{2}) \ = \ \tfrac{1}{4\pi}\left(\lambda + s\tfrac{\mathrm{i}}{2}\right)\tanh\pi\lambda$$

Supersymmetric Hilbert space

- Asymptotic behaviour : not vanishing at the asymptotic boundary as

$$\psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k} \sim \mathrm{e}^{\pm\mathrm{i}(k+\frac{1}{2})\theta - s\mathrm{i}\lambda\eta} \upsilon_{(s)}\,, \qquad \frac{1}{2}(1\pm\sigma_1)\upsilon_{(\pm)} = \upsilon_{(\pm)}$$

- Nevertheless, the SUSY eigenfunctions form a delta-function orthonormal basis as

$$\langle \psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k} | \psi^{\pm}_{\lambda'+s\frac{\mathrm{i}}{2},k'} \rangle \equiv \pm\mathrm{i} \int \mathrm{d}\eta \mathrm{d}\theta \sqrt{g} (\psi^{\pm}_{\lambda+s\frac{\mathrm{i}}{2},k})^T C \psi^{\pm}_{\lambda'+s\frac{\mathrm{i}}{2},k'} = \delta_{kk'}\delta(\lambda-\lambda')$$

  - We define the appropriate inner product without using hermitian conjugate [Osterwalder, Schrader '72], "Euclidean inner product".

  - The projection condition of $\upsilon_{(\pm)}$ cancels the dominant term in there inner product, making the inner product well-defined.

Supersymmetric Hilbert space

- Compatible with asymptotic boundary condition.
  Asymptotic fall-off behavior of the fluctuation of fields is dictated by variational principle.

$\delta S \equiv 0$ at on-shell saddle requires

$$\delta\phi = \delta\phi_{(0)} \mathrm{e}^{-\Delta_\phi \eta} + \cdots \,, \qquad \Delta_\phi > \tfrac{1}{2} \,,$$
$$\delta\psi = \delta\psi_{(0)} \mathrm{e}^{-\Delta_\psi \eta} + \cdots \,, \qquad \Delta_\psi > 0 \,.$$

- Note that behavior of the SUSY basis, $\phi_{\lambda,k} \sim \mathrm{e}^{-\frac{1}{2}\eta} \,, \quad \psi_{\lambda+s\frac{\mathrm{i}}{2},k} \sim \mathrm{e}^{-0\cdot\eta}$,
  saturate the bounds, being able to span all fluctuations above the bound.

  cf. For the case $1/2 \geq \Delta_\psi > 0$, standard basis having 1/2 damping may not span the corresponding fluctuation.

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1-loop in SUSY Hilbert space

- Let us compare 1-loop partition function using SUSY and non-SUSY basis.
  - Focus on contribution of spinors having kinetic term,

$$-\mathrm{i}\overline{\psi}(\not{D}+M_\psi)\psi$$

  - A difference: SUSY Hilbert space does not suffer from zero modes
  - If there are zero modes, one need to separate out their regularized contribution

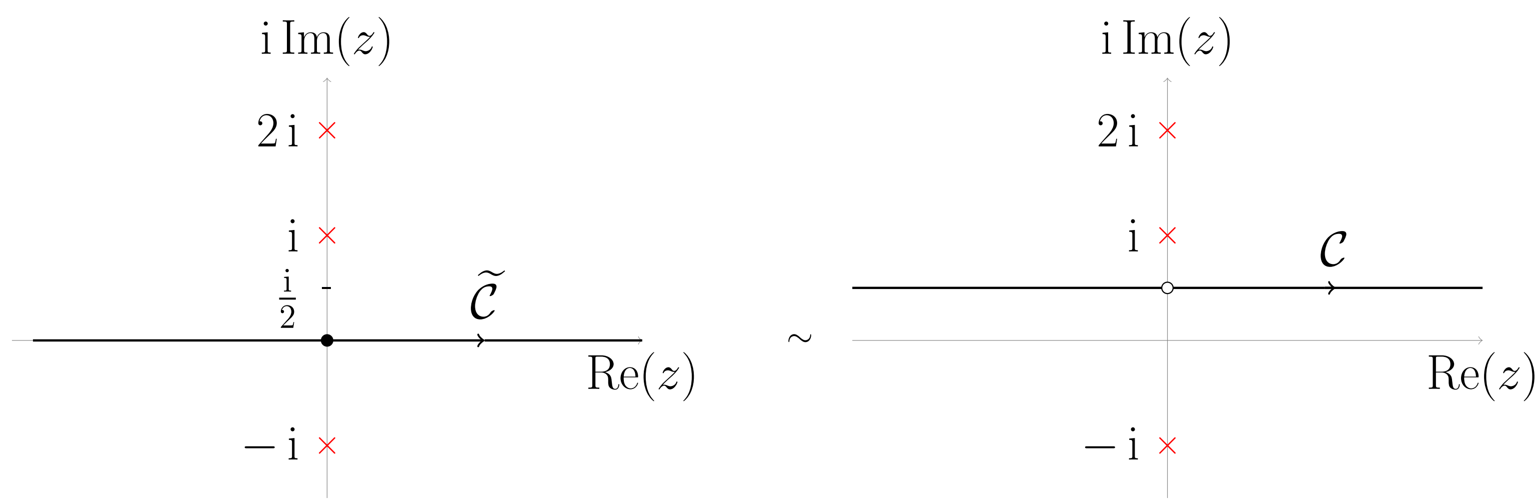
$$Z_{\text{1-loop}} = Z_{\text{zm}} Z'_{\text{1-loop}}$$

- Use the heat kernel method.

$$\begin{aligned}\log Z_\psi &= \frac{1}{2}\int_{\epsilon/L^2}^{\infty}\frac{\mathrm{d}\overline{s}}{\overline{s}}\,K_\psi(\overline{s}) \;=-\frac{1}{2}\int_{\epsilon/L^2}^{\infty}\frac{\mathrm{d}\overline{s}}{\overline{s}}\,\mathrm{Tr}\exp\Big[-\overline{s}\big(\mathrm{i}L(\not{D}+M_\psi)\big)^2\Big],\\ &=\int_{\epsilon/L^2}^{\infty}\frac{\mathrm{d}\overline{s}}{\overline{s}}\int\mathrm{d}^2x\sqrt{g}\int_{\mathcal{C}}\mathrm{d}z\,\mu_\psi(z)\exp\Big[-\overline{s}(z+\mathrm{i}LM_\psi)^2\Big]\,.\end{aligned}$$

1-loop in SUSY Hilbert space

- Standard basis vs. SUSY basis



The shift of the contour does not cross any pole and thus does not change the result of the trace of heat kernel.

- Local contribution of the heat kernel is unchanged, whereas the global contribution (zero mode contribution) can be different.

1-loop in SUSY Hilbert space

- Argument on how 1-loop study of black hole entropy using standard basis matches with supersymmetric result.
  - Black hole near horizon geometry has additional internal geometry to AdS_2. e.g. $AdS_2 \times S^2$.
  - Dirac operator along S^2 does not give zero mass in the Kaluza-Klein tower, thus there is no zero mode even in the standard basis.

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Concluding remarks

- We have constructed supersymmetric basis for scalar and spinor field on $EAdS_2$ by complexifying the spectrum of Dirac operator.

- In quantum study of **supersymmetric theories** on **Euclidean** $\mathbf{AdS_2}$, the spectrum of the quantum fluctuation of fields should be **complexified**.

- We expect: complexified spectrum is pervasively necessary:
  - For higher dimensions, i / 2 shift
  - Vector, graviton multiplets should have complexified spectrum.

Discussion

- Structure of SUSY Hilbert space for vector, graviton multiplet is an interesting problem: understanding the zero modes in vector, graviton, gravitino would be relevant for understanding of quantum entropy function.

- Physical implication this boundary condition to black hole? Relation to Atiyah-Patodi-Singer(APS) boundary condition?

- Group theoretic understanding of the spectrum? In terms of principal series representation of SL(2,R) ~ SO(1,2)?

- We could consider this boundary condition without reference to the SUSY.

Thank you for your attention.