Large N Master Field Optimization for Multi-Matrix Systems

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Plan of the talk

Why Matrices

- Intermediate vector bosons (W^+, W^-, Z^0) and gluons are matrix valued gauge fields:

$$\boxed{[A_\mu(x)]_{ab}}$$

$$a, b = 1, 2 \; - \; SU(2) \; - \text{ weak interactions}$$
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- QCD cousin: $\mathcal{N} = 4$ super Yang-Mills theory ($i = 1, ..., 6$ matrix scalars)

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu X_i D^\mu X_i + \frac{g_{YM}^2}{4}[X_i, X_j]^2 + ...$$

Why Large Matrices

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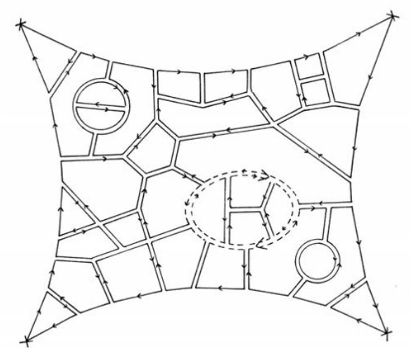
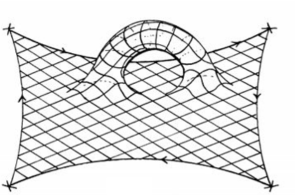
- [Wilke van der Schee, 2011]

□ Large N field theory





Planar limit:

$\lambda = g^2 N$ fixed

$N \to \infty$

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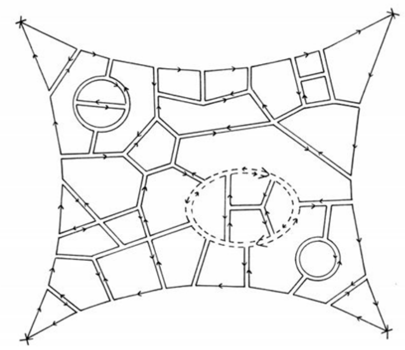

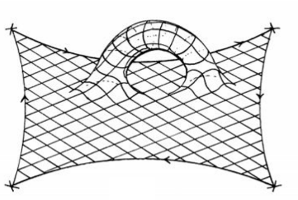

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- Example: GS energy $E = N^{2-2g} f(\lambda)$. QCD string ?

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Large N, gauge invariance and loop equations

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- Wilson loops

$$\phi(C) = \mathrm{Tr}\left(P\, e^{i \oint_C A_\mu dx^\mu}\right)\ ,\quad \phi(C, x_1, x_2) = \bar{\psi}(x_1) P\, e^{i \int_{x_1}^{x_2} A_\mu dx^\mu} \psi(x_2)$$

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- Large N factorization of gauge invariant operators (loops from now on):

$$< \phi(C_1)\phi(C_2) >_{N\rightarrow\infty} = < \phi(C_1) >< \phi(C_2) > +1/N^2 ...$$

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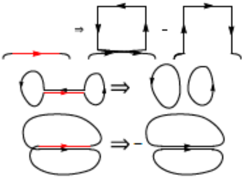
- Migdal-Makeenko equations [1979] (Schwinger-Dyson equations). On the lattice [Kazakov and Zheng, 2022]:



(Path) integral or quantum mechanics of finite number of Matrices - reduced models

- Compactified gauge theories - QCD motivated
  - [Luscher, 1982-1984] - Y-M theory on a torus
  - [Eguchi and Kawai, 1982] - Loop equations from $\mu = 1, \ldots, 4$ unitary matrices
  - [Bhanot, Heller and Neuberger, 1982] - Quenched unitary matrices
  - [Gross and Y. Kitazawa, 1982] - Large N quenched prescription for hermitian matrices
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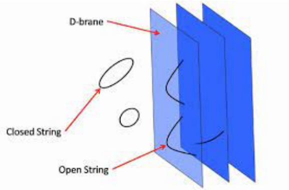
- $D-$ branes [Polchinski, 1995]
  - [Banks, Fischler, Shenker, Susskind, 1997] - SS QM of 9 hermitian matrices (D0's) - M-theory!
  - [Ishibashi, Kawai, Kitazawa, Tsuchiya, 1997] - 10d SYM
  - [Maldacena, Gubser, Klebanov, Polyakov, Witten, 1998-1999] - AdS/CFT
  - [Berenstein, Maldacena, Nastase, 2002] - Scalars of $\mathcal{N} = 4$ SYM
  - [tzhaki, Maldacena, ...] - Black holes and matrix QM



Constraints in loop space

- Can one study matrix systems directly in the large N limit? Loop space is highly non-trivial and difficult to parametrize (more later). And is subject to positivity constraints!

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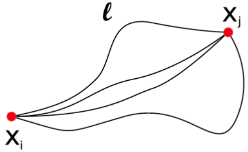
- Recently re-discovered [Anderson and Kruczenski, 2017]
- Consider set of open Wilson lines $C_l, l = 1, ..., L$ from x_1 to x_2, and U^l the corresponding product of unitary matrices along the curves. For an arbitrary set of coefficients c_l, define $A = \sum_1^L c_l U^l$. Since $\mathrm{Tr} A^\dagger A \geq 0$ for any c_l, one must have

$$\rho_{ll'} = \frac{1}{NL} < \mathrm{Tr}\left[(U^l)^\dagger U^{l'}\right] > \succeq 0$$

  Semi-definite programming can then be used. Wording bootstrap is associated with existence of constraints and parameter "scanning".
- Recent interest [H. Lin, 2020; Han, Hartnoll Krutho, 2020; Kazakov and Z. Zheng 2022; Koch, Jevicki, Liu, Mathaba, Rodrigues, 2022]



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- The idea is to implement a change of variables from the original variables of the theory, generically denoted by $X_{\mathcal{A}}$, to the invariant set of operators (the collective fields) $\phi(C)$, and to require explicit hermiticity of the collective field Hamiltonian. This change of variable is accompanied by a Jacobian J. In general J is not known explicitly, but it satisfies the following equation

$$\sum_{C'} \frac{\partial \ln J}{\partial \phi^\dagger(C')} \Omega(C', C) = w(C) - \sum_{C'} \frac{\partial \Omega(C', C)}{\partial \phi^\dagger(C')}.$$

This is sufficient to obtain explicitly the collective field Hamiltonian in terms of $\phi(C)$ and its canonical conjugate $\pi(C)$.

Collective Field Hamiltonian and constraints

- In general,

$$\Omega(C, C') = \sum_{\mathcal{A}} \frac{\partial \phi^\dagger(C)}{\partial X^\dagger_{\mathcal{A}}} \frac{\partial \phi(C')}{\partial X_{\mathcal{A}}}, \quad w(C) = \sum_{A} \frac{\partial^2 \phi(C)}{\partial X^\dagger_{\mathcal{A}} \partial X_{\mathcal{A}}} .$$

  $\Omega(C, C')$ joins two loops into a sum of single loops, and $w(C)$ splits a given loop into a sum of two (in general smaller) loops.

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- The collective field Hamiltonian H_{col} is ideally suited to a numerical approach based on minimisation of the effective potential V_{eff}, in a truncated loop space $H_{col} \to H^{trunc}_{col}$. Already some time ago [Jevicki, Karim, Rodrigues, Levine,1983, 1984] this approach was successfully implemented for $2+1$ lattice gauge theories.

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- Systems of unitary matrices have a phase transition between a strong and weak phase, and it was then established that in the weak coupling phase the minimization has to be accompanied by a constraint:

$$\begin{cases} \text{Minimize} \;\; V^{trunc}_{eff}, \\ \Omega(C, C') \succeq 0. \end{cases}$$

  In other words, the large N expectation values of the loop variables $\phi(C)$ must satisfy the constraint that the matrix $\Omega(C, C')$ is semi-positive definite, with a number of eigenvalues saturating to zero in the weak coupling regime. This was shown to also be the case when considering loop equations [Rodrigues, 1985] .

Constraint and density of eigenvalues

- This constraint is not difficult to understand: the large N limit of the single unitary matrix integral has a well known third order phase transition [Gross, Witten, 1980], described in terms of the density of its (phases of) eigenvalues $\rho(\theta)$ as:

$$\begin{cases} \rho(\theta) = \frac{1}{2\pi}(1 + \frac{2}{\lambda}\cos\theta), & -\pi \leq \theta \leq \pi \quad \text{for } \lambda \geq 2, \\ \left\{ \begin{array}{ll} \rho(\theta) = \frac{2}{\pi\lambda}\cos\frac{\theta}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{\theta}{2}}, & |\theta| < 2\sin^{-1}\frac{\lambda}{2} \\ \rho(\theta) = 0, & 2\sin^{-1}\frac{\lambda}{2} \leq |\theta| \leq \pi \end{array} \right\} & \text{for } \lambda \leq 2. \end{cases}$$

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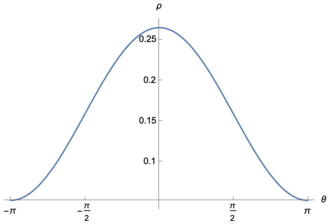
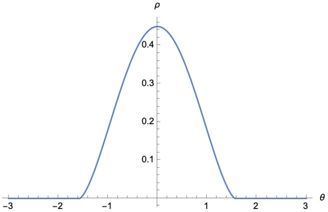
- For a single hermitian $N \times N$ matrix M, with invariants $\phi_k = \mathrm{Tr}(e^{-ikM})$, the density of eigenvalues is simply its Fourier transform.Then $\Omega(x, y) = \partial_x\partial_y\left(\phi(x)\delta(x - y)\right)$, and $\Omega(x, y)$ is seen to have zero eigenvalues when the density matrix $< x|\hat{\phi}|y >= \phi(x)\delta(x - y)$ has zero eigenvalues, or when $\phi(x) = 0$. For single matrix systems then, this constraint on Ω is easily related to the requirement that the density is non-negative.

Density of eigenvalues - another look



(a) Density of eigenvalues at $\lambda = 3$



(b) Density of eigenvalues at $\lambda = 1$

Quantum mechanics of two massless Y-M coupled matrices

- Our system is then [Mathaba, Mulokwe, Rodrigues, 2306.00935 [hep-th]]

$$\hat{H} = \frac{1}{2}\sum_{A=1}^{2} \mathrm{Tr} P_A^2 - \frac{g_{YM}^2}{N} \mathrm{Tr}[X_1, X_2]^2 = \frac{1}{2}\sum_{A=1}^{2} \mathrm{Tr} P_A^2 + \mathrm{Tr}(V(X_A)).$$

P_A is canonical conjugate to X_A. In our convention, 't Hooft's coupling λ is $\lambda = g_{YM}^2$

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- The $U(N)$ invariant loops are single traces of products of the matrices X_A, up to cyclic permutations:

$$\phi(C) = \mathrm{Tr}(\dots X_1^{m_1} X_2^{m_2} X_1^{n_1} X_2^{n_2} \dots)\,.$$

  For instance, with two matrices one has $[1\,1] = \mathrm{Tr}(X_1^2)\,, [1\,2] = \mathrm{Tr}(X_1 X_2)$, $[2\,2] = \mathrm{Tr}(X_2^2)$, with three matrices $[1\,1\,1] = \mathrm{Tr}(X_1^3)\,, [1\,1\,2] = \mathrm{Tr}(X_1^2 X_2)$, $[1\,2\,2] = \mathrm{Tr}(X_1 X_2^2)\,, [2\,2\,2] = \mathrm{Tr}(X_2^3)$, etc.

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- We let $\phi(C) \to \phi(C)/N^{\frac{l(C)}{2}+1}$ and then

$$H_{col} = \frac{1}{2N^2}\sum_{C,C'} \pi^\dagger(C)\Omega(C,C')\pi(C') + N^2 V_{eff}(\phi)$$

$$V_{eff}(\phi) \equiv \frac{1}{8}\sum_{C,C'} w(C)\Omega^{-1}(C,C')w^\dagger(C') + V(\phi).$$

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- $V_{eff}^{trunc}(\phi(C), C = 1, ..., N_{\text{loops}}) = \frac{1}{8} \sum_{C,C'=1}^{N_\Omega} w(C)\Omega^{-1}(C, C')w^\dagger(C') + V(\phi)$

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$l_{\max}$	N_Ω	N_{loops}
8	23	93
10	37	261
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- How is the constraint enforced ?

Master Variables

- To minimize V_{eff}^{trunc} subject to the constraint $\Omega(C, C') \succeq 0$, we introduce master variables ϕ_α that explicitly satisfy the constraint:

$$\Omega(C, C') = \sum_\alpha \frac{\partial \phi^\dagger(C)}{\partial \phi_\alpha} \frac{\partial \phi(C')}{\partial \phi_\alpha} \succeq 0$$

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$$\Omega(C,C') = \sum_{\alpha} \frac{\partial \phi^\dagger(C)}{\partial \phi_\alpha} \frac{\partial \phi(C')}{\partial \phi_\alpha} \succeq 0$$

- Specifically, we choose X_1 to be diagonal and X_2 a $N \times N$ hermitian matrix. The master field then has $N^2 + N$ real components ϕ_α, $\alpha = 1, 2, ..., N(N+1)$.

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$$\frac{\partial V_{eff}^{trunc}}{\partial \phi_\alpha} \equiv \sum_{C=1}^{N_{\rm loops}} \frac{\partial V_{eff}^{trunc}}{\partial \phi(C)} \frac{\partial \phi(C)}{\partial \phi_\alpha}\Big|_{\phi_\alpha^0} = 0, \;\; \alpha = 1, 2, ..., N(N+1)$$

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- We have chosen a truncation with $l_{\max} = 14$, that is, 2615 $N_{\rm loops}$ and a 93×93 Ω matrix. For the master field, we took $N = 51$, corresponding to 2652 master variables.

Scaling properties in the massless limit

- Recall

$$\hat{H} = \frac{1}{2}\sum_{A=1}^{2} \mathrm{Tr} P_A^2 - \frac{g_{YM}^2}{N} \mathrm{Tr}[X_1, X_2]^2 \,.$$

P_A is canonical conjugate to X_A , $A = 1, 2$.

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$$e = \Lambda_e\, g_{YM}^{2/3}\,, \quad \mathrm{Tr} X_1^2 = \Lambda_{[11]}\, g_{YM}^{-2/3}\,, \quad \mathrm{Tr} X_1^4 = \Lambda_{[1111]}\, g_{YM}^{-4/3}\,, \quad \text{etc.},$$

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  where e is any energy of the system.
- We considered 15 values of g_{YM}, ranging from 1 to 12, chosen so that they are reasonably distributed over this range in both a linear and logarithmic scale:

g_{YM}														
1	1.28403	1.64872	2	2.6	3.25	4	5	6	7	8	9	10	11	12

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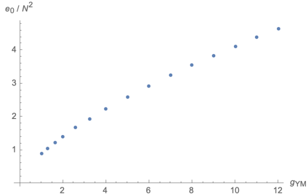
- Working directly in this massless limit, the optimization algorithm exhibited remarkable stable convergence to the system's minimum for all g_{YM}.

Planar quantities - ground state energy

- Plot of large N ground state energies versus g_{YM}:



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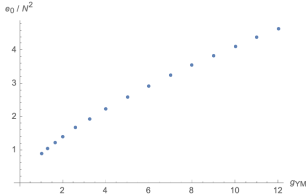

- We fit the data to the curve

$$e_0/N^2 = A_0\, g_{YM}^p\,,$$

by performing a least squares fit to the logarithmic plot, with result. We find:

$$\ln A_0 = -0.117625(8)\,, \qquad p = 0.666671(5)\,.$$

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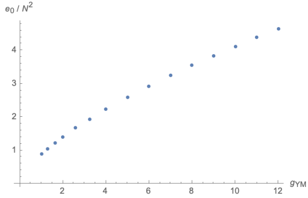

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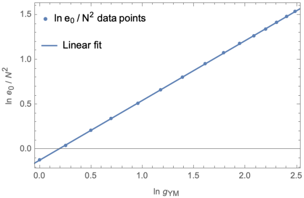
- This linear fit is shown below



Planar quantities

- The accuracy with which the interpolation matches the exact scaling $p = 2/3$ at this level of truncation is remarkable. We are then justified in setting $p = 2/3$ and fit the data to the scaling function

$$e_0/N^2 = \Lambda_0\, g_{YM}^{2/3}\,, \quad \left(= \Lambda_0\, \lambda^{1/3}\right) \tag{1}$$

with result

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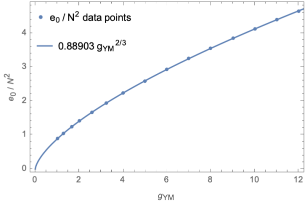
- The fit of the large N planar ground state energies to the scaling function is shown below. The level of accuracy with which the numerically obtained planar ground state energies match the scaling behaviour at this level of truncation is again remarkable.



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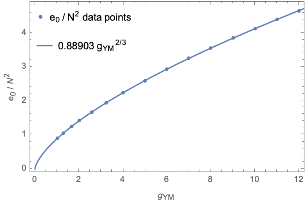

- Taking into account possible truncation dependent errors, the final scaling dependence on 't Hooft's coupling for the planar ground state energy of the massless system as:

$$\boxed{e_0/N^2 = 0.8890(2)\,\lambda^{1/3}}$$

Planar quadratic correlators

- We consider the correlator $\mathrm{Tr}(Z^\dagger Z)/N^2 = (\mathrm{Tr}X_1^2 + \mathrm{Tr}X_2^2)/N^2$, $(Z \equiv X_1 + iX_2)$ and do same analysis.

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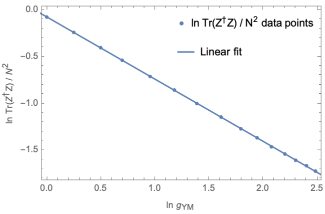
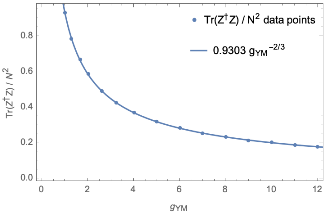
- The results are presented in the table and figures below

Parameters of (log) linear fit		$p = -2/3$ fixed	Final scaling function
$\ln A_{Z^\dagger Z}$	p	$\Lambda_{Z^\dagger Z}$	$\mathrm{Tr}(Z^\dagger Z)/N^2$
-0.07219(7)	-0.66672(4)	0.93027(3)	$0.930(1)\,\lambda^{-1/3}$



(e) Linear fit of $\ln \mathrm{Tr}Z^\dagger Z/N^2$ versus $\ln g_{YM}$



(f) Fit of $\mathrm{Tr}Z^\dagger Z/N^2$ to scaling function $0.9303\ g_{YM}^{-2/3}$

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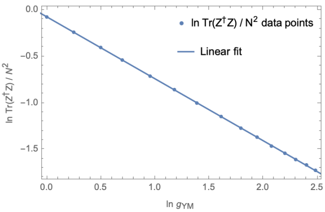

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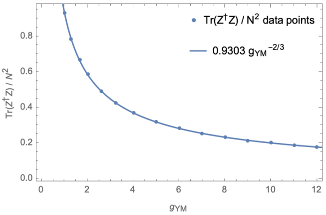


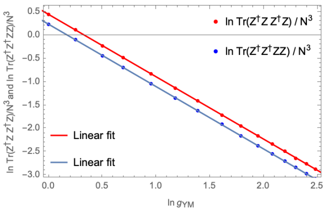
(h) Fit of $\mathrm{Tr}Z^\dagger Z/N^2$ to scaling function $0.9303\ g_{YM}^{-2/3}$

- The scaling power for the large N planar correlator is again predicted with a high level of accuracy, and their numerical values match with a high level of precision the scaling behaviour.

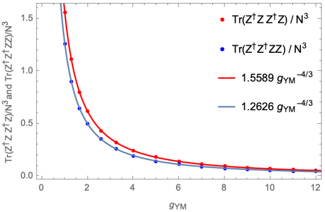
Quartic correlators

- For invariant loops with 4 matrices, we consider the loops $\mathrm{Tr}(Z^\dagger ZZ^\dagger Z)/N^3$ and $tr(Z^\dagger Z^\dagger ZZ)/N^3$, and carry out the same analysis, summarized in table and figures below.

	Log linear fit		$p = -4/3$	Final
	ln A	p	Λ	Scaling function
$\mathrm{Tr}(Z^\dagger ZZ^\dagger Z)/N^3$	0.4441(1)	-1.33340(6)	1.55895(8)	$1.559(8)\,\lambda^{-2/3}$
$\mathrm{Tr}(Z^\dagger Z^\dagger ZZ)/N^3$	0.2333(1)	-1.33341(8)	1.26261(8)	$1.263(6)\,\lambda^{-2/3}$



(i) Linear fit of the log of 4 matrices loop expectation values versus ln g_{YM}



(j) Fit of loops of 4 matrices to scaling functions $A\ g_{YM}^{-4/3}$

Similar remarks concerning the high level of accuracy of the numerical results apply.

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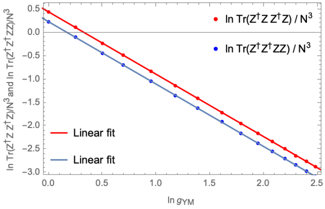

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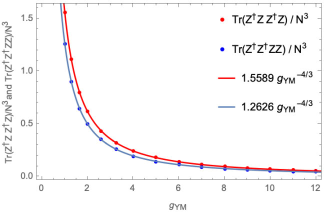

(l) Fit of loops of 4 matrices to scaling functions $A\ g_{YM}^{-4/3}$

Similar remarks concerning the high level of accuracy of the numerical results apply.

- Finally, we consider an "angle" defined to be

$$\mathcal{A} \equiv N\frac{\mathrm{Tr}X_1^2X_2^2 - \mathrm{Tr}X_1X_2X_1X_2}{\mathrm{Tr}X_1^2\mathrm{Tr}X_2^2} = -\frac{N}{2}\frac{\mathrm{Tr}[X_1,X_2]^2}{\mathrm{Tr}X_1^2\mathrm{Tr}X_2^2}\,.$$

and obtain

$$\boxed{\mathcal{A} = 0.685(2)}$$

Spectrum

- Master variables can be used to obtain the spectrum of the $O(N^0) = O(1)$ quadratic collective Hamiltonian [Koch, Jevicki, Liu, Mathaba, Rodrigues, 2022] (based on [Jevicki and Rodrigues, 1984]). The "mass matrix" is a $N_{\rm Loops} \times N_{\rm Loops}$ matrix, with $N_{\rm Loops} - N_\Omega$ unphysical zero modes.

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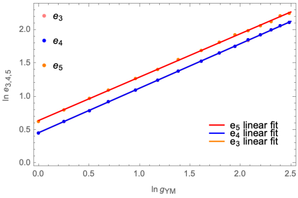
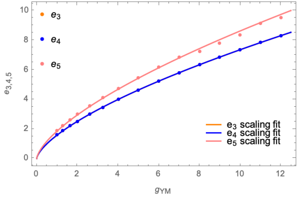
- Same analysis as before is carried out. Levels $3-15$ are shown in the table below

	Log linear fit		$p = 2/3$ fixed	Final
n	$\ln A_n$	p	Λ_n	Scaling function
e_3	0.4624(1)	0.66657(7)	1.58767(9)	1.588(1) $\lambda^{1/3}$
e_4	0.4627(1)	0.66656(6)	1.58806(8)	1.588(1) $\lambda^{1/3}$
e_5	0.645(6)	0.650(3)	1.862(8)	1.86(3) $\lambda^{1/3}$
e_6	0.873(6)	0.660(4)	2.373(8)	2.37(3) $\lambda^{1/3}$
e_7	0.885(3)	0.661(2)	2.406(5)	2.41(3) $\lambda^{1/3}$
e_8	1.09(1)	0.651(6)	2.91(2)	2.91(11) $\lambda^{1/3}$
e_9	1.112(7)	0.652(4)	2.98(1)	2.98(10) $\lambda^{1/3}$
e_{10}	1.159(3)	0.663(2)	3.170(5)	3.17(2) $\lambda^{1/3}$
e_{11}	1.167(2)	0.662(1)	3.191(5)	3.19(2) $\lambda^{1/3}$
e_{12}	1.34(2)	0.62(1)	3.57(5)	3.57(18) $\lambda^{1/3}$
e_{13}	1.336(6)	0.660(3)	3.77(1)	3.77(6) $\lambda^{1/3}$
e_{14}	1.361(5)	0.657(3)	3.85(1)	3.85(7) $\lambda^{1/3}$
e_{15}	1.382(7)	0.655(4)	3.92(2)	3.92(8) $\lambda^{1/3}$

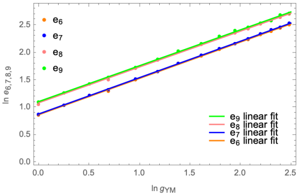
Spectrum patterns



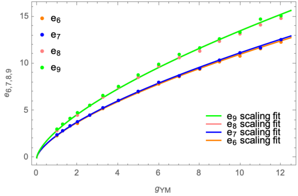
(m) Linear fit of the log of $e_{3,4,5}$ versus $\ln g_{YM}$. $e_{3,4}$ form a $l = \pm 2$ doublet, e_5 is a $l = 0$ singlet



(n) Fit of the $n = 3, 4, 5$ masses to scaling functions $\Lambda_{3,4,5}\ g_{YM}^{2/3}$



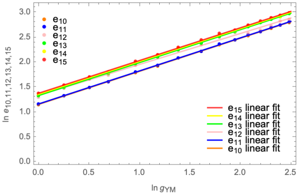
(o) Linear fit of the log of $e_{6,7,8,9}$ versus $\ln g_{YM}$. $e_{6,7}$ form a $l = \pm 3$ doublet and $e_{8,9}$ a $l = \pm 1$ doublet



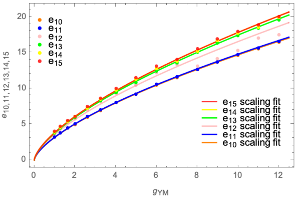
(p) Fit of $n = 6, 7, 8, 9$ masses to scaling function $\Lambda_{6,7,8,9}\ g_{YM}^{2/3}$

More on Spectrum

- Further spectrum energies



(q) Linear fit of the log of $e_{10,\dots,15}$ versus $\ln g_{YM}$.They form 3 $l = \pm 4, \pm 2, \pm 0$ doublets.



(r) Fit of the $n = 10, \dots, 15$ masses to scaling function $\Lambda_{10,11,12,13,14,15}\ g_{YM}^{2/3}$

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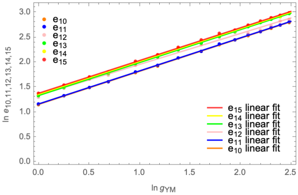

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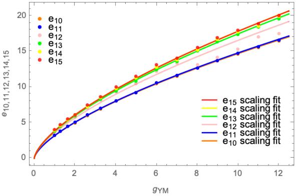
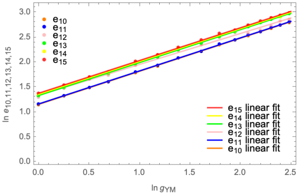


(t) Fit of the $n = 10, \ldots, 15$ masses to scaling function $\Lambda_{10,11,12,13,14,15}\ g_{YM}^{2/3}$

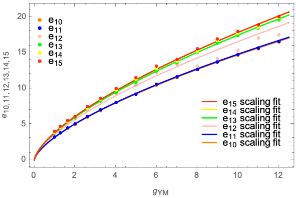
- For the lowest excited sates e_1 and e_2, numerically, their masses do not increase with the coupling, and remain very small compared with the other massive excited states. These are the $U(N)$ traced fundamental single particle states $\mathrm{Tr}X_1$ and $\mathrm{Tr}X_2$, and we associate them with the non interacting (free) $U(1) \times U(1)$ subgroup of the Hamiltonian. Numerically, one should recall that the eigenvalues of the mass matrix include $N_{\mathrm{loops}} - N_\Omega$ unphysical zero eigenvalues, so these modes will mix with physical zero modes if present in the system.

More on Spectrum

- Further spectrum energies



(u) Linear fit of the log of $e_{10,\ldots,15}$ versus $\ln g_{YM}$.They form 3 $l = \pm 4, \pm 2, \pm 0$ doublets.



(v) Fit of the $n = 10, \ldots, 15$ masses to scaling function $\Lambda_{10,11,12,13,14,15}\ g_{YM}^{2/3}$

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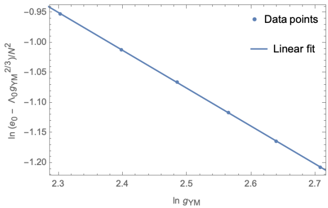
- In order to confirm numerically that, indeed, our interpretation that e_1 and e_2 are decoupled zero mass states, we "switch on" masses in the Hamiltonian and seek evidence that indeed e_1 and e_2 remain decoupled states with masses equal to their "bare" masses. This will also allow us to compare our results with the few planar results available in the literature.

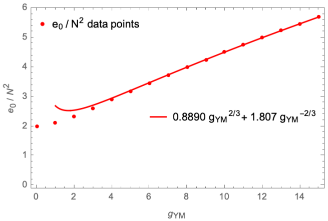
Y-M coupled matrices with masses - planar quantities

- Given that the leading large g_{YM} behaviour of the large N energy, that of the massless limit, has been established, we can obtain the next, mass dependent, power dependence on g_{YM}. The least squares fit result for the exponent is $-0.630(2)$, in other words $p = -2/3$ to a high degree of accuracy. Setting $p = -2/3$, we obtain at this truncation level:

$$e_0/N^2 = 0.8890(2)\,\lambda^{1/3} + 0.4518(1)\,\frac{m^2}{\lambda^{1/3}} + \ldots$$



(w) Strong coupling linear fit to $\ln(e_0 - \Lambda_0 g_{YM}^{2/3})/N^2$ versus $\ln g_{YM}$



(x) Fit of e_0/N^2 to mass corrected scaling function ($m = 2$)

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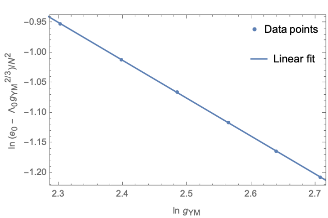

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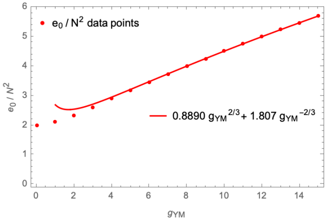

(z) Fit of e_0/N^2 to mass corrected scaling function ($m = 2$)

- The following table compares our large N planar results to those available in the literature.

	This article	[Morita,Yoshida, 2020]	[Han, Hartnoll,Krutho, 2020]
e_0/N^2	$0.8890(2)\lambda^{1/3} + 0.4518(1)\frac{m^2}{\lambda^{1/3}} + \ldots$	$0.882\lambda^{1/3} + \ldots$	$0.882\lambda^{1/3} + 0.401\frac{m^2}{\lambda^{1/3}} + \ldots$
$\mathrm{Tr}Z^\dagger Z/N^2$	$0.930(1)\lambda^{-1/3} + \ldots$	$0.913\lambda^{-1/3} + \ldots$	$0.968\lambda^{-1/3} + \ldots$

Spectrum with masses

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- For the next 3 states, we display the mass corrected large g_{YM} scaling function

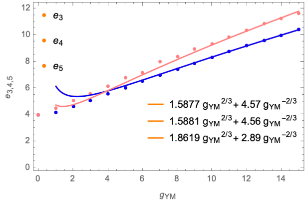

Figure: Numerical results for the masses $e_{3,4,5}$ and fit to mass corrected scaling functions.

3 massless matrices - preliminary results

- We considered 9 values of 't Hooft's coupling $\lambda = e^{-4}, e^{-3}, ..., e^{3}, e^{4}$, and a truncation with $l_{\max} = 10$, corresponding to 9503 N_{loops} and a 225×225 Ω matrix. For the master field, we took $N = 69$, corresponding to $2N^2 + N = 9591$ master variables.

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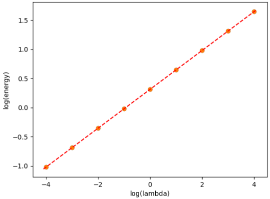
- The accuracy of the planar $O(3)$ symmetry is illustrated below

	$\lambda = e^{-4}$	$\lambda = 1$	$\lambda = e^{4}$
[11]	1.6622	0.4380	0.1155
[22]	1.6625	0.4382	0.1156
[33]	1.6625	0.4381	0.1156
[1111]	5.5667	0.3866	0.0269
[2222]	5.5687	0.3868	0.0269
[3333]	5.5683	0.3867	0.0269
[1122]	2.5855	0.1796	0.0125
[1133]	2.5856	0.1796	0.0125
[2233]	2.5848	0.1796	0.0125
[1212]	0.3985	0.0276	0.0019
[1313]	0.3984	0.0276	0.0019
[2323]	0.3981	0.0276	0.0019

Table: Discrete symmetries

- As for the 2 matrix case, we fit the planar ground state energies to the curve $e_0/N^2 = A_0\,\lambda^p$ by performing a least squares fit to the logarithmic plot, with result. We find $\ln A_0 = 0.3131\,, \quad p = 0.3333.$
- This linear fit is shown below



3 massless matrices - planar quantities and scaling

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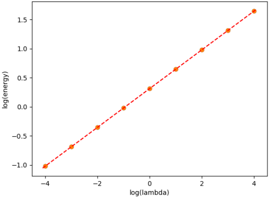

- Similarly for $\mathrm{Tr}X_1^2 + \mathrm{Tr}X_2^2 + \mathrm{Tr}X_3^2 = A_2\,\lambda^p$, we find $\ln A_2 = 0.2753\,, p = -0.3333$. The linear fit is shown below:

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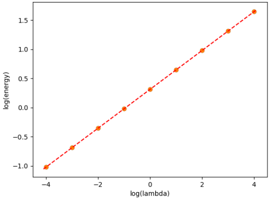

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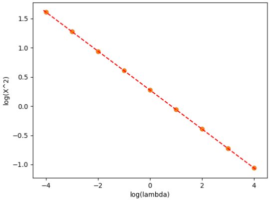

3 massless matrices - lowest bound states

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- Lowest non-zero spectrum states at 3 different coupling values:

	$\lambda = e^{-4}$	$\lambda = 1$	$\lambda = e^{4}$
e_4	0.0972	1.671	14.08
e_5	0.2427	3.500	50.10
e_6	0.2429	3.501	50.14
e_7	0.2430	3.503	50.18
e_8	0.2431	3.504	50.21
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- A clear pattern of degeneracies is evident, with a lowest mass singlet and a quintuplet (corresponding to a traceless symmetric matrix).

Summary

- We studied the large N dynamics of two massless Yang-Mills coupled matrix quantum mechanics, by minimization of a loop truncated Jevicki-Sakita effective collective field Hamiltonian.
- The loop space constraints are handled by the use of master variables.
- The method is successfully applied directly in the massless limit for a range of values of the Yang-Mills coupling constant, and the scaling behaviour of different physical quantities derived from their dimensions are obtained with a high level of precision.
- We consider both planar properties of the theory, such as the large N ground state energy and multi-matrix correlator expectation values, and also the spectrum of the theory.
- For the spectrum, we establish that the $U(N)$ traced fundamental constituents remain massless and decoupled from other states, and that bound states develop well defined mass gaps, with the mass of the two degenerate lowest lying bound states being determined with a particularly high degree of accuracy.
- Similar preliminary 3 matrix results were presented.

Open questions

- More matrices
- Quenched eigenvalues and $3d$ physics?
- BMN
- Supersymmetry
- More gravity properties?
- Finite temperature, ...

Thank you !