Spread Complexity and PT Symmetry

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based on [C Beetar, N Gupta, SS Haque, J Murugan, HJRvZ, 2312.15790]

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Talk Layout

Complexity

- Central question: How hard is it to synthesize a desired target state with the gates at your disposal?
- Need, $|\phi_r\rangle$, $|\phi_t\rangle$, $\{U_1, U_2, \cdots, U_n\}$, $g(U_1, U_2, \cdots, U_n)$
- Discrete notion of complexity closely related to quantum computational setups
- We will, however, be interested in a continuous notion of complexity

Nielsen Complexity

- Accessible gates are taken to be from some symmetry group

  [Nielsen, quant-ph/0502070]

- E.g. $SU(1,1)$: Gates $U = e^{\frac{s_1+is_2}{2}a^\dagger a^\dagger - \frac{s_1-is_2}{2}aa + i\frac{s_3}{4}(a^\dagger a + aa^\dagger)}$
- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$

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- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$
- We have a manifold of target states on which one can define a metric
- Complexity = shortest distance connecting points

Coherent States

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- Manifold of states $\Leftrightarrow$ group elements of G/H

[Perelemov, math-ph/0203002]

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  [Perelemov, math-ph/0203002]
- $U = e^{\frac{s_1+is_2}{2}a^\dagger a^\dagger - \frac{s_1-is_2}{2}aa + i\frac{s_3}{4}(a^\dagger a + aa^\dagger)} = e^{\frac{z}{2}a^\dagger a^\dagger} e^{\frac{z'}{4}(a^\dagger a + aa^\dagger)} e^{\frac{z''}{2}aa}$
- $U|0\rangle = N e^{\frac{z}{2}a^\dagger a^\dagger}|0\rangle \qquad |z| < 1$

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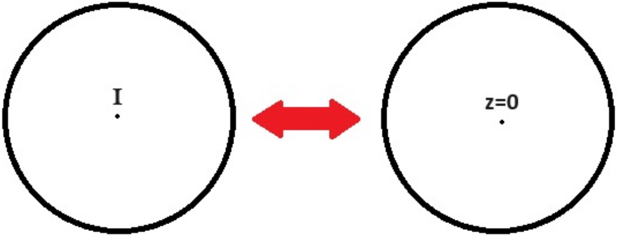
Coherent States



Spread Complexity

- A notion of complexity without the need to specify gates

  [Parker, Cao, Avdoshkin, Scaffidi, Altman, 1812.08657]

- Given a Hamiltonian and reference state one first builds the basis $|O_n) = H^n|\phi_r\rangle$

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- A notion of complexity without the need to specify gates

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- Given a Hamiltonian and reference state one first builds the basis $|O_n) = H^n|\phi_r\rangle$
- From a Gram-Schmidt process one then obtains the Krylov basis $|K_n\rangle$
- The K-complexity (or spread complexity) of a state is then given by $C_K = \sum_n n\langle\phi_t|K_n\rangle\langle K_n|\phi_t\rangle \equiv \langle\phi_t|\hat{K}|\phi_t\rangle$
- The Krylov basis provides an **ordered** basis for the Hilbert space of target states

Spread Complexity

- Given some basis for the Hilbert space of target space in increasing complexity $|B_n\rangle$
- We can define complexity as $C = \sum_n c_n \langle\phi_t|B_n\rangle\langle B_n|\phi_t\rangle$
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- With c_n strictly increasing
- The choice $|B_n\rangle = |K_n\rangle$ minimises the complexity of the time-evolved reference state [Balasubramanian, Caputa Magan, Wu, 2202.06957]

Spread Complexity

- Spread Complexity is a function of the target state, reference state and Hamiltonian $C(|\phi_t\rangle; |\phi_r\rangle, H)$
- Invariant under unitary transformations $C(|\phi_t\rangle; |\phi_r\rangle, H) = C(U|\phi_t\rangle; U|\phi_r\rangle, UHU^\dagger)$
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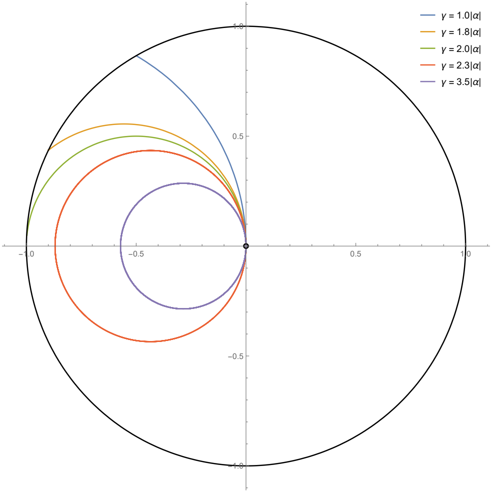
- If the Hamiltonian is an element from some symmetry algebra, the time-evolved reference state can be represented entirely on the manifold of coherent states

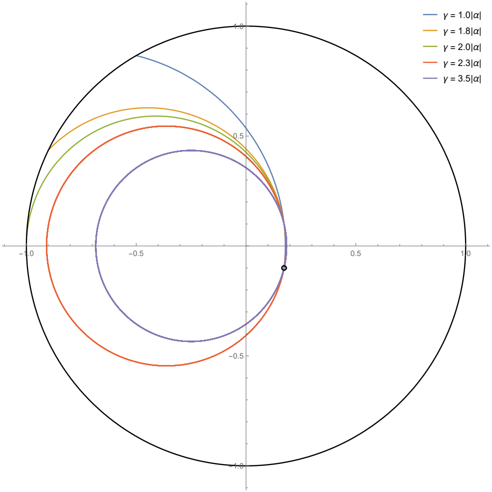
Single Oscillator

- $H = \frac{\alpha}{2} a^\dagger a^\dagger + \frac{\bar{\alpha}}{2} aa + \frac{\gamma}{4}(a^\dagger a + aa^\dagger)$



Single Oscillator

- $|K_0\rangle = Ne^{\frac{z_0}{2}a^\dagger a^\dagger}|0\rangle$



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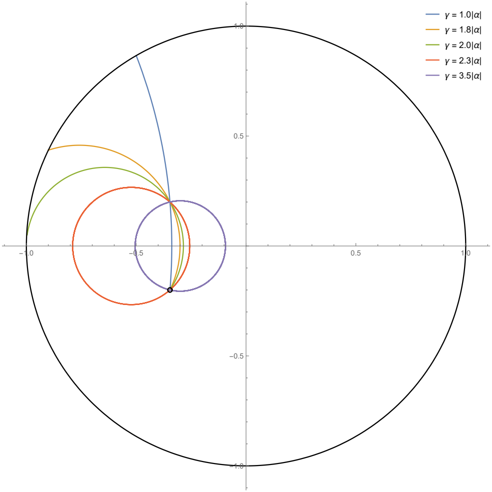

Orbits

- The orbits generated by the elements of $su(1,1)$ can be classified into hyperbolic, parabolic and elliptical

  [De Alfaro, Fubini, Furlan, 1976]

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- This is unchanged under unitary transformations
- Spread complexity is a field on the manifold of coherent states, given by $C(|z\rangle, |z_0\rangle) = \frac{(z-z_0)(\bar{z}-\bar{z}_0)}{(1-|z|^2)(1-|z_0|^2)}$

  [Chattopadhyay, Mitra, HJRvZ, 2302.10489]

- This is special (to rank 1 algebras) where the Krylov basis is always $|K_n\rangle \sim (a^\dagger)^{2n}|0\rangle$ [Caputa, Magan, Patramanis, 2109.03824]

Return Amplitude

- $R(t) = \langle K_0 | e^{-itH} | K_0 \rangle$
- Lanczos coefficients:
  $H|K_n\rangle = a_n|K_n\rangle + b_n|K_{n-1}\rangle + b_{n+1}|K_{n+1}\rangle$
- Probability amplitudes: $\phi_n(t) = \langle K_0 | e^{-itH} | K_n \rangle$
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  $it\partial_t\phi_n(t) = a_n\phi_n(t) + b_{n-1}\phi_n(t) + b_{n+1}\phi_{n+1}(t)$
- Can also generate these from the return amplitude

$$
\begin{aligned}
\phi_n(t) &= \sum_{m=0}^{n+1} k_{m,n+1}\partial_t^m R(t) \\
k_{m,n+1} &= \frac{ik_{m-1,n} - a_n k_{m,n} - b_n k_{m,n-1}}{b_{n+1}}
\end{aligned}
$$

Coupled Oscillators

- We will consider the e.o.m. [Bender, Gianfrida, 2013]

$$\begin{aligned} \ddot{x} + \omega^2 x + \mu \dot{x} &= -\epsilon y \\ \ddot{y} + \omega^2 y - \nu \dot{y} &= -\epsilon x \end{aligned}$$

- Coupled oscillators that are damped (x) and anti-damped (y)
- Balanced loss and gain achieved when $\mu = \nu \equiv 2\gamma$
- These equations can be derived from the Hamiltonian $H = p_x p_y + \gamma(y p_y - x p_x) + \varkappa xy + \frac{\epsilon}{2}(x^2 + y^2)$
- $\varkappa \equiv \omega^2 - \gamma^2$
- PT symmetric $P : x \leftrightarrow -y \;,\; p_x \leftrightarrow -p_y \qquad T : p_a \rightarrow -p_a$
- PT symmetric systems with balanced loss gain exhibit phase transitions

Coupled Oscillators

- Equations of motion

$$\begin{aligned} \ddot{x} + \omega^2 x + 2\gamma\dot{x} &= -\epsilon y \\ \ddot{y} + \omega^2 y - 2\gamma\dot{y} &= -\epsilon x \end{aligned}$$

- At the classical level solutions of the form $x = e^{i\lambda t}$ satisfies
- $\lambda^4 - 2(\omega^2 - 2\gamma^2)\lambda^2 + (\omega^4 - \epsilon^2) = 0$
- Different phases depending on the realness of the frequencies - on the quantum level this leads to phases of unbroken or broken PT symmetry

Coupled Oscillators

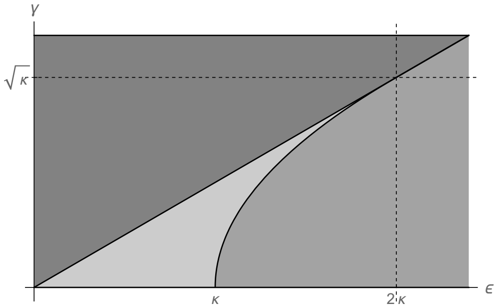

Figure: In the underdamped case there are three distinct regimes for the frequencies. If $\epsilon > 2\sqrt{\varkappa}\gamma$ and $\epsilon < \gamma^2 + \varkappa$ the frequencies are real - this corresponds to the lightest shading above. This window closes as γ approaches $\sqrt{\varkappa}$ and after $\gamma > 2\sqrt{\varkappa}$ it is no longer possible to obtain two real frequencies. ($\varkappa \equiv \omega^2 - \gamma^2$)

Coupled Oscillators

- $H = p_x p_y + \gamma(y p_y - x p_x) + \varkappa xy + \frac{\epsilon}{2}(x^2 + y^2)$
- Can define oscillators $a = \frac{x + i p_x}{\sqrt{2}} \quad ; \quad b = \frac{y + i p_y}{\sqrt{2}}$
- Hamiltonian is an element of the algebra spanned by $\left\{aa, a^\dagger a^\dagger, a^\dagger a + a a^\dagger\right\} \cup \left\{bb, b^\dagger b^\dagger, b^\dagger b + b b^\dagger\right\} \cup \left\{ab, a^\dagger b, b^\dagger a, a^\dagger b^\dagger\right\}$
- Manifold of coherent states $|z_{aa}, z_{bb}, z_{ab}\rangle = N e^{\frac{z_{aa}}{2} a^\dagger a^\dagger} e^{\frac{z_{bb}}{2} b^\dagger b^\dagger} e^{z_{ab} a^\dagger b^\dagger} |0, 0\rangle$
- $e^{-itH}|z_{aa}, z_{bb}, z_{ab}\rangle = |z_{aa}(t), z_{bb}(t), z_{ab}(t)\rangle$

Results

- Return amplitude is of the schematic form

$$\langle z_{aa}, z_{bb}, z_{ab}|e^{-itH}|z_{aa}, z_{bb}, z_{ab}\rangle \sim (1+\sum_{a,b=\pm} c_{ab}e^{i(a\omega_{+}+b\omega_{-})t})^{-\frac{1}{2}}$$

- $\omega_{\pm} = \sqrt{\varkappa - \gamma^2 \pm \sqrt{\epsilon^2 - 4\gamma^2\varkappa}}$
- The frequencies **do not** depend on the choice of reference coherent state
- The class of orbit is determined solely by the frequencies
- Reference state: $e^{-\frac{i}{4}\log(\Omega)(xp_x+p_x x+yp_y+p_y y)}|0,0\rangle$

Underdamped, strong coupling

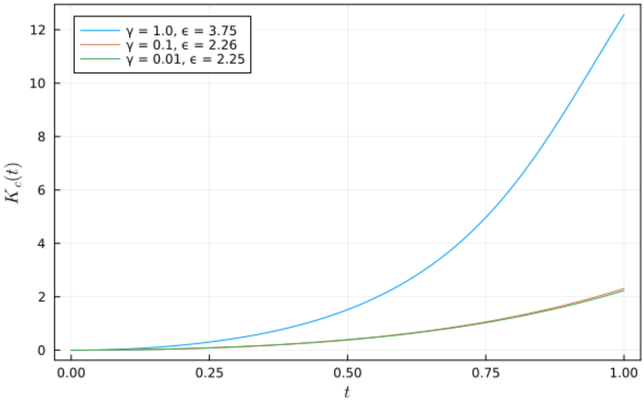

Figure: The spread complexity for $\varkappa = 1.5$, $\Omega = 1$, and $N = 80$ for various values of γ. At least, 99% of the probability is captured.

Underdamped, Rabi Oscillations

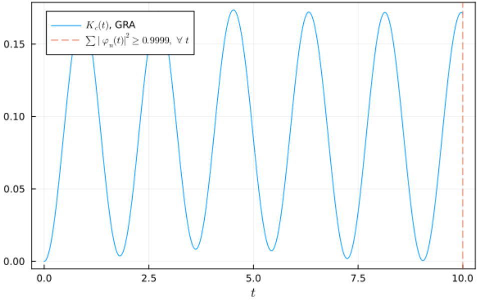

Figure: The spread complexity approximated with the first 40 probability amplitudes for $\varkappa = 2.0, \ \Omega = 1.0, \gamma = 0.01, \epsilon = 1.014$. At least 99.99% of the time-evolved reference state is captured.

Underdamped, Weak Coupling

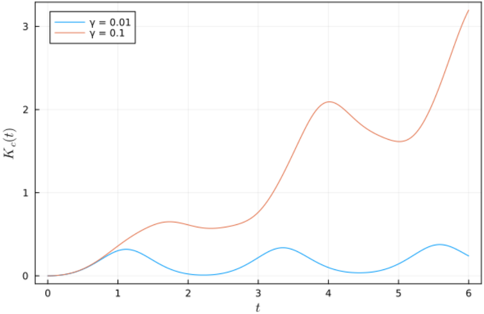

Figure: The spread complexity of the time-evolved reference state as a function of time with $\varkappa = 2.0$, $\epsilon = 0.0$, and varying values for γ for the under-damped weakly coupled system. The first 50 probability amplitudes are included in the summation, capturing at least 95% of the probability.

Conclusions

- Spread Complexity is capable of differentiating between the PT-symmetric and broken PT-symmetric phases of coupled oscillators
- In the PT-symmetric phase, the spectrum is real and eigenfunctions are normalisable - this gives rise to elliptical orbits on the manifold of coherent states. The complexity is **bounded**
- In fact, spread complexity can also differentiate between the weakly coupled and strongly coupled broken PT-symmetric phases
- The bounded nature of complexity may be a generate property unbroken PT-symmetric phases
- Would be fascinating to compare these results with an open system description

Thank you for your attention!

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