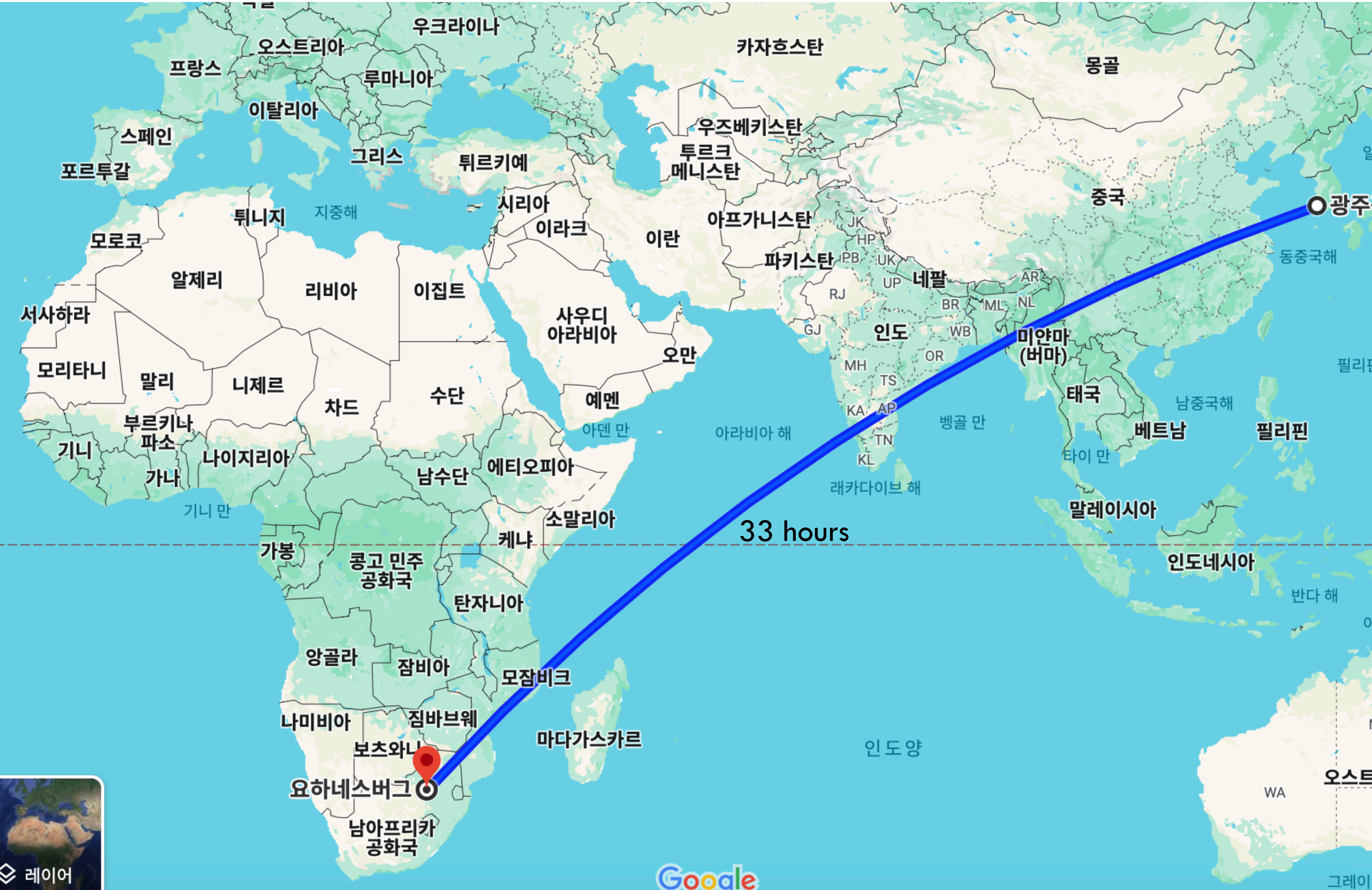
Sixth Mandelstam Theoretical Physics School and Workshop 2024

Comments on Quantum Chaos and Complexity

2024.01.15

Keun-Young Kim



33 hours
광주공
요하네스버그
우크라이나
카자흐스탄
몽골
오스트리아
프랑스
루마니아
이탈리아
스페인
포르투갈
그리스
튀르키예
우즈베키스탄
투르크메니스탄
중국
일본
지중해
튀니지
모로코
시리아
이라크
이란
아프가니스탄
파키스탄
네팔
동중국해
알제리
리비아
이집트
서사하라
사우디 아라비아
오만
인도
미얀마 (버마)
모리타니
말리
니제르
차드
수단
예멘
태국
남중국해
필리핀
부르키나 파소
기니
가나
나이지리아
아덴 만
아라비아 해
벵골 만
베트남
타이 만
남수단
에티오피아
래카다이브 해
말레이시아
기니 만
소말리아
케냐
가봉
콩고 민주 공화국
인도네시아
반다 해
탄자니아
앙골라
잠비아
모잠비크
나미비아
짐바브웨
보츠와나
마다가스카르
인도양
남아프리카 공화국
오스트레일리아
WA
레이어

Comments on Quantum Chaos and Complexity

- Considering a broad range of the audience
  I would like to take this opportunity to introduce these very interesting topics (to me) but boring looking subjects (chaos?, complexity?) to hep-th/gr people.
  In the end of my talk, however, I hope you get to like them like me.

- I will try to convey motivations, history, basic ideas of the topics instead of too much technical details.

Comments on Quantum Chaos and Complexity

Both are not well defined yet.

So, the current status of the research strategies is based on educated guess, imagination, and trial and errors, but not the rigorous proof.

Comments on Quantum Chaos and Complexity

For fun,

Let’s ask AI (Artificial Intelligence)

AI-painter "MidJourney" wins 1st prize at Colorado State Fair

"Space Opera Theater" by Midjourney

Quantum chaos and complexity

Midjourney

Quantum Chaos

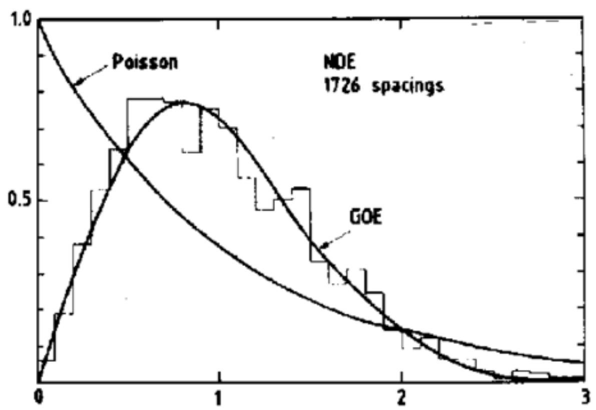
$$|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}$$

$$-\langle [q^i(t), p^j(0)]^2 \rangle_\beta,$$

$$-\langle [V(t), W(0)]^2 \rangle_\beta \quad \sim e^{\lambda t}$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Random Matrix Theory

Thermalization
(ETH, Quantum device)
Quantum black holes
Quantum gravity

FO What is quantum chaos?

ChatGPT

Quantum chaos is a field of physics that studies the behavior of quantum systems that are classically chaotic. Classically chaotic systems are those that exhibit sensitive dependence on initial conditions, meaning small changes in the initial conditions lead to vastly different outcomes over time.

In quantum mechanics, chaotic behavior can manifest in the statistical properties of the system's energy spectrum or in the time evolution of its wave function. The study of quantum chaos is concerned with understanding how the behavior of the system changes as classical chaos is introduced or increased, and how this behavior can be observed and measured experimentally.

Quantum chaos has applications in various fields, including solid-state physics, quantum computing, and quantum information theory. It has also led to new insights into the fundamental nature of quantum mechanics and the connections between classical and quantum physics.

Comments on Quantum Chaos and Complexity

Now

Let’s get more serious

A Universal Operator Growth Hypothesis

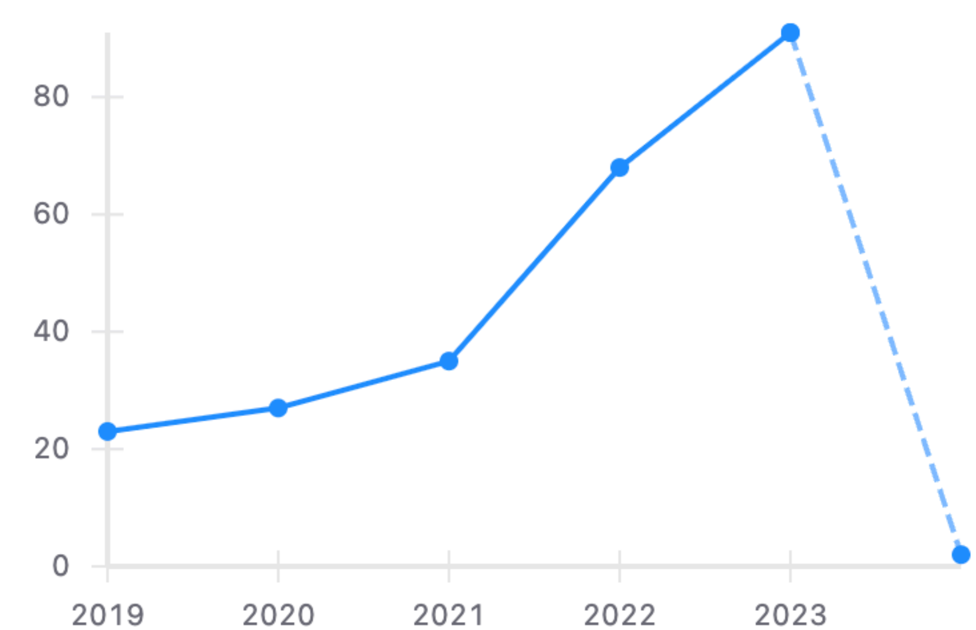
Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley)

Dec 20, 2018

29 pages
Published in: *Phys.Rev.X* 9 (2019) 4, 041017
Published: Oct 24, 2019
e-Print: 1812.08657 [cond-mat.stat-mech]
DOI: 10.1103/PhysRevX.9.041017 (publication)
View in: ADS Abstract Service

Krylov Complexity: a new diagnose of quantum chaos

Citations per year



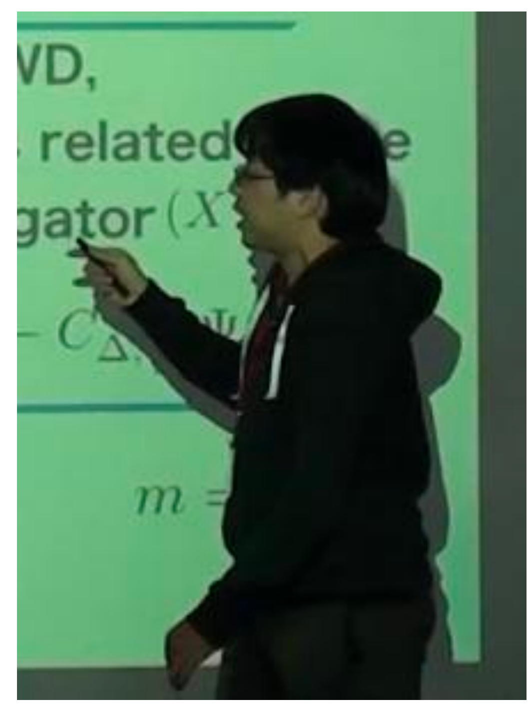
arXiv > hep-th > arXiv:2212.14702

High Energy Physics – Theory

[Submitted on 30 Dec 2022 (v1), last revised 25 Jan 2023 (this version, v2)]

Krylov Complexity in Free and Interacting Scalar Field Theories with Bounded Power Spectrum

Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, Mitsuhiro Nishida



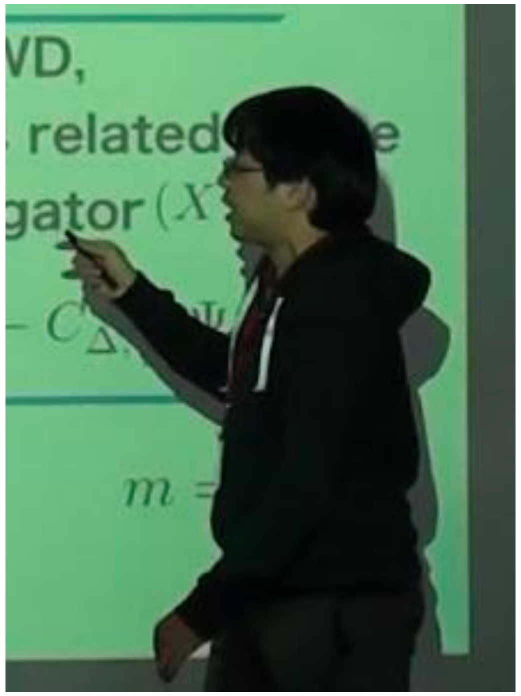
arXiv > hep-th > arXiv:2306.11632

High Energy Physics – Theory

[Submitted on 20 Jun 2023]

Spectral and Krylov Complexity in Billiard Systems

Hugo A. Camargo, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, Mitsuhiro Nishida



[Reminder]
Quantum Chaos
Why?

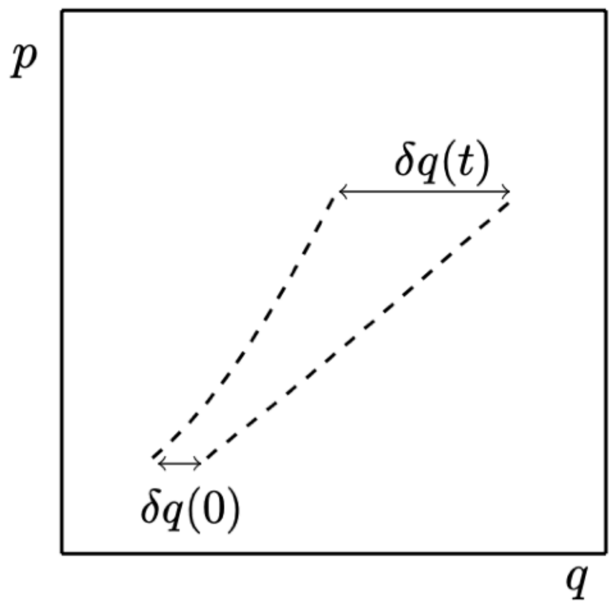
[Reminder]
Quantum Chaos
Why?

$\hbar \to 0$

Classical Physics ← Quantum physics

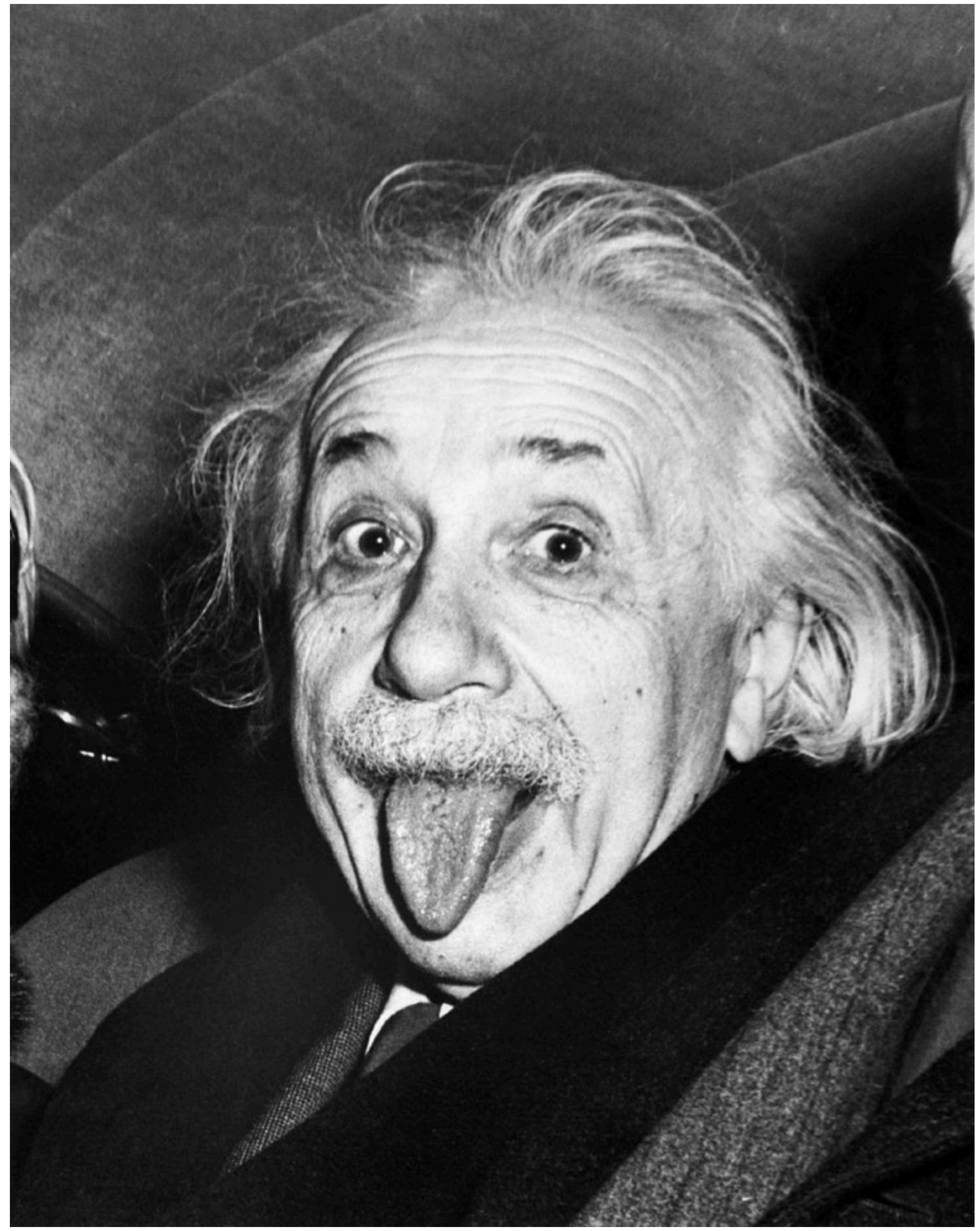
$\hbar \to 0$

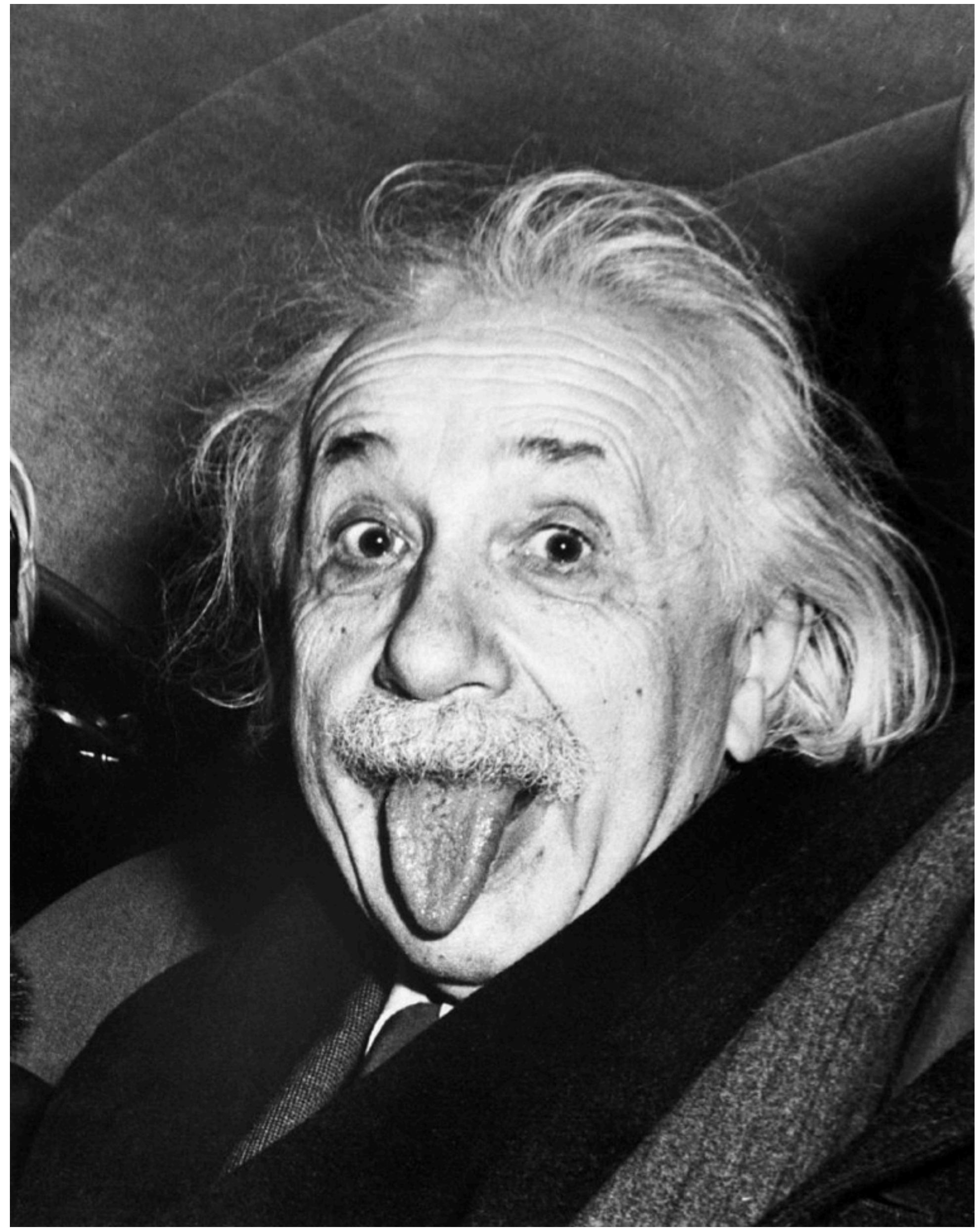
Classical Chaos ← Quantum chaos



- "Quantum" is more fundamental. "Classical" is approximation.
- Quantum chaos may exist even without classical counter part.
- We do not need to stick to classical concept.

Who is the first raising the issue of quantum chaos
In physics literature?

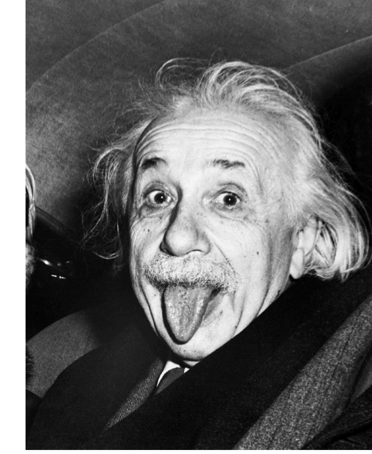




I have thought a hundred times as much about the quantum problems as I have about general relativity.

- 1905: Photon concept
- 1916: Quantum theory of radiation
- 1917: Quantum chaos
- 1925: Bose-Einstein condensation
- 1935: EPR paradox

First identification of the problem of quantizing chaotic motion



A. Einstein, *Zum quantensatz von sommerfeld und epstein, Deutsche physikalische Gesellschaft, Verhandlungen* **19** (1917) 82–92.

Before Schrödinger equation (1926)

434 DOC. 45 QUANTUM THEOREM

Doc. 45

[p. 82] **On the Quantum Theorem of Sommerfeld and Epstein**

by A. Einstein

(Presented at the session of May 11)

(cf. above, p. 79)

§1. Previous Formulation. There is hardly any more doubt that the quantum condition for periodic mechanical systems with one degree of freedom is (after SOMMERFELD
[1] and DEBYE)

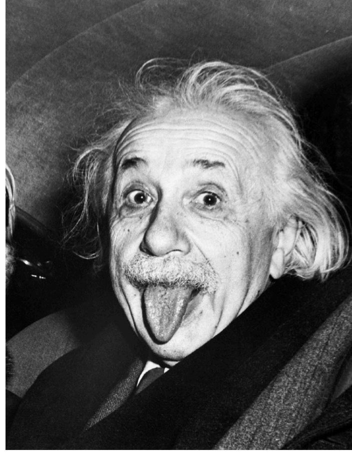
$$\int p dq = \int p \frac{dq}{dt} dt = nh. \qquad (1)$$

Non-integrable (chaotic) ⟶ How to quantize?

Non- chaotic ⟶ How to thermalize in quantum system?

Not ergodic?

First identification of the problem of quantizing chaotic motion



A. Einstein, *Zum quantensatz von sommerfeld und epstein*, *Deutsche physikalische Gesellschaft, Verhandlungen* **19** (1917) 82–92.

Forgotten for 55 years
And
Rediscovered (independently) in 1971

M. C. Gutzwiller, *Periodic orbits and classical quantization conditions*, *Journal of Mathematical Physics* **12** (1971) 343.

Regular and irregular spectra

I C Percival
Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, Colorado, 80302†
Physics Department, University of Stirling, Stirling, Scotland‡

Received 6 August 1973

$$I_k = (n_k + \text{constant})\hbar.$$

Apart from the phase space formulation and the possibility of a non-zero constant (Keller 1958) this result is due to Einstein (1917).

History of quantum chaos

A. Einstein, *Zum quantensatz von sommerfeld und epstein*, *Deutsche physikalische Gesellschaft, Verhandlungen* **19** (1917) 82–92.

M. C. Gutzwiller, *Periodic orbits and classical quantization conditions*, *Journal of Mathematical Physics* **12** (1971) 343.

Michael Berry

History of quantum chaos

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Berry Tabor conjecture

M.V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. Roy. Soc. A* **356** (1977) 375-394.

M.V. Berry, Quantizing a classically ergodic system: Sinai's billiard and the KKR method, *Ann. Phys.* **131** (1981) 163-216.

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BGS conjecture

O. Bohigas, M.-J. Giannoni and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.* **52** (1984) 1-4.

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Look at the energy level instead of the classical path

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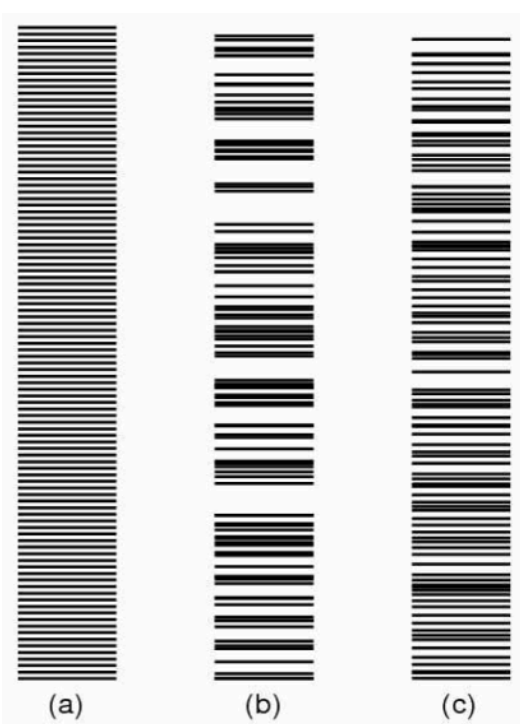
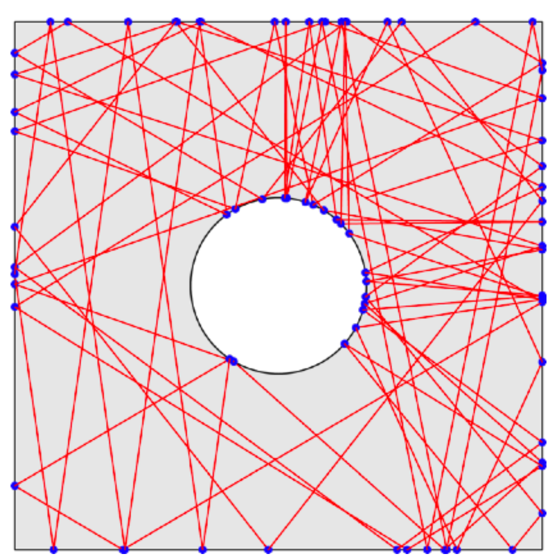
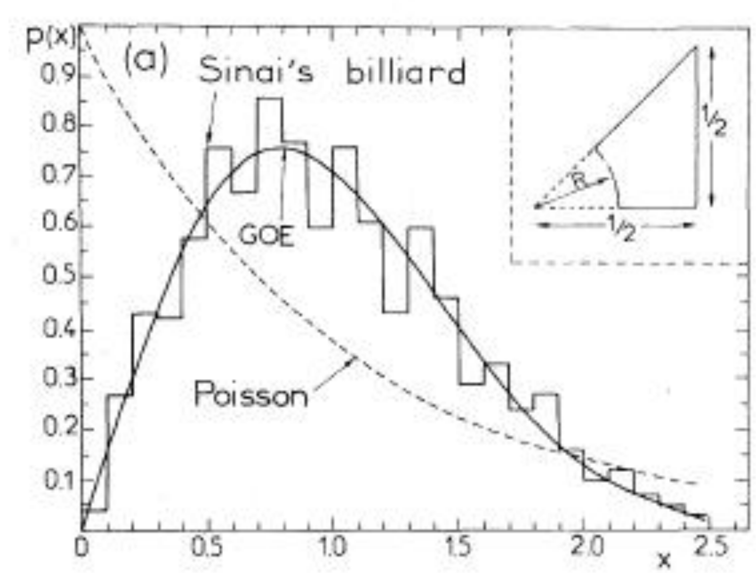


Sinai Billiard (classical)



Sinai Billiard (quantum)



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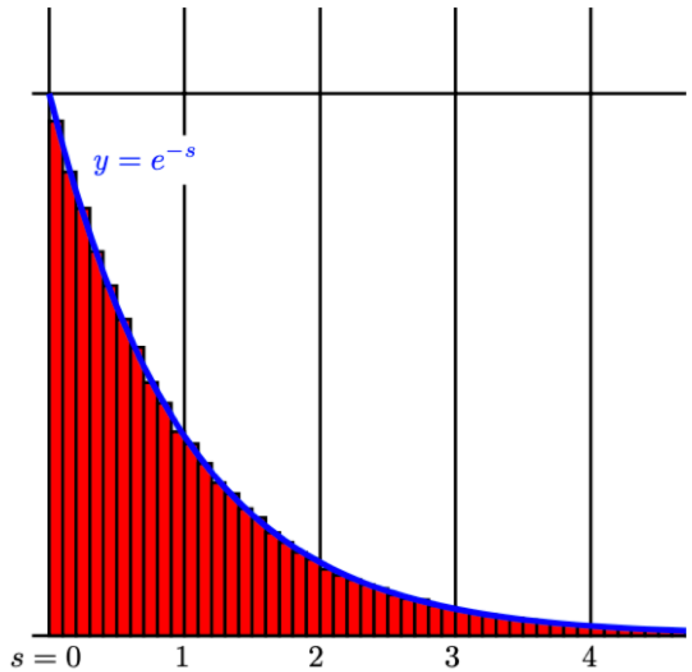
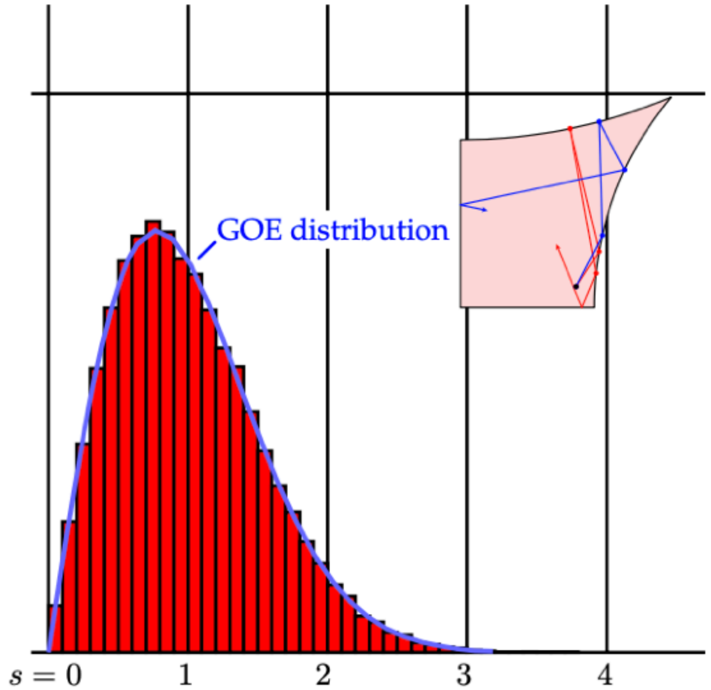
Look at the energy level instead of the classical path

Single particle in a cavity





Z. Rudnick, Notices of the AMS 55 (2008), p. 32.

History of quantum chaos

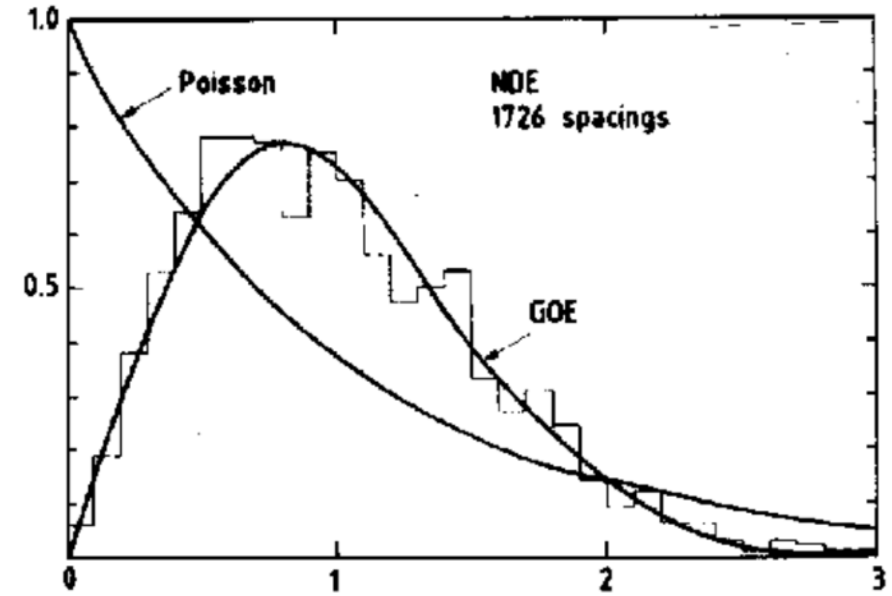
A. Einstein, *Zum quantensatz von sommerfeld und epstein, Deutsche physikalische Gesellschaft, Verhandlungen* **19** (1917) 82–92.

Heavy nuclei

Wigner E. Wigner, Ann. of Math. 62 (1955), pp. 548–564

- Hopeless to predict the exact energy levels of complex systems such as large nuclei
- Focus on statistical property
- Property of random matrix

chaotic?



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Holography

Infinitely strong interaction ~ universality ~ black hole

Maximally chaotic ~ universality ~ black hole

~ random (matrix)

Quantum Chaos

$$|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}$$

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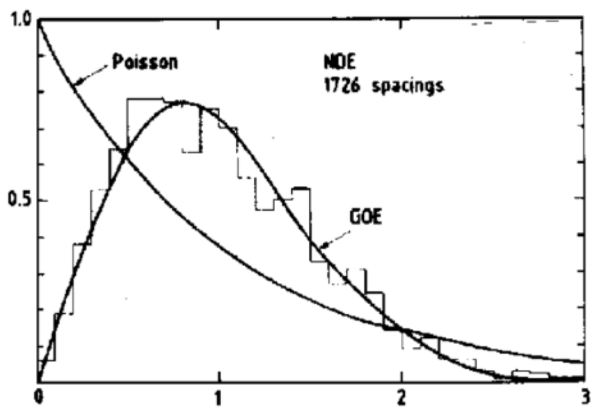
$$-\langle [V(t), W(0)]^2 \rangle_\beta \quad \sim e^{\lambda t}$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Random Matrix Theory

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(ETH, Quantum device)
Quantum black holes
Quantum gravity

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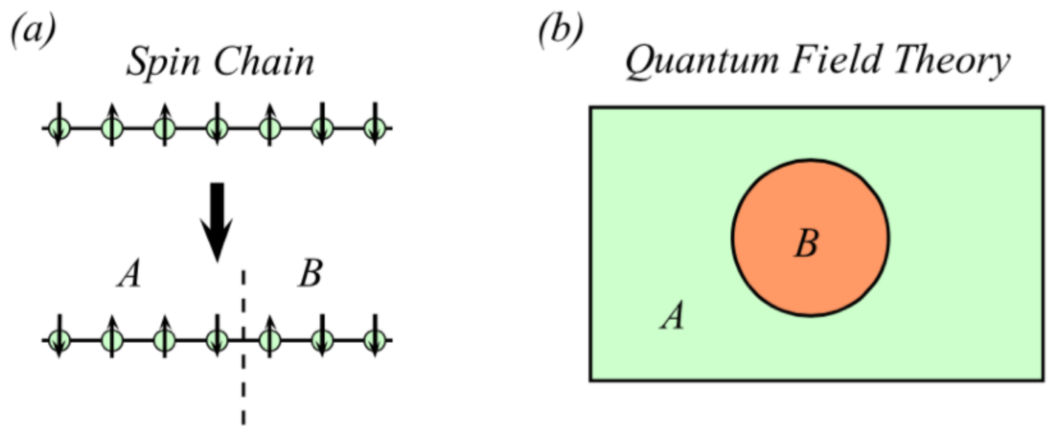
Krylov complexity?

Quantum complexity
why?

Quantum complexity
why?

Entanglement is not enough.
What else do we need?

Entanglement Entropy



$$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A$$

$$\rho_A = \mathrm{tr}_B\ \rho_{tot}$$

The Simplest Example: two spins (2 qubits)

(i) $|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\ \right]$

$$\Rightarrow \rho_A = \mathrm{Tr}_B\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right].$$

Not Entangled

$$S_A = 0$$

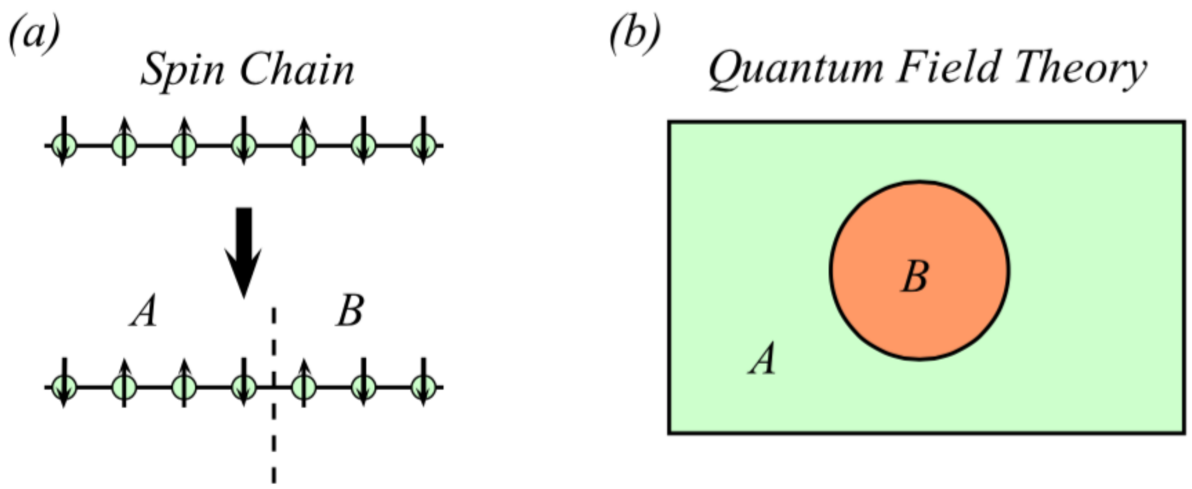
(ii) $|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\ \right] / \sqrt{2}$

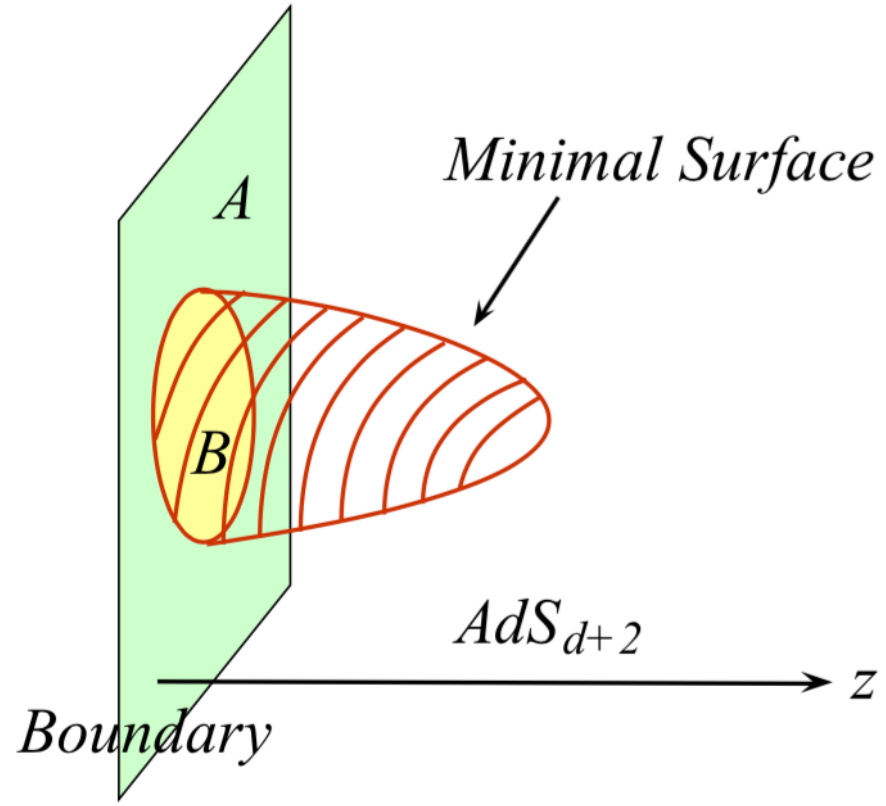
$$\Rightarrow \rho_A = \mathrm{Tr}_B\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right]$$

Entangled

$$S_A = \log 2$$

Entanglement Entropy

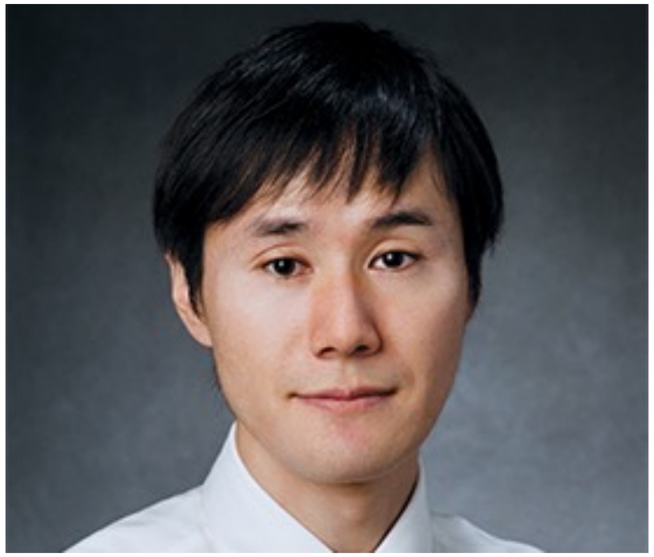




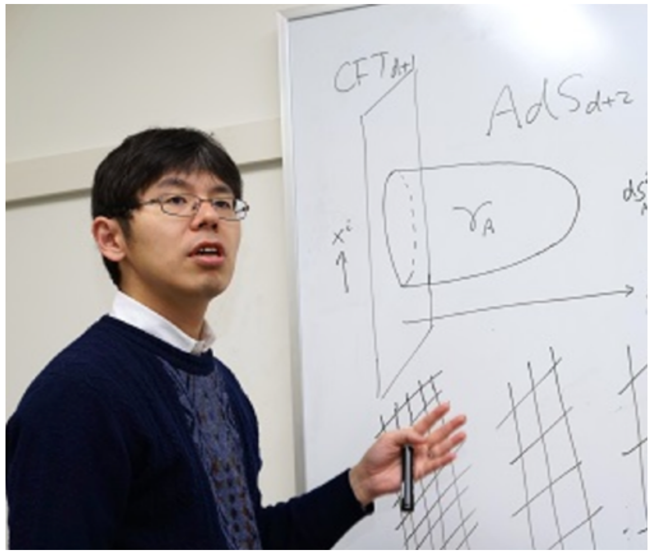
$$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A$$

$$\rho_A = \mathrm{tr}_B\ \rho_{tot}$$

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{(d+2)}}$$



Shines Ryu



Tadashi Takayanagi

Successful agreements
with field theory computation

Complexity

"Distance" between two sates?

(inner-product) distance: $d_{AB} = \arccos |\langle B|A\rangle|$ (closest) $0 \sim \pi/2$ (farthest)

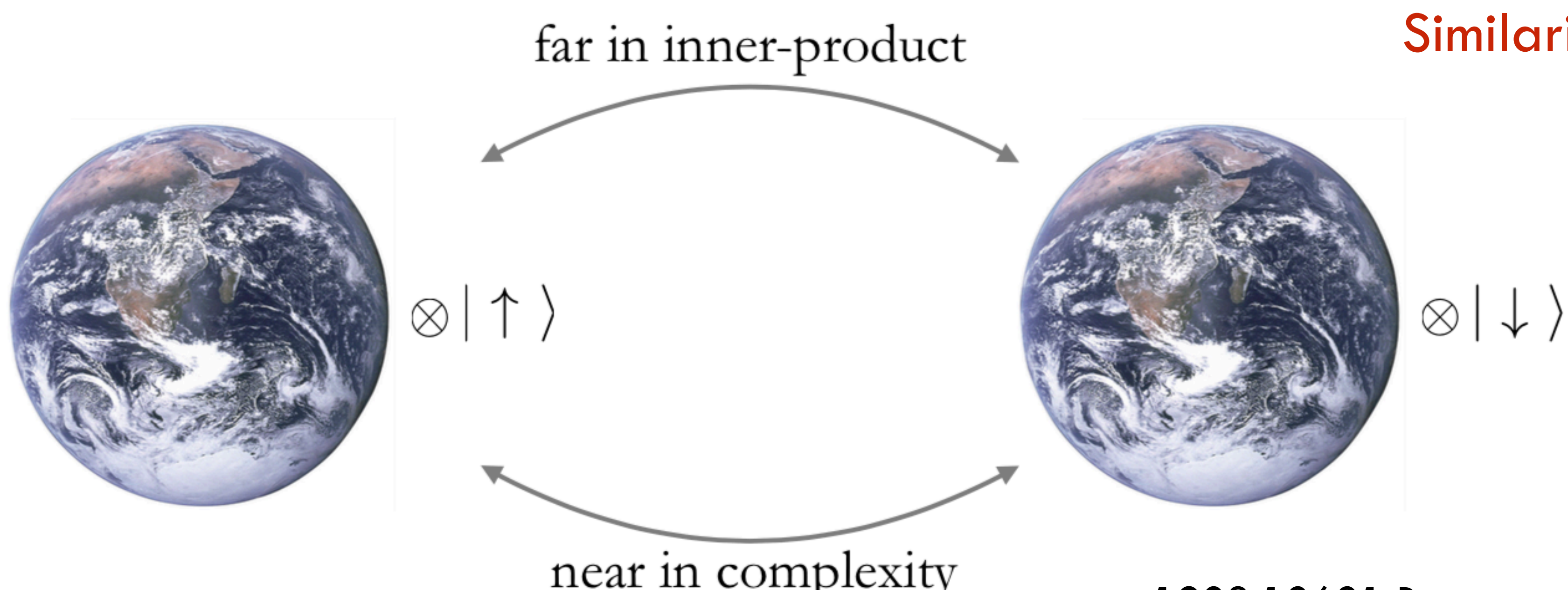
Are these close or far? $|0000000000\rangle \longrightarrow |0000000001\rangle$

Far in the inner-product sense

However, in some sense they are close → "easy" or "difficult" transform

Need a new distance reflecting this sense: "Complexity distance?"

Similarity vs Distance?



1903.12621 Brown and Susskind

Complexity

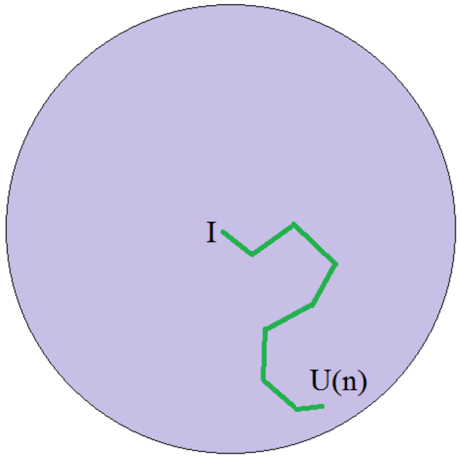
Complexity of quantum states New distance in Hilbert space

For given states $|\psi_T\rangle = U|\psi_R\rangle$ ~How hard (minimal number of gates) from the reference to target state

Complexity of operator (unitary transformation) New distance in Unitary group

For a given operator $U = g_n g_{n-1} \cdots g_2 g_1$ ~ minimum number of gates

$$\mathbb{I} \longrightarrow U$$



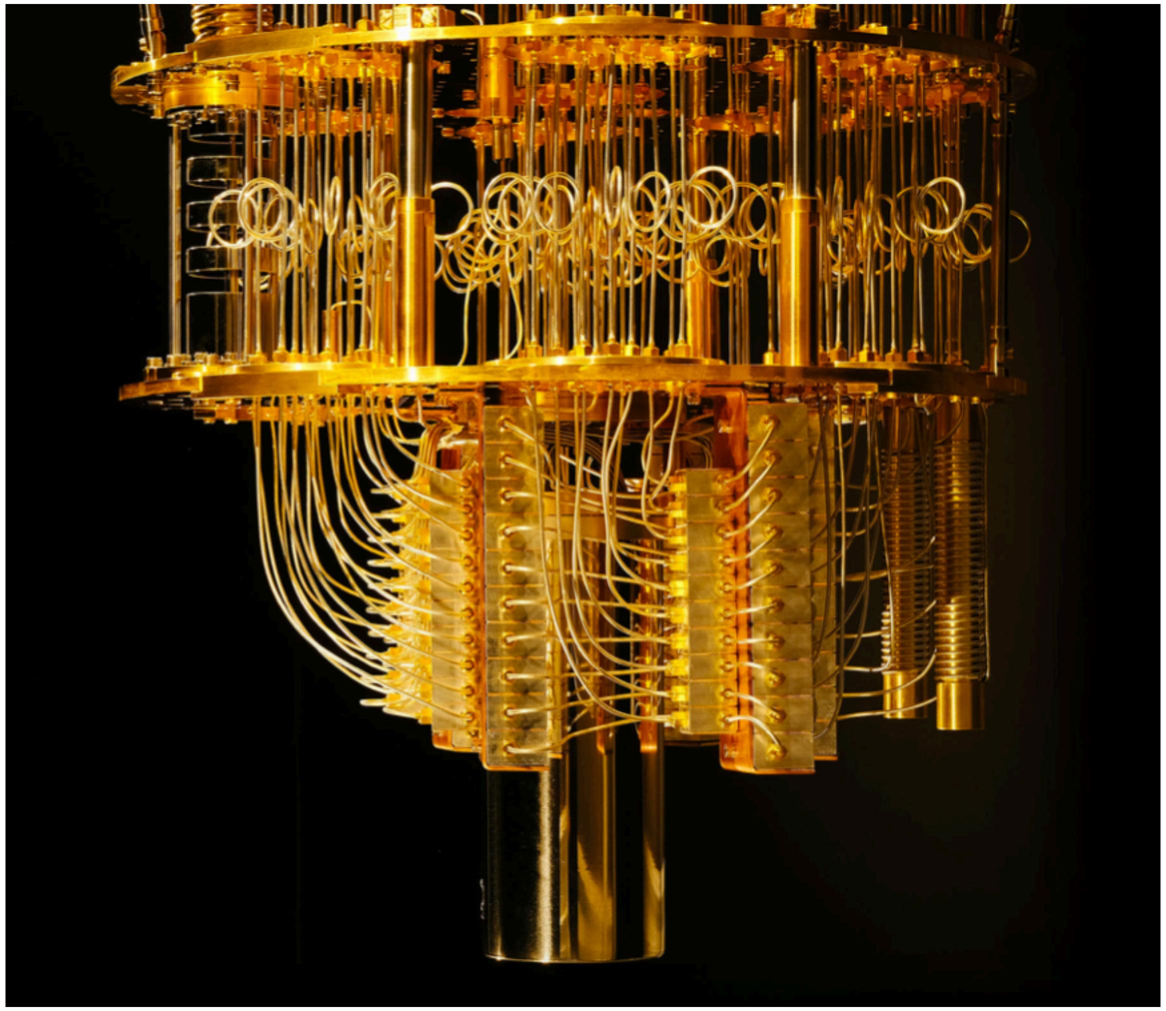
Relation between two

$$\mathcal{C}(|\psi_1\rangle, |\psi_2\rangle) = \min \left\{ \mathcal{C}(U) \mid \forall\, \hat{U} \in \mathcal{O}, \quad |\psi_2\rangle = \hat{U}|\psi_1\rangle \right\}$$

Complexity

(Computational) complexity [Computer science] quantifying the difficulty of carrying out a task.

Quantum Computer

Input state →  → output state

Complexity

(Computational) complexity [Computer science] quantifying the difficulty of carrying out a task.

(Circuit) complexity

Quantum Computer ~ Quantum Circuit

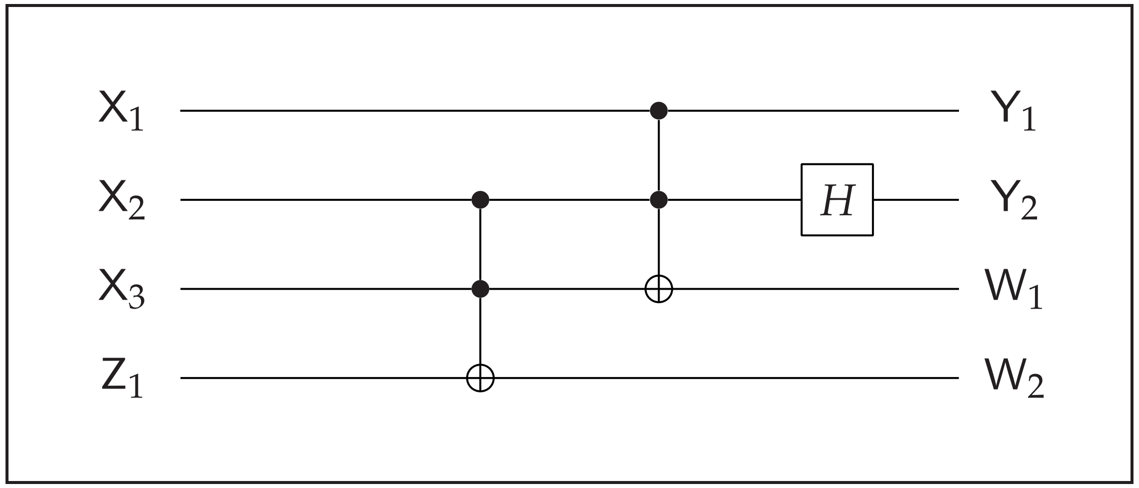
Minimal number of gates for the transformation from the reference to target state

Ambiguity

$$|\psi_T\rangle = U|\psi_R\rangle = g_n g_{n-1} \cdots g_2 g_1 |\psi_R\rangle$$

Universal gate sets = {a,b,c,d,e,f}

ex)



$$G = dbe$$
$$G = ceab$$
$$G = abefa$$

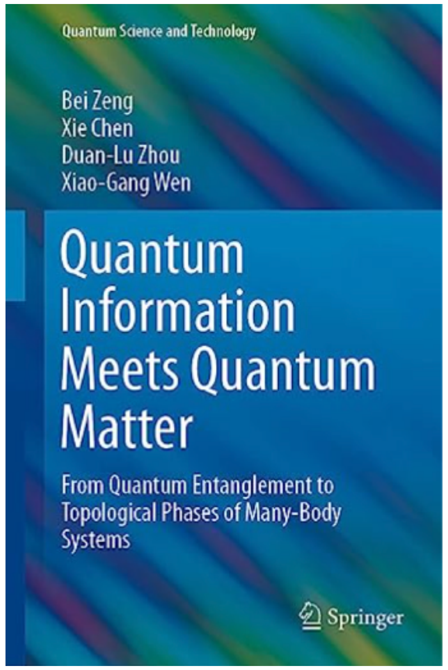
complexity = 3

Why chaos and complexity?

Chaos

What is thermalization?
Infinitely strong interaction ~ universality ~ black hole
Maximally chaotic. ~ universality ~ black hole

Complexity



Is quantum theory explored enough?
Entanglement?
Complexity?

- Complexity by definition has nothing to do with Chaos
- Complexity in principle has nothing to do with Hamiltonian

Complexity and chaos

Complexity: how much things are complex
Chaos: how fast things get complex
~ fast increase of complexity

Krylov Complexity

Krylov complexity

A Universal Operator Growth Hypothesis

Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley)
Dec 20, 2018

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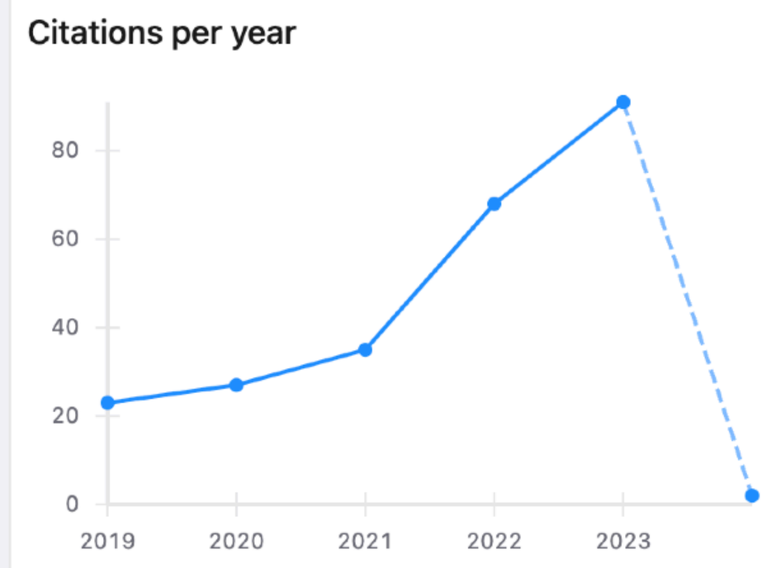

Circuit complexity in quantum field theory

Ro Jefferson (Perimeter Inst. Theor. Phys. and Amsterdam U.), Robert C. Myers (Perimeter Inst. Theor. Phys.)
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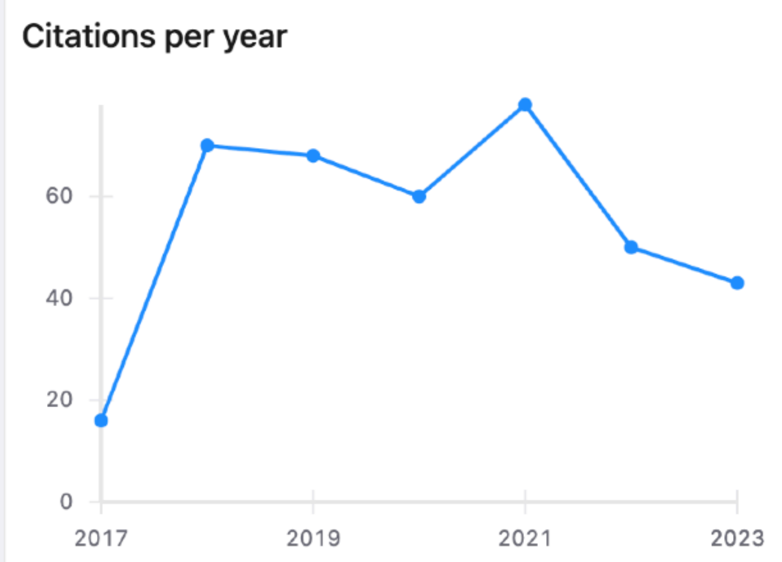
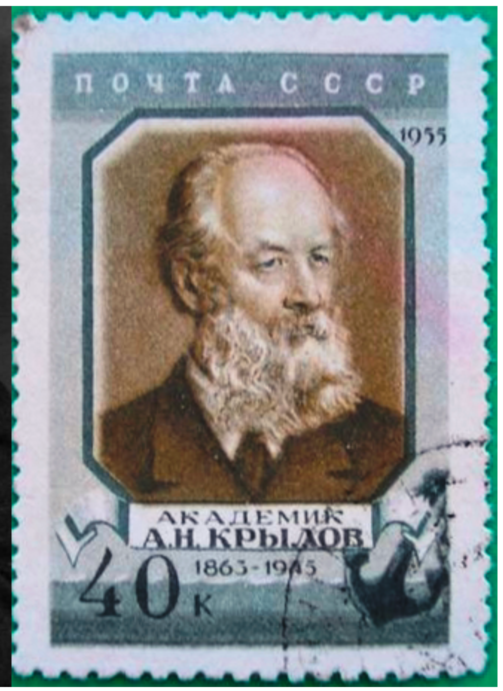


Circuit complexity is not well-defined

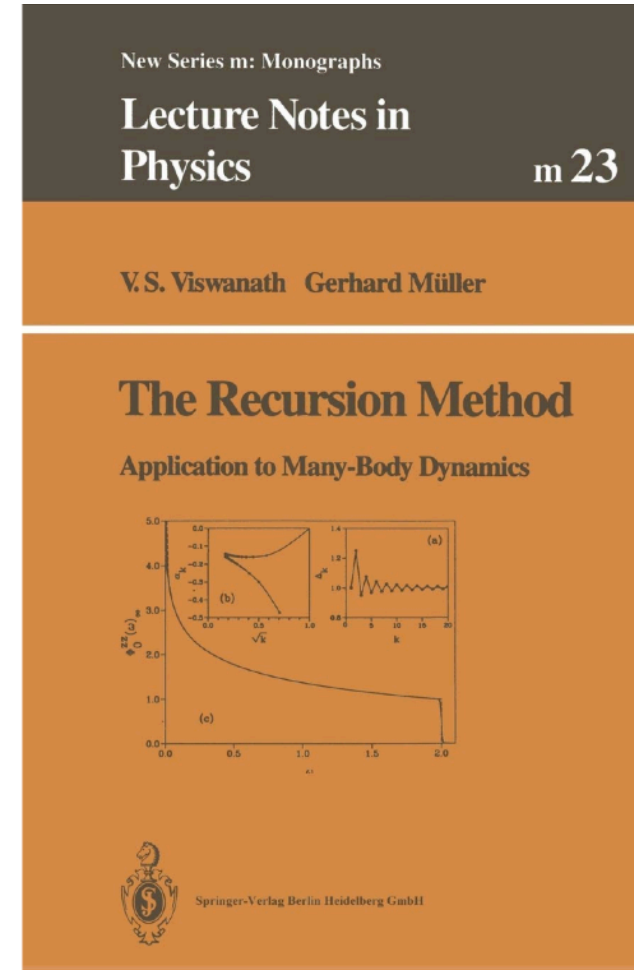
"Krylov complexity" is a well-defined concept
proposed as a diagnose of quantum chaos (which is not-well defined)

Contents

Aleksey Nikolaevich Krylov (1863 –1945)
a Russian naval engineer, applied mathematician



- Short Review on Krylov Complexity
  - Operator growth
  - Krylov space
  - Lanczos coefficient
  - Krylov complexity
- Success in lattice systems
- Towards field theory
  - Too good to be true
  - How to extract info from the power spectrum (IR/UV cutoff effect)



1994

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish mathematician and physicist.

Operator growth

1812.08657: Parker et al.

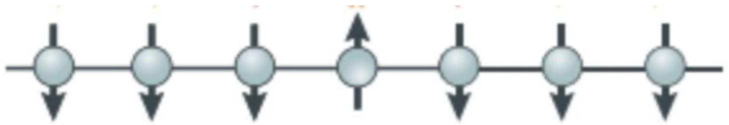
The time evolution of an operator O by a time independent Hamiltonian *H*

$$\partial_t \mathcal{O}(t) = i\ [H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\ \mathcal{O}(0)\ e^{-itH}$$

Baker-Campbell-Hausdorff (BCH) formula $e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_X^n Y}{n!}$

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

ex) 1D spin chain 

$$H = -\sum (Z_i Z_{i+1} + gX_i + hZ_i)$$

$$Z_1(t) = Z_1 + it[H,Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$\begin{aligned}
[H, Z_1] &\sim Y_1 \\
[H,[H, Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H, Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H, Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \\
&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

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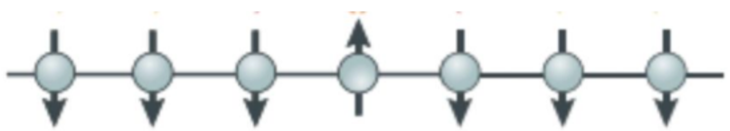
$$\mathcal{O}(t) = \mathcal{O}_0 + it[H, \mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \ .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!} \tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

ex) 1D spin chain



$$H = -\sum (Z_i Z_{i+1} + g X_i + h Z_i)$$

$$Z_1(t) = Z_1 + it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$[H, Z_1] \sim Y_1$$

$$[H,[H,Z_1]] \sim Y_1 + X_1 Z_2$$

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$$[H,[H,[H,[H,Z_1]]]] \sim X_1 + Y_1 + Z_1 + X_1X_2 + Y_1Y_2 + Z_1Z_2 + X_1Z_2 + Z_3Y_1 + Y_1Z_2Y_2 + Z_1X_2X_1 + X_2Z_3X_1$$

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$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}\left(e^{-\beta H}\right)$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

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$$\partial_t|\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\;\mathcal{O}(0)$$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t)|\mathcal{O}_n)$$

$$\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

"probability amplitudes"

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

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$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

Krylov complexity

Discrete "Schrodinger equation"

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

"probability amplitudes"

$$\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) |\mathcal{O}_n) \qquad \varphi_n(t) := i^{-n} (\mathcal{O}_n|\mathcal{O}(t))$$

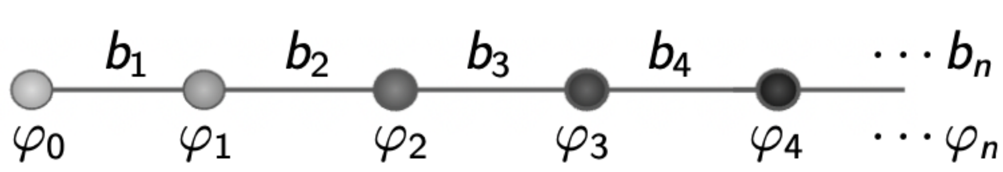
$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = b_n \delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

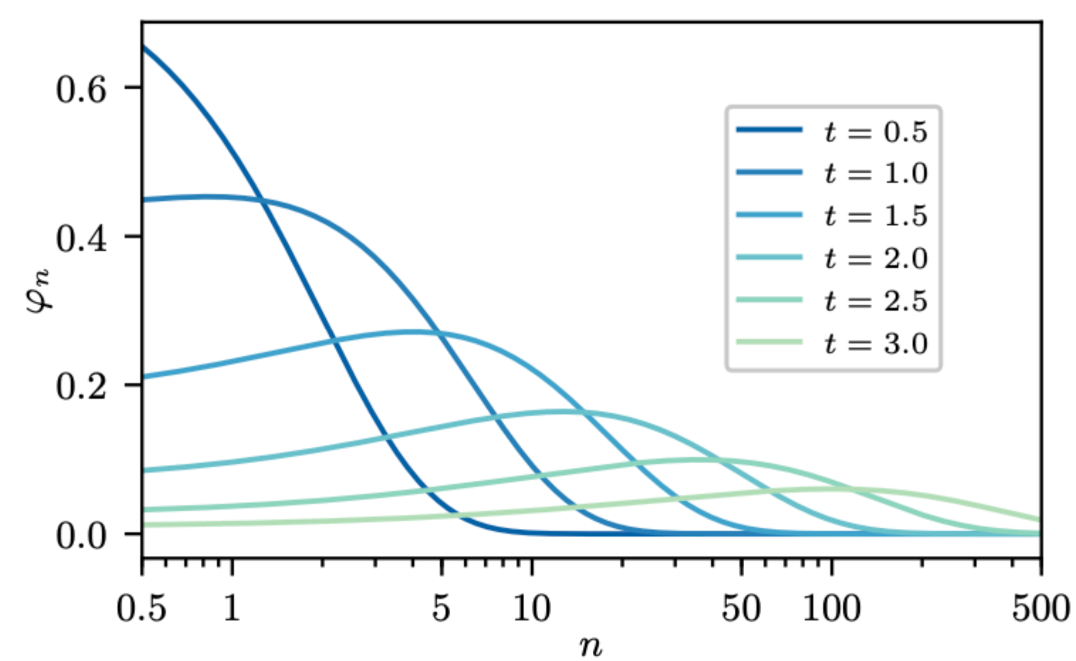
$$\frac{\mathrm{d}\,\varphi_n(t)}{\mathrm{d}t} = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0} \qquad \varphi_{-1}(t) \equiv 0 \equiv b_0$$

$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot{\varphi}_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

a quantum-mechanical particle on a 1- dimensional chain.

b_n = hopping amplitudes



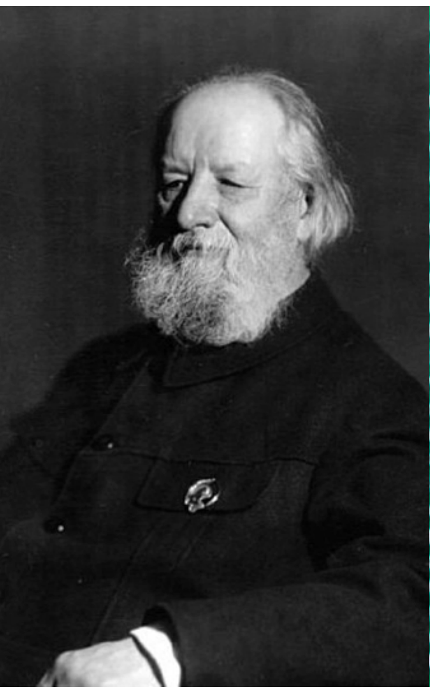


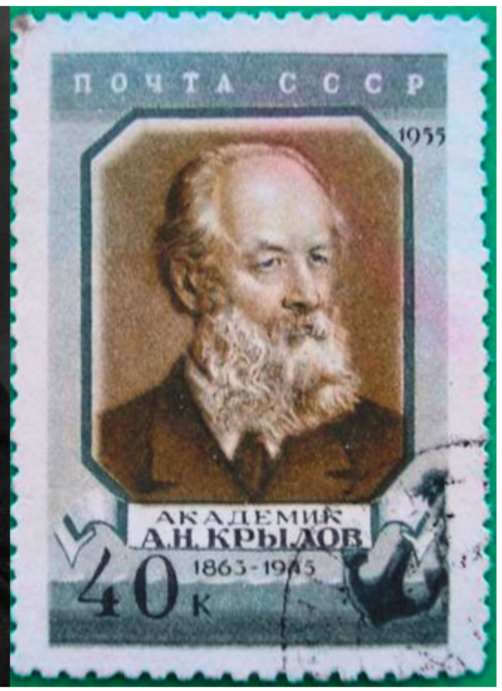
Krylov complexity

average position over the chain

$$K_{\mathcal{O}}(t) := (\mathcal{O}(t)|n|\mathcal{O}(t)) = \sum_{n=0}^{\infty} n|\varphi_n(t)|^2$$

Aleksey Nikolaevich Krylov (1863 –1945)
a Russian naval engineer, applied mathematician





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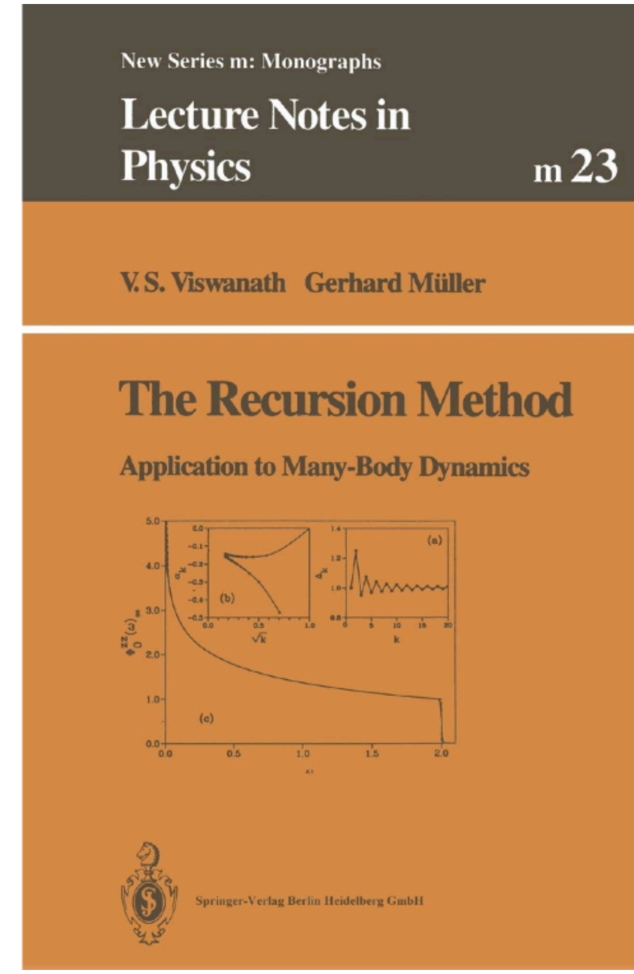

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Cornelius (Cornel) Lanczos (1893-1974):
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K-complexity from the auto-correlation function

1812.08657: Parker et al.

Auto-correlation function

$C(t) = \Pi^W(t) = \varphi_0(t)$

$$C(t) := (\mathcal{O}(t)|\mathcal{O}(0)) = \varphi_0(t)$$
$$= \langle e^{i\,(t-i\,\beta/2)H}\mathcal{O}^\dagger(0)e^{-i\,(t-i\,\beta/2)H}\mathcal{O}(0)\rangle_\beta$$
$$= \langle \mathcal{O}^\dagger(t-i\beta/2)\mathcal{O}(0)\rangle_\beta =: \Pi^W(t)\ .$$

$$\langle\cdots\rangle_\beta = \mathrm{Tr}(e^{-\beta H}\cdots)/\mathrm{Tr}(e^{-\beta H})$$

Power spectrum

$f^W(\omega)$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

Moments

μ_{2n}

$$\Pi^W(t) := \sum_{n=0}^{\infty}\mu_{2n}\frac{(it)^{2n}}{(2n)!} \qquad \mu_{2n} := \frac{1}{i^{2n}}\frac{\mathrm{d}^{2n}\Pi^W(t)}{\mathrm{d}t^{2n}}\bigg|_{t=0}$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\omega^{2n} f^W(\omega)$$

Lanczos coefficients from moments

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_2 = b_1^2\,, \quad \mu_4 = b_1^4 + b_1^2 b_2^2\,, \quad \cdots\,.$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j,\ldots,n\,,$$
$$M_{2l}^{(0)} = \mu_{2l}\quad , \quad b_{-1}\equiv b_0 := 1 \quad , \quad M_{2l}^{(-1)} = 0\,.$$

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$\Pi^W(t) =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$C(t) = \Pi^W(t) = \varphi_0(t)$ $\qquad$ b_n

K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$

$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$

$$\vdots$$

$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$

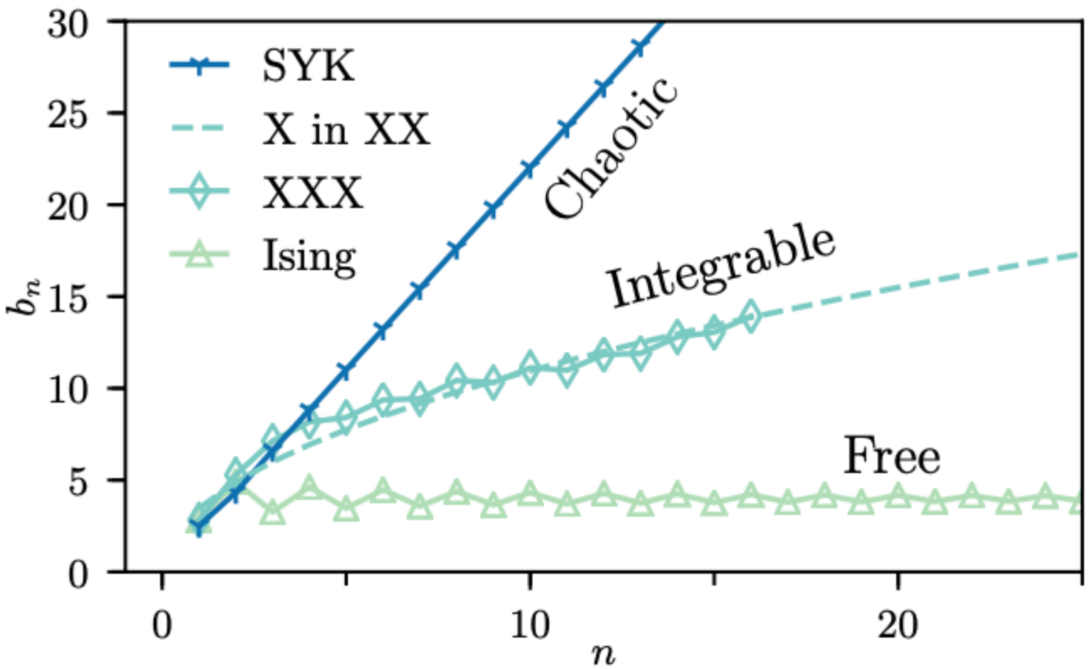
Success in lattice systems

Universal operator growth hypothesis

1812.08657: Parker et al.

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$\delta \leq 1$



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Signatures of chaos in time series generated by many-spin systems at high temperatures

Tarek A. Elsayed, Benjamin Hess, and Boris V. Fine
Phys. Rev. E **90**, 022910 – Published 20 August 2014

$f^W(\omega) \sim e^{-\frac{\omega}{\omega_0}}$ Is a signature of classical chaos

Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

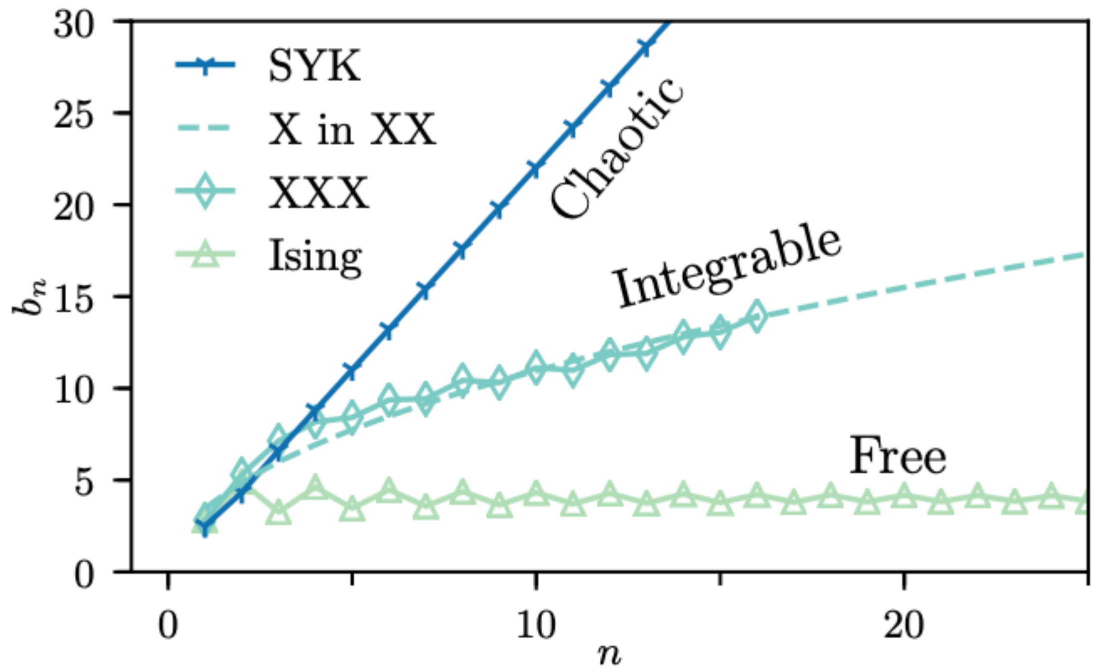
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

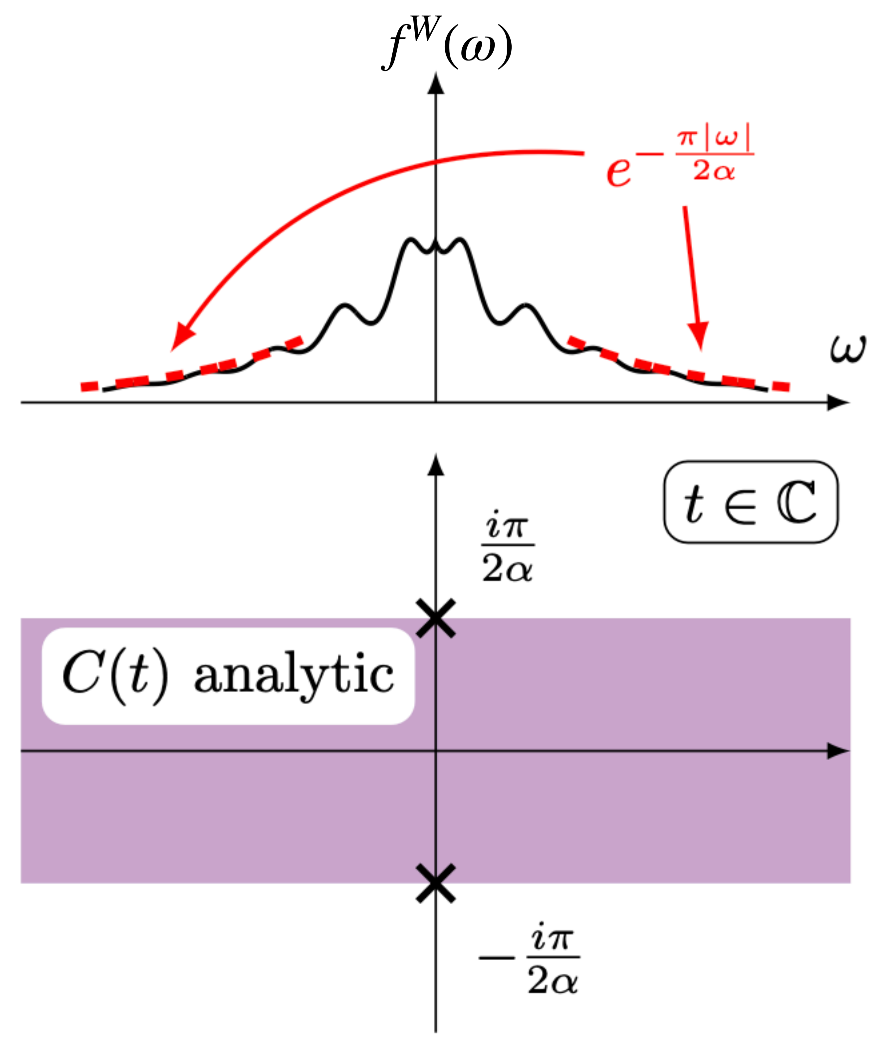
Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$
$$\delta \leq 1$$





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Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \Longleftrightarrow \quad b_n \sim \alpha n \quad \Longleftrightarrow \quad K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Towards Field theory

Free massive scalar in d-dimensions

2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \ ,$$

$$\Pi^W(\omega,\mathbf{k}) := \int \mathrm{d}t \int \mathrm{d}^{d-1}\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\, \Pi^W(t,\mathbf{x})$$

$$\Pi^W(\omega,\mathbf{k}) = \frac{1}{\sinh[\beta\omega/2]}\rho(\omega,\mathbf{k}).$$

$$\rho(\omega,\mathbf{k}) = \frac{N}{\epsilon_k}[\delta(\omega-\epsilon_k) - \delta(\omega+\epsilon_k)],$$

$$\epsilon_k := \sqrt{|\mathbf{k}|^2 + m^2},$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t} = \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \Pi^W(\omega,\mathbf{k}).$$

$$f^W(\omega) = N(m,\beta,d)\,\frac{(\omega^2 - m^2)^{(d-3)/2}}{|\sinh\left(\frac{\beta\omega}{2}\right)|}\Theta(|\omega| - m)$$

$$\int \frac{\mathrm{d}\omega}{2\pi} f^W(\omega) = 1$$

m=0, d=4

$$f^W(\omega) = \frac{\beta^2\omega}{\pi\sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

$$f^W(\omega) \longrightarrow \mu_{2n} \longrightarrow b_n$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) \qquad b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

Counter example in QFT

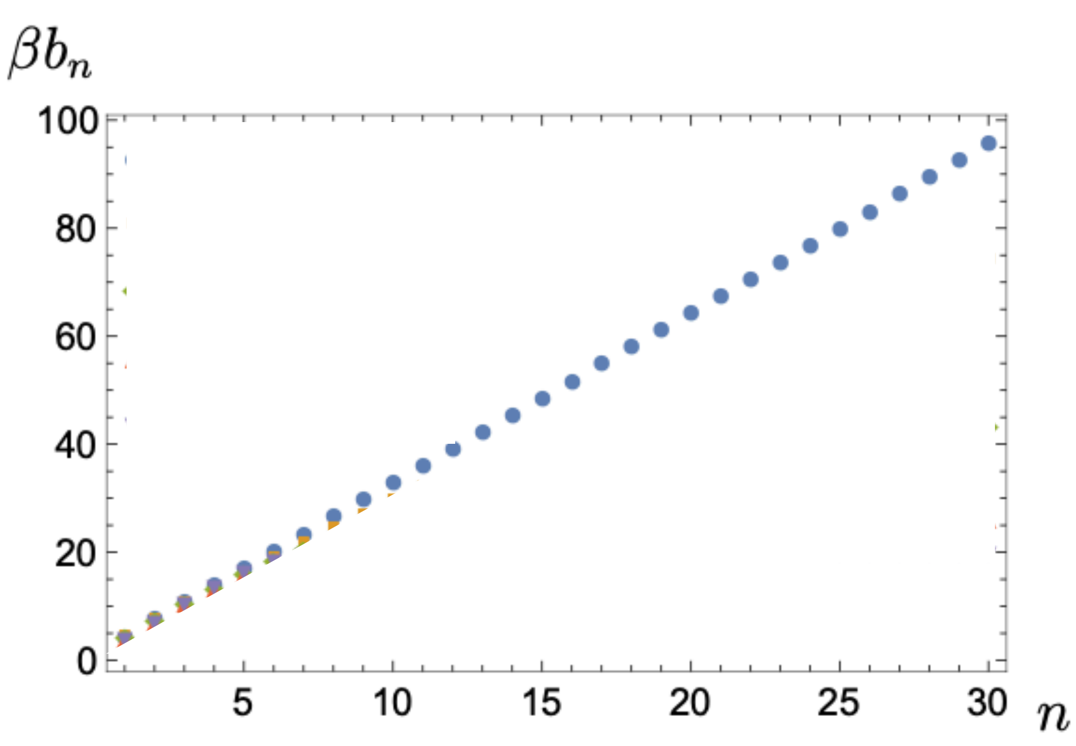
2212.14429: Avdoshikin, Dymarsky, Smolkin
2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Power spectrum (m=0, d=4)

$$f^W(\omega) = \frac{\beta^2\omega}{\pi \sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$



Free theory is chaotic?

~~In a chaotic quantum system~~ In free QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

? $f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$

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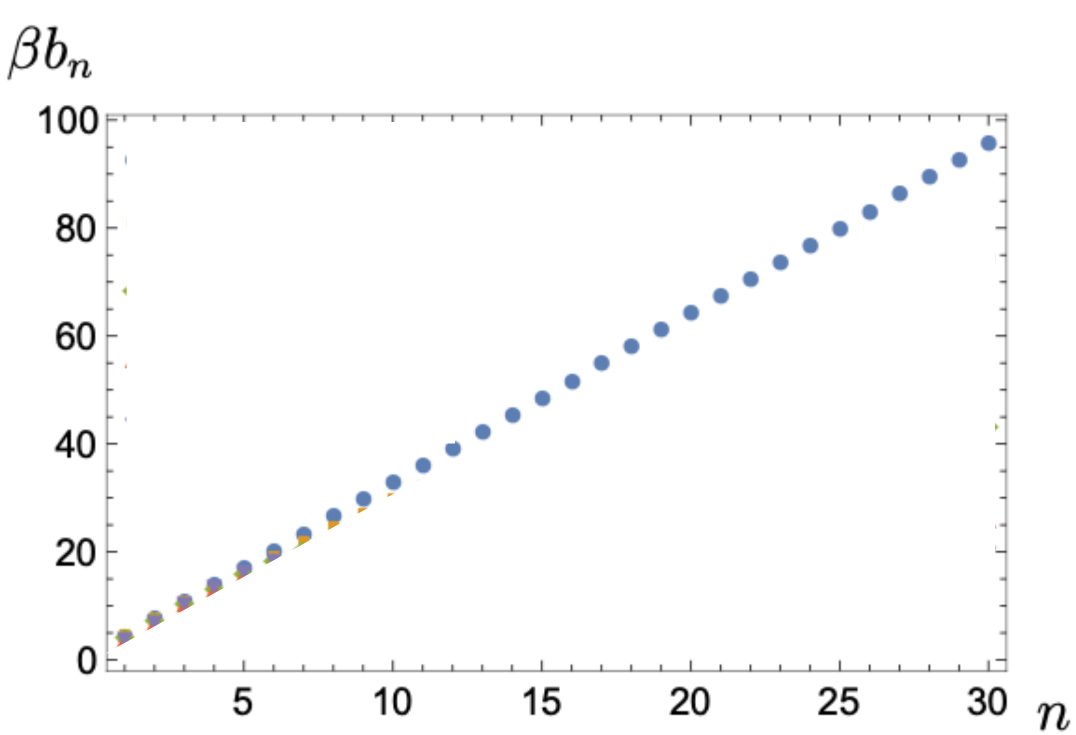


Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle\phi(t - i\beta/2,\mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \qquad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum $\quad C(t) = \Pi^W(t,\mathbf{0})$

$$f^W(\omega) := \int \mathrm{d}t\, C(t)e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0})e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$

2104.09514: Dymarsky, Smolkin

General QFT is chaotic? No

~~In a chaotic quantum system~~ In general QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible!

$$b_n \sim \alpha n \sim \frac{\pi}{\beta}n$$

?

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_\mathcal{O}(t) \sim e^{2\alpha t}$$

Too good to be true

Counter example:

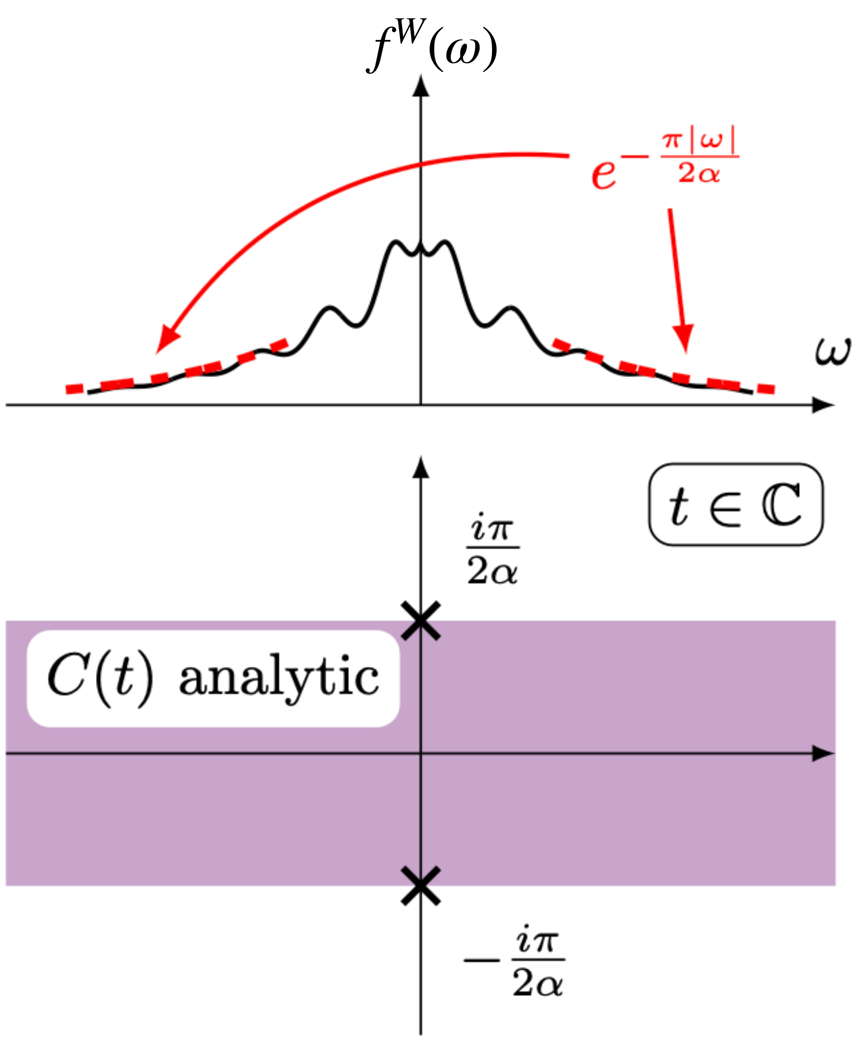
- Field theory
- Krylov complexity in saddle-dominated scrambling (2203.03534: Bhattacharjee, Cao, Nandy, Pathak)

Too good to be true

Chaos $\Longleftrightarrow$ $f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$

Only if b_n is a smooth function of n, Otherwise

Chaos $\Longleftrightarrow$? $f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow$? $b_n \nsim \alpha n \Longleftrightarrow$? $K_{\mathcal{O}}(t) \sim e^{2\alpha t}$



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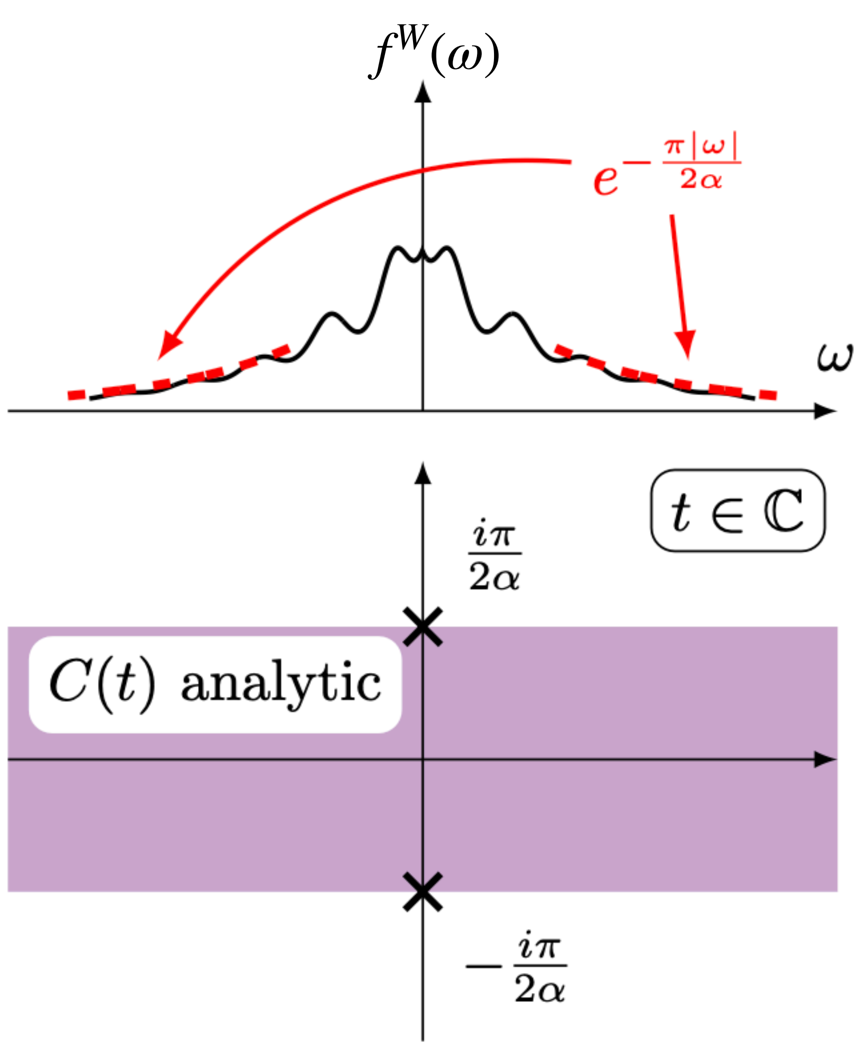
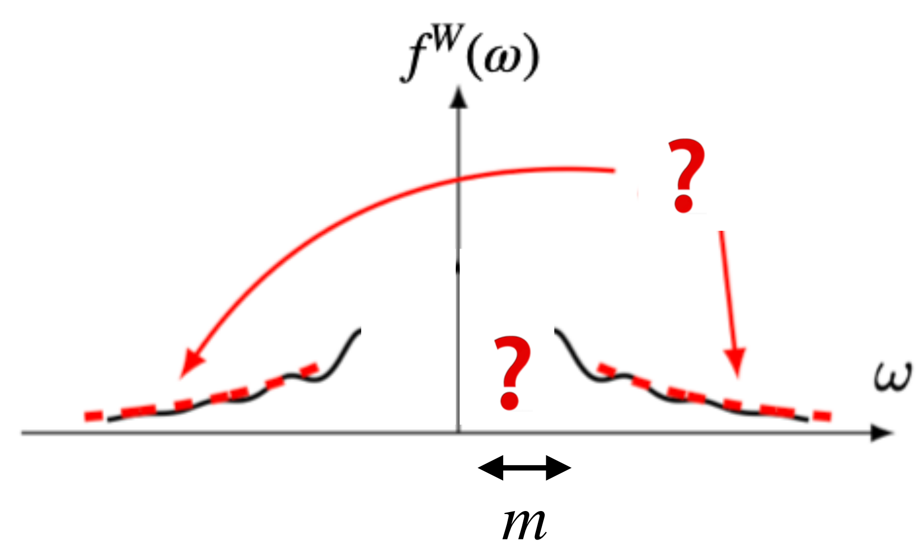
Chaos $\overset{?}{\Leftrightarrow}$ $\quad f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \overset{?}{\Longleftrightarrow} b_n \nsim \alpha n \overset{?}{\Longleftrightarrow} K_{\mathcal{O}}(t) \sim e^{2\alpha t}$

Need to investigate these relations further.

How to extract (chaotic) information from the power spectrum?





Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$C(t) = \Pi^W(t) = \varphi_0(t) \qquad\qquad b_n$$

K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$

$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$

$$\vdots$$

$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

2212.14429: Avdoshikin, Dymarsky, Smolkin
2212.14702: Camargo, Jahnke, KYK, Nishida

Power spectrum

$\beta m \gg 1$

$$f^W(\omega) \approx N(m,\beta,d)\, e^{-\beta|\omega|/2}\left(\omega^2 - m^2\right)^{(d-3)/2}\Theta(|\omega| - m) \qquad N(m,\beta,d) = \frac{\pi^{3/2}\beta^{(d-2)/2}}{2^{d-2}\, m^{(d-2)/2} K_{\frac{d-2}{2}}\left(\frac{m\beta}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$$

$K_n(z)$ is the modified Bessel function of the second kind

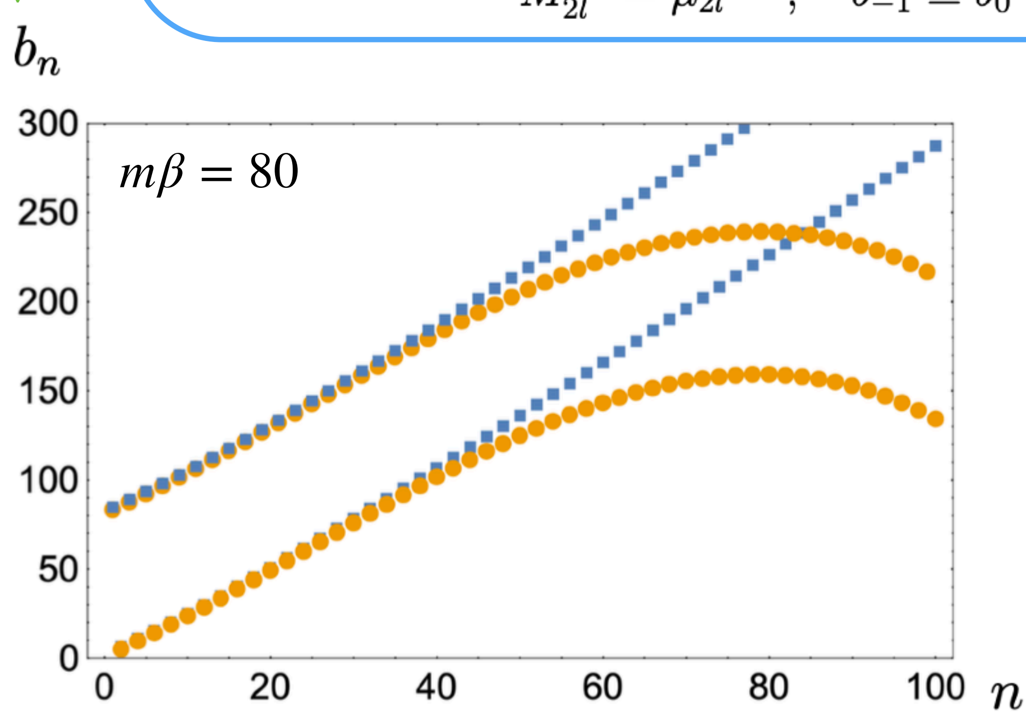
Moments to Lanczos coefficients (d=5)

$\tilde{\Gamma}(n,z)$ is the incomplete Gamma function.

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) = \frac{2^{-2} e^{\frac{m\beta}{2}}}{2 + m\beta}\left(\frac{2}{\beta}\right)^{2n}\left[-m^2\beta^2\,\tilde{\Gamma}\left(2n+1, \frac{m\beta}{2}\right) + 4\tilde{\Gamma}\left(2n+3, \frac{m\beta}{2}\right)\right]$$

$$b_n = \sqrt{M_{2n}^{(n)}}\ , \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j, \ldots, n\ ,$$

$$M_{2l}^{(0)} = \mu_{2l}\ , \quad b_{-1} \equiv b_0 := 1\ , \quad M_{2l}^{(-1)} = 0\ .$$

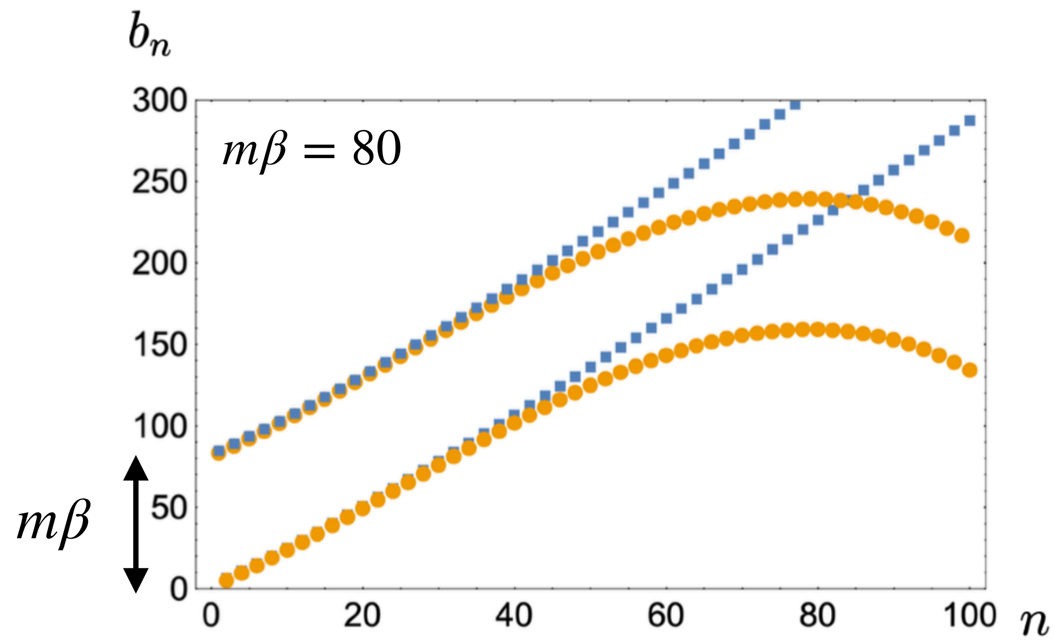


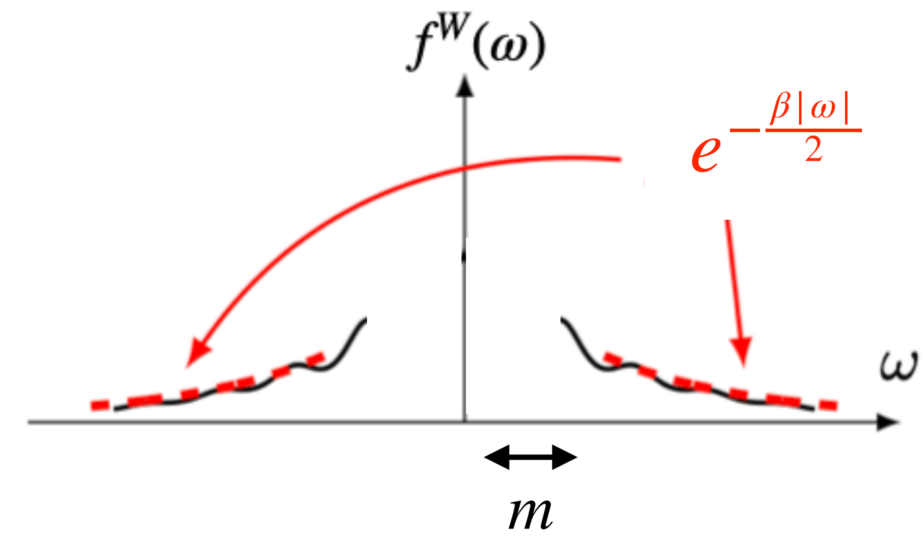
$$\beta^2 b_n^2 = m^2\beta^2 \begin{cases} 1 + 4\frac{1+n}{m\beta} + 8\frac{(n+1)^2}{m^2\beta^2} + 12\frac{(n+1)^3}{m^3\beta^3} + \cdots\ , & \text{for } n \text{ odd}\ , \\ 4\frac{n(n+2)}{m^2\beta^2} + 8\frac{n(n+1)(n+2)}{m^3\beta^3} + \cdots\ , & \text{for } n \text{ even}\ , \end{cases}$$

Staggering: two families for even *n* and odd *n*

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

Non-trivial mass (IR-cutoff) effect: staggering





Staggering

$$b_n \sim \alpha_{\mathrm{odd}}\, n + \gamma_{\mathrm{odd}} \quad (\text{odd } n)$$
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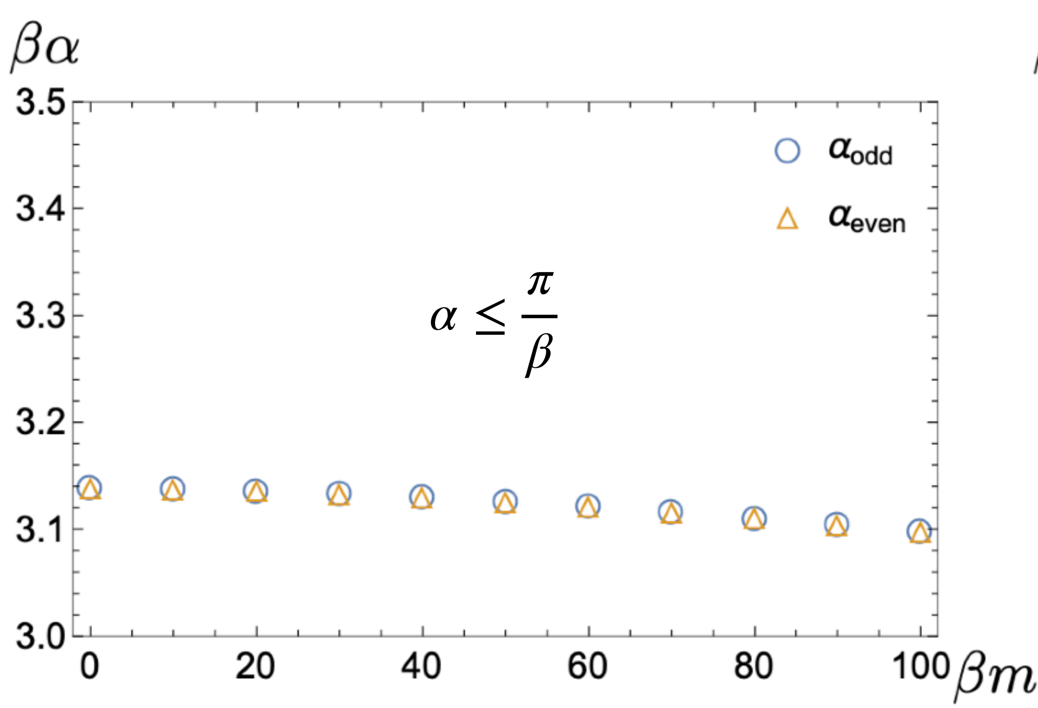
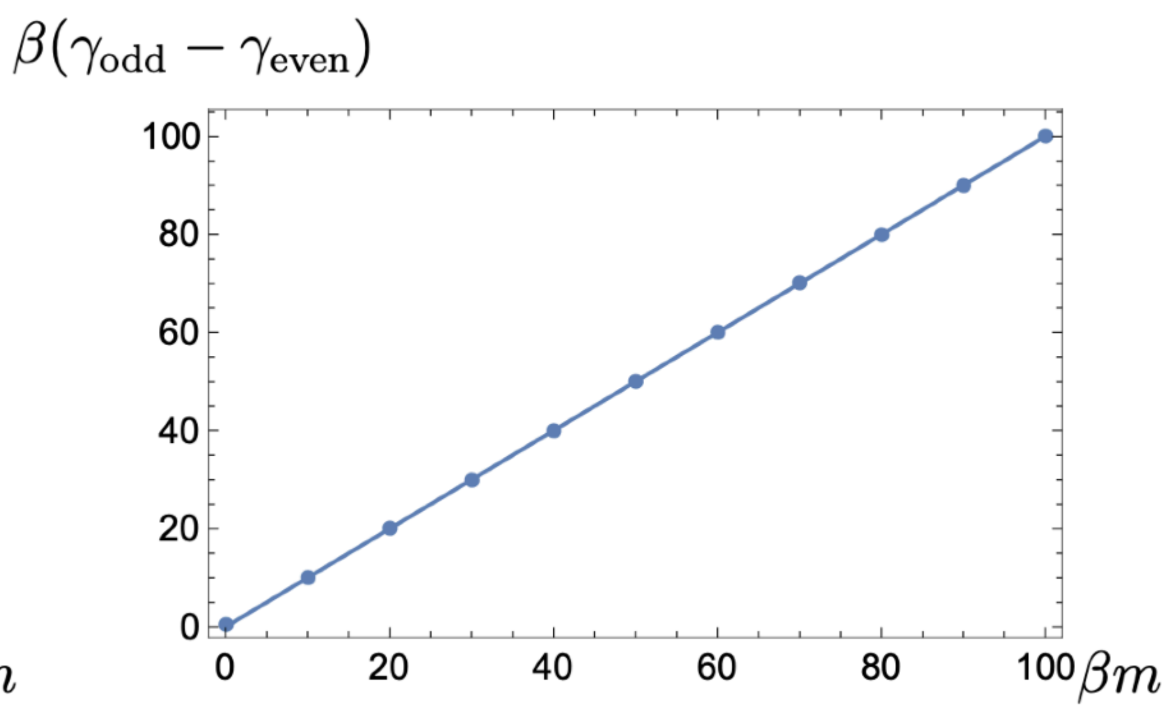


(a) Mass-dependence of α_{odd} and α_{even}



(b) Mass-dependence of $\gamma_{\mathrm{odd}} - \gamma_{\mathrm{even}}$

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

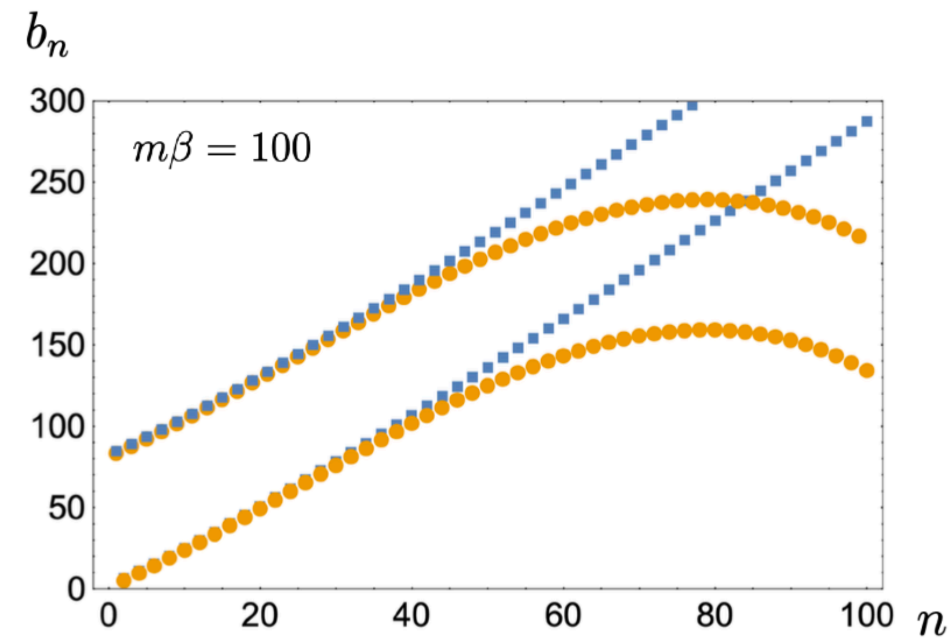
$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n} \cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$\mu_{2n} \longrightarrow b_n$

$$C(t) = \Pi^W(t) = \varphi_0(t)$$

$$C^{(d)}(t) \equiv \varphi_0^{(d)}(t) = c_1^{(d)}(t)\left(c_2^{(d)}(t)\sin(m\,t) + c_3^{(d)}(t)\cos(m\,t)\right)$$



K-complexity

$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$

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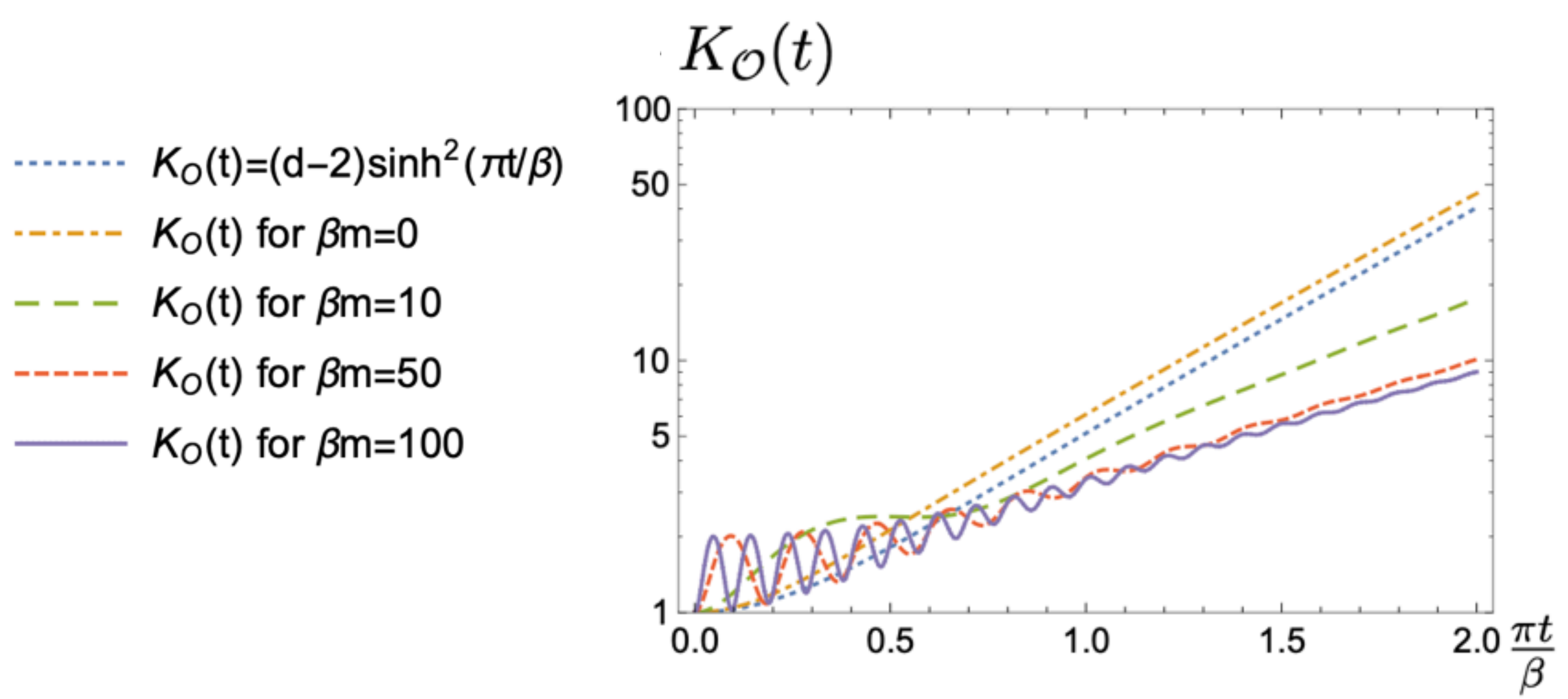

Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

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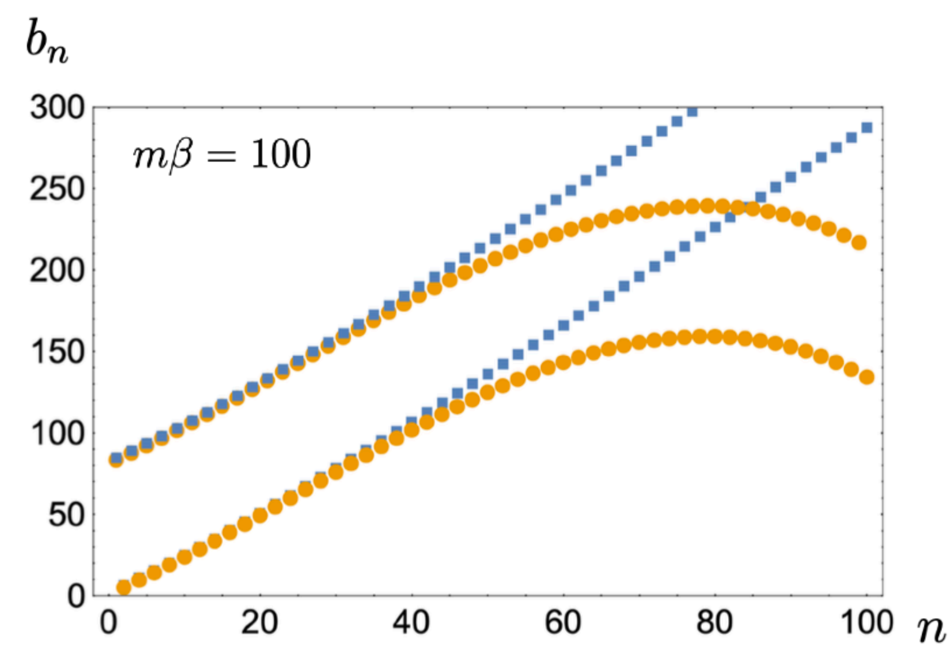
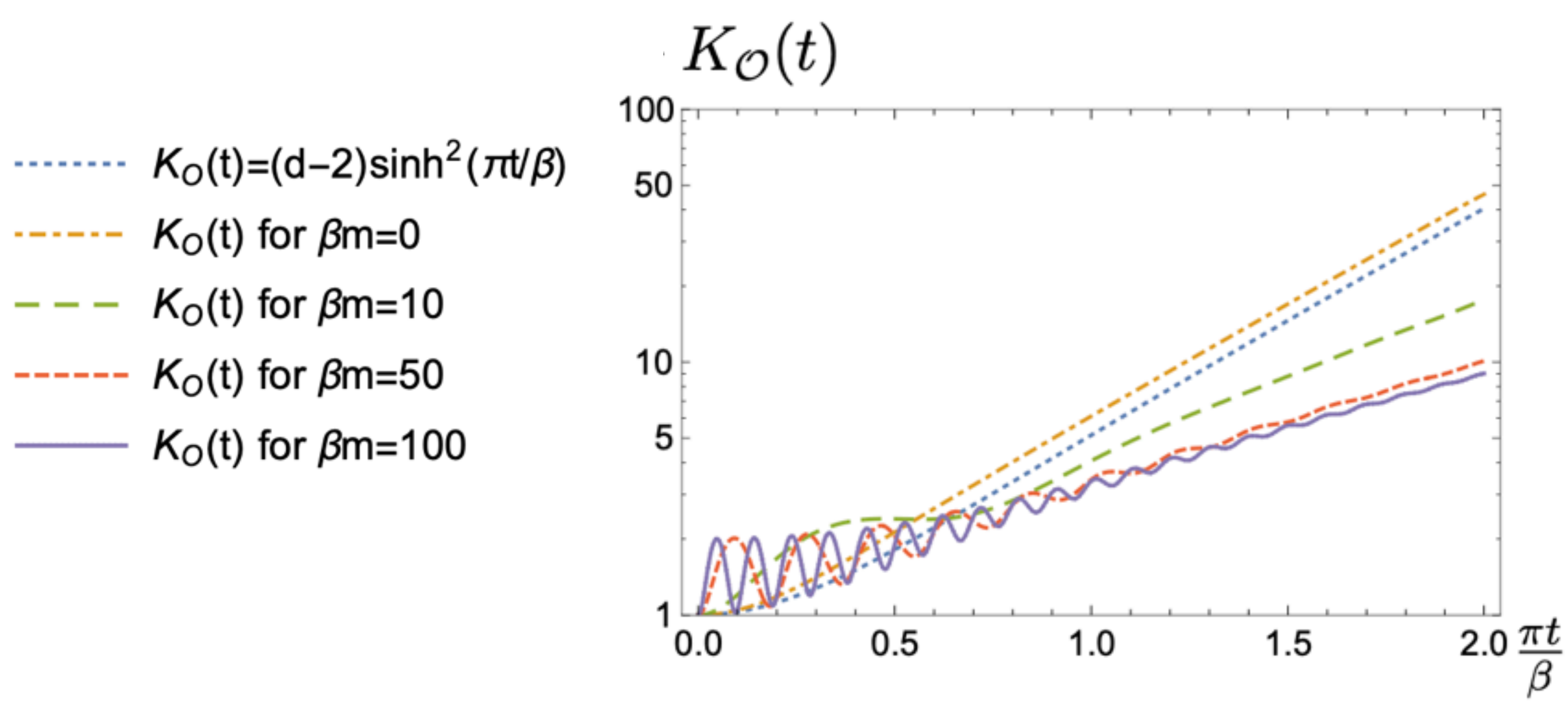


K-complexity

- Early time: oscillation:
  - larger m, shorter period
- Late time: oscillation disappears
  - cancelation due to large n
- Exponential increase
  - larger m, slower increase
  - mass effect



Non-trivial mass (IR-cutoff) effect: K-complexity

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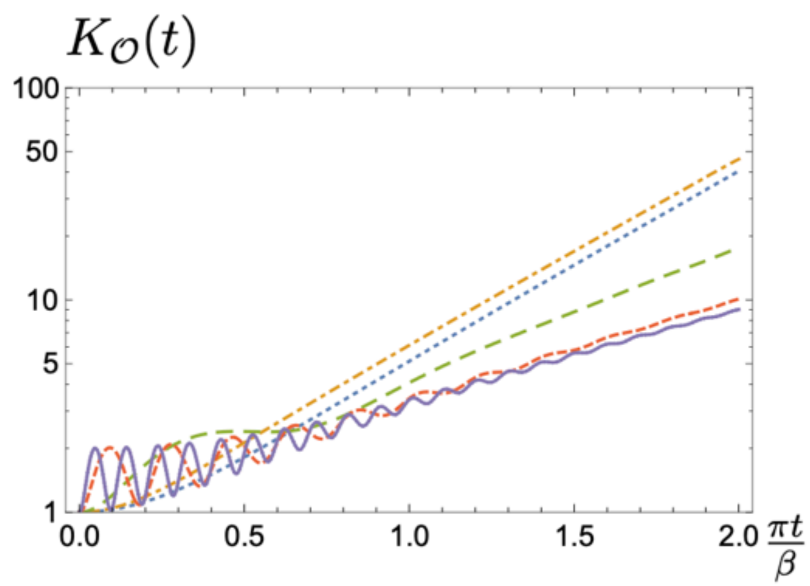
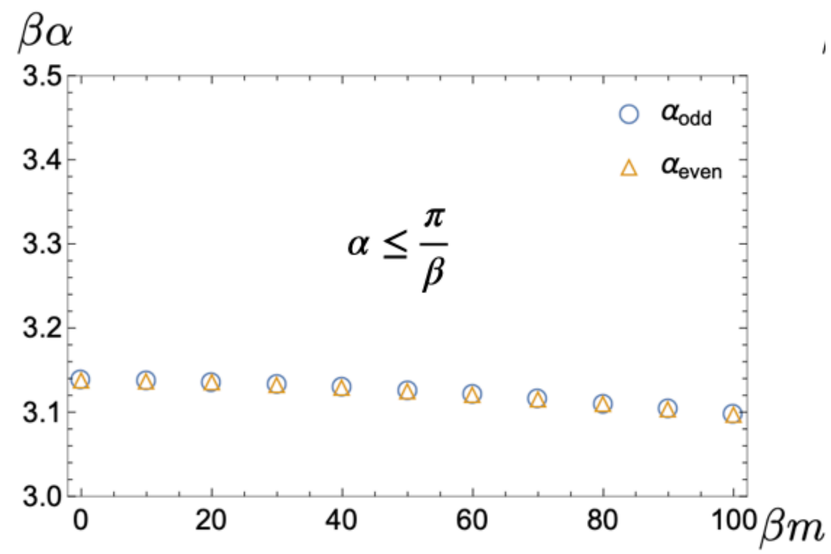
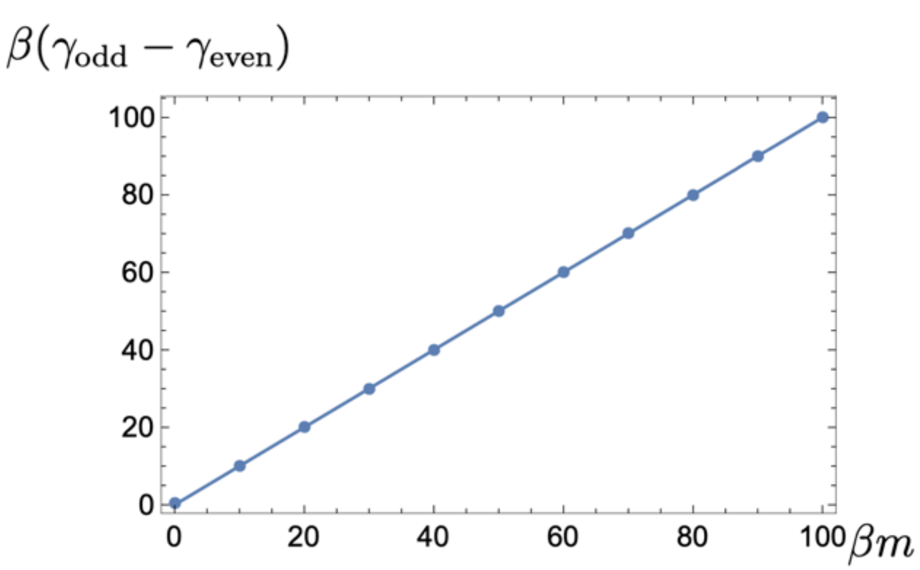
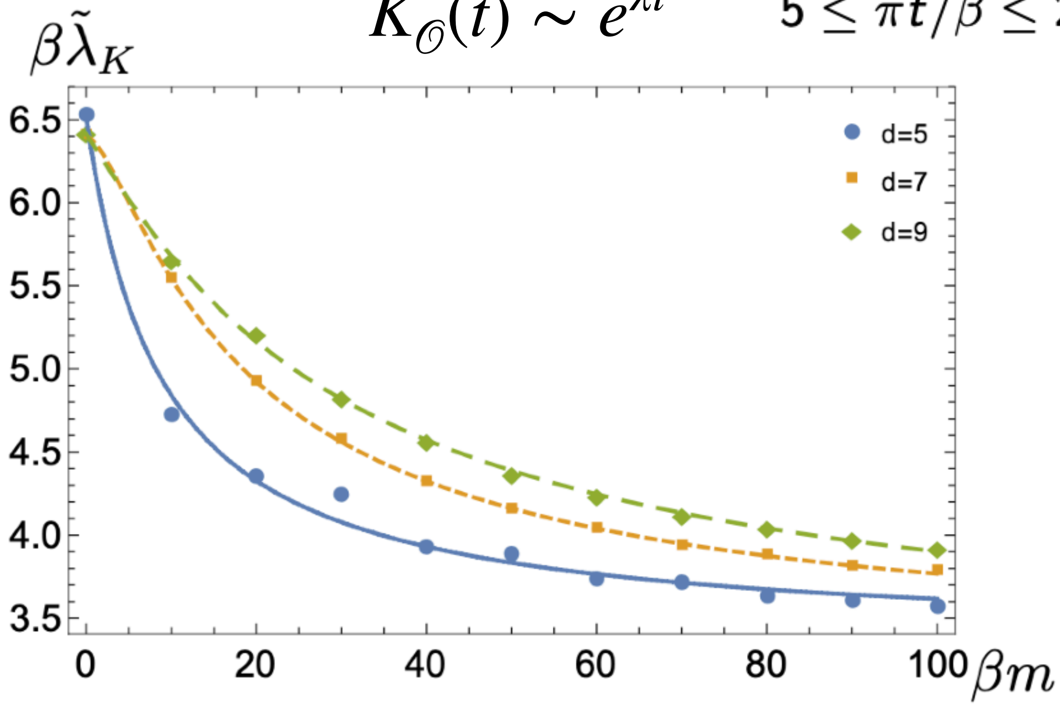






$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$ $\quad 5 \leq \pi t/\beta \leq 2.$



$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\rm odd} + \alpha_{\rm even}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\rm odd} - \gamma_{\rm even}|} - \frac{1}{k_3^{(d)}}\right) + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\rm odd} - \gamma_{\rm even}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$

Staggering

$$b_n \sim \alpha_{\rm odd}\, n + \gamma_{\rm odd} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\rm even}\, n + \gamma_{\rm even} \quad (\text{even } n)$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Complexity of state

Complexity of a state

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \qquad |\psi(t)\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\psi_n\rangle \qquad |\psi_n\rangle = H^n|\psi(0)\rangle.$$

$$|A_{n+1}\rangle = (H-a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

$$a_n = \langle K_n|H|K_n\rangle\,, \quad b_n = \langle A_n|A_n\rangle^{1/2}$$

$$|K_0\rangle = |\psi(0)\rangle\,, \quad |K_n\rangle = b_n^{-1}|A_n\rangle\,, \quad b_0 \equiv 0\,.$$

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$|\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$$

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

Spread complexity

$$C(t) \equiv \sum_n n p_n(t) = \sum_n n|\psi_n(t)|^2\,.$$

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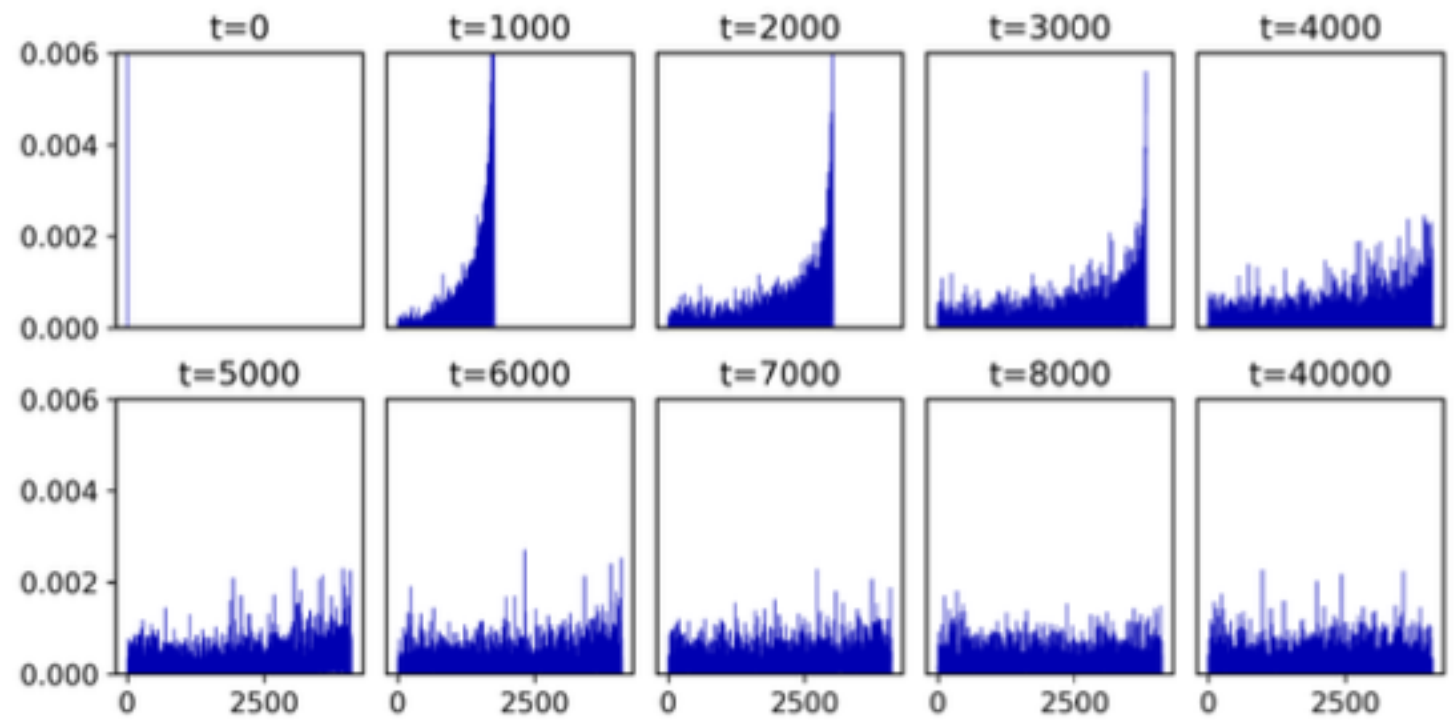

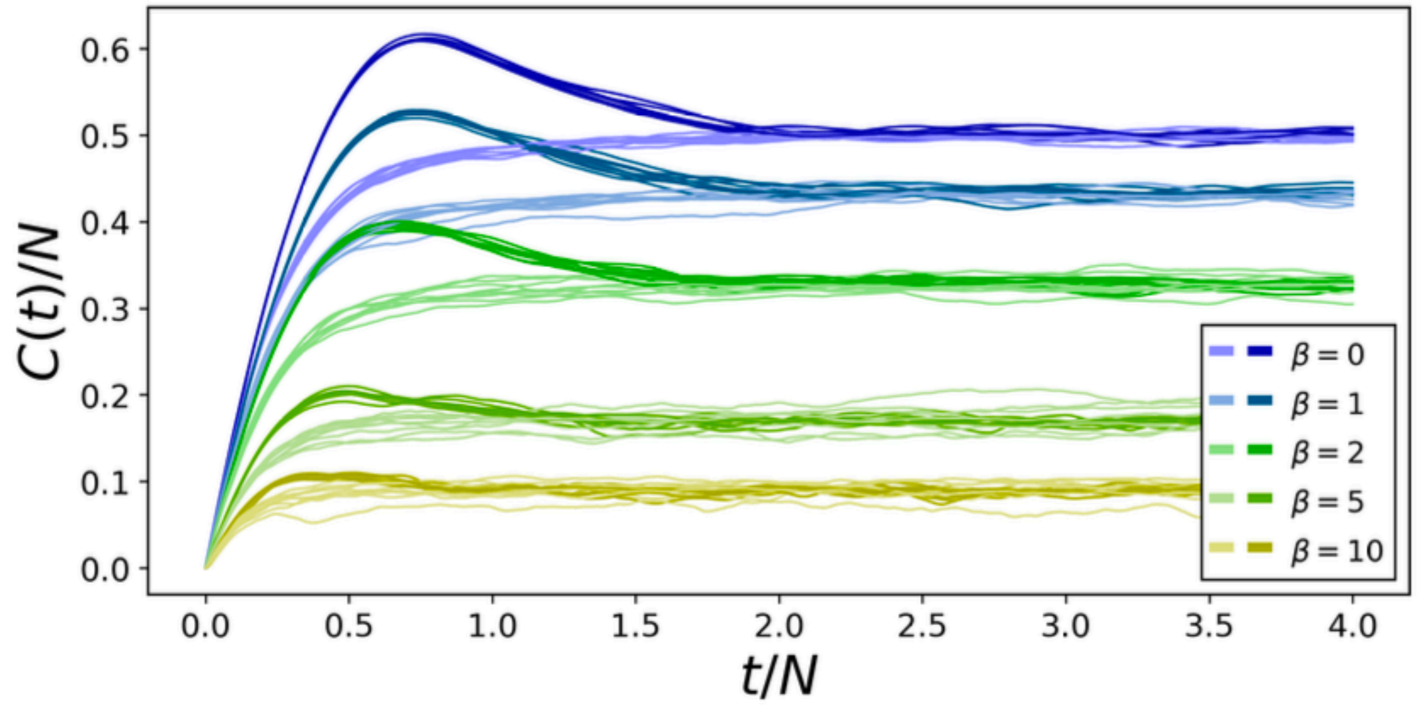

Complexity of a state

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Spread complexity

$$C(t) \equiv \sum_n n p_n(t) = \sum_n n|\psi_n(t)|^2 \,.$$

Survival amplitude

$$S(t) = \langle\psi(t)|\psi(0)\rangle = \langle\psi(0)|e^{iHt}|\psi(0)\rangle$$

$$\mu_n = \frac{d^n}{dt^n}S(t)\Big|_{t=0} = \langle\psi(0)|\frac{d^n}{dt^n}e^{iHt}|\psi(0)\rangle\Big|_{t=0}$$
$$= \langle K_0|(iH)^n|K_0\rangle \,.$$

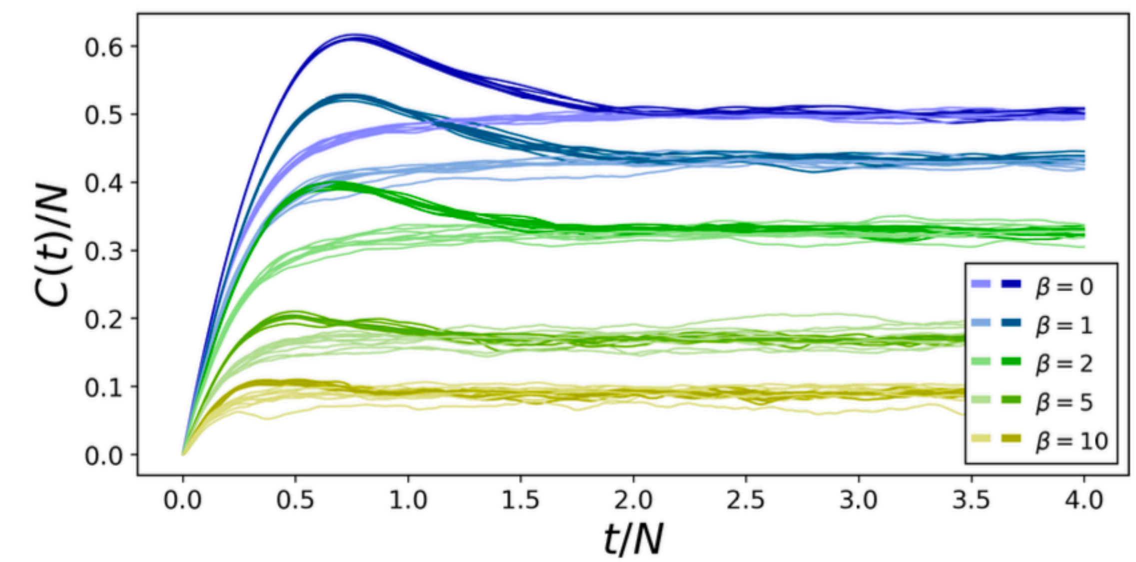
Thermo Field Double (TFD) state

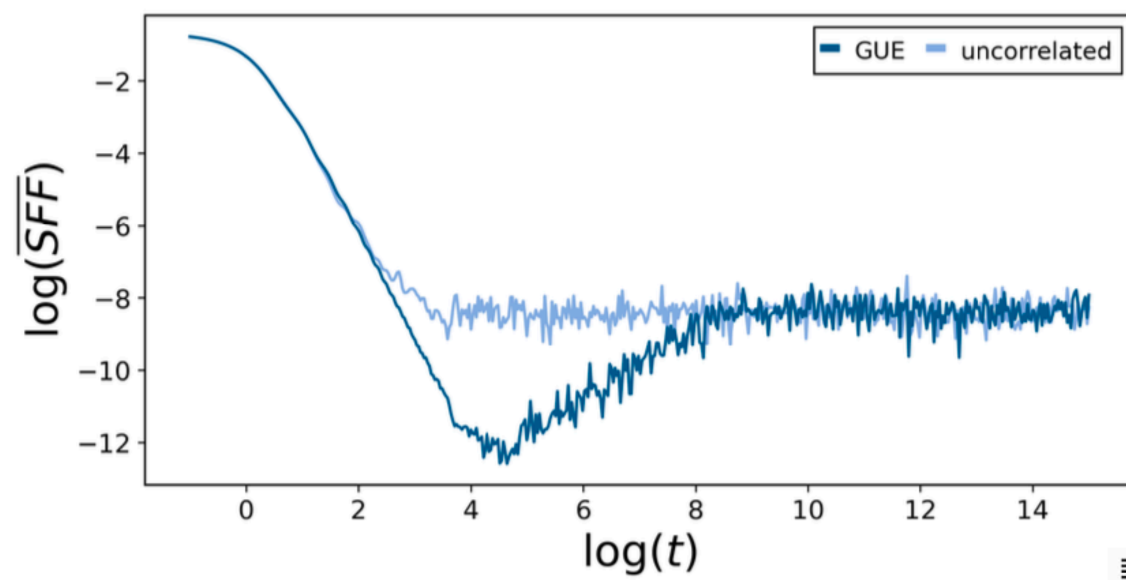
$$|\psi_\beta\rangle \equiv \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle$$

$$|\psi_\beta(t)\rangle = e^{-iHt}|\psi_\beta\rangle = |\psi_{\beta+2it}\rangle$$

$$S(t) = \langle\psi_{\beta+2it}|\psi_\beta\rangle = \frac{Z_{\beta-it}}{Z_\beta}$$

$$SFF_{\beta-it} \equiv \frac{|Z_{\beta-it}|^2}{|Z_\beta|^2}.$$





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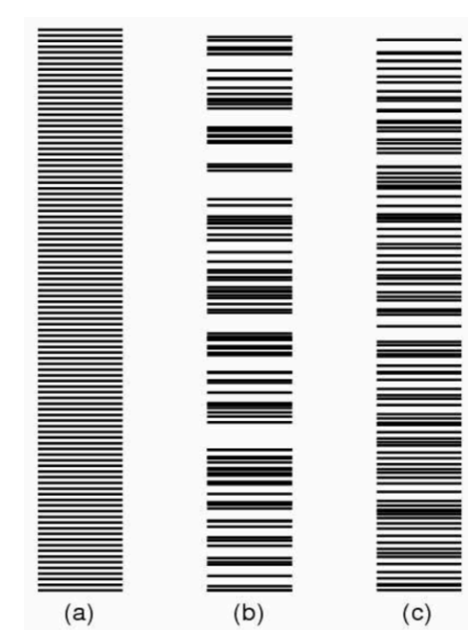
$$= \langle K_0|(iH)^n|K_0\rangle .$$

Observations for RMT, SYK

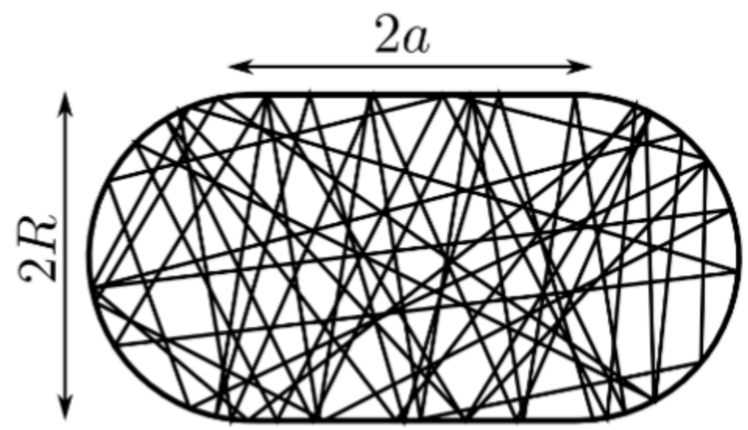
Universal for Maximal chaos? Why?

What if not TFD

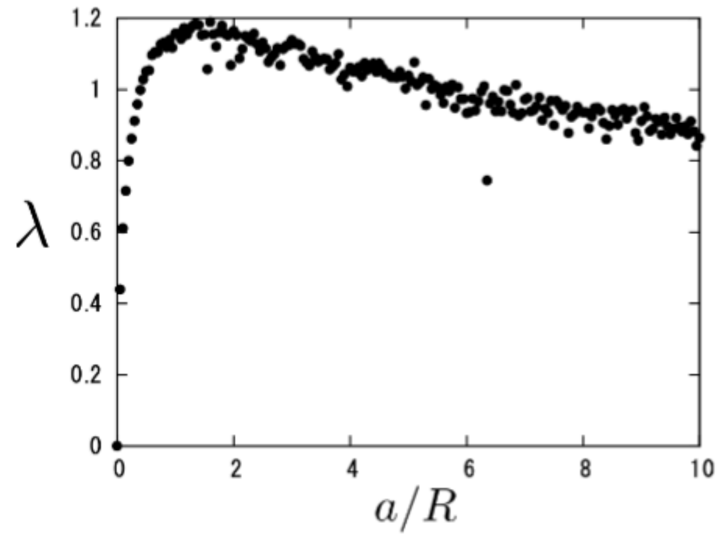


Other ways?

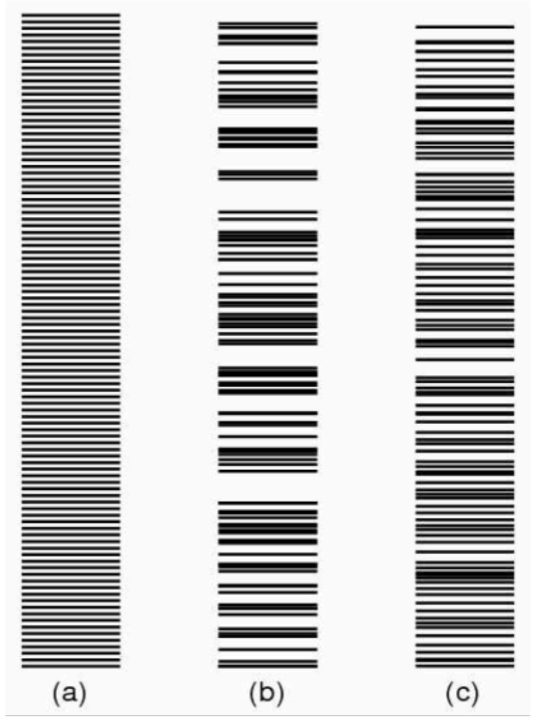
Fluctuations of the Lanczos coefficients

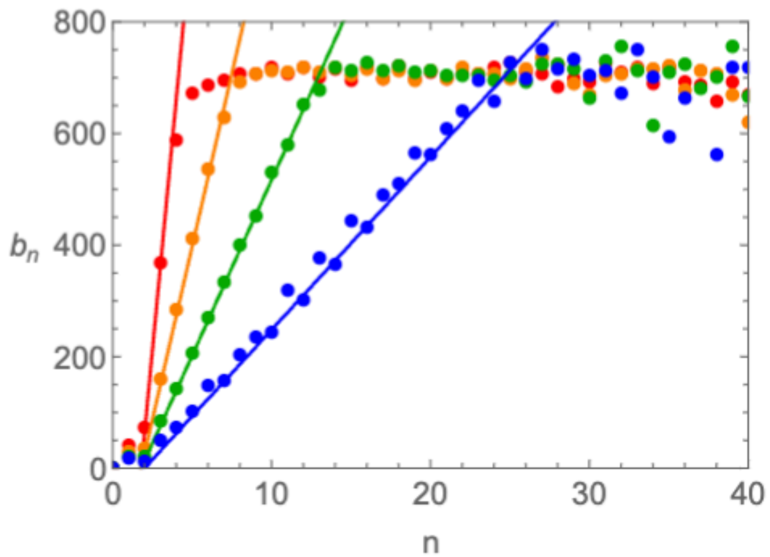


(a) A typical trajectory

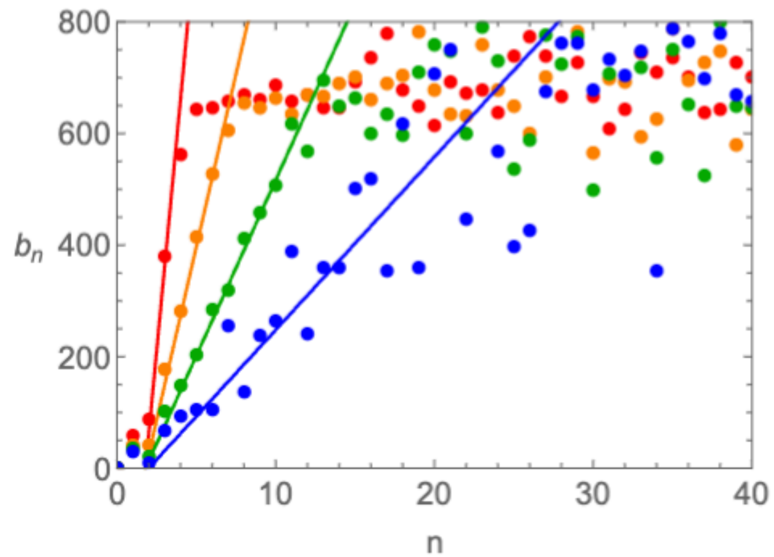


(b) Lyapunov exponent





(a) Stadium billiard ($a/R = 1$)



(b) Circle billiard ($a/R = 0$)

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

Variation of Lanczos coefficients?

Spectral rigidity?

2306.11632: Camargo, Jahnke, Jeong, KYK, Nishida

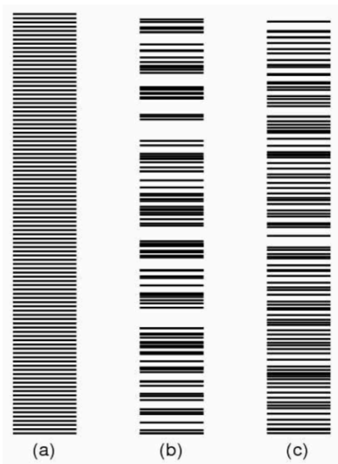
2305.16669: Hashimoto, Murata, Tanahashi, Ryota Watanabe

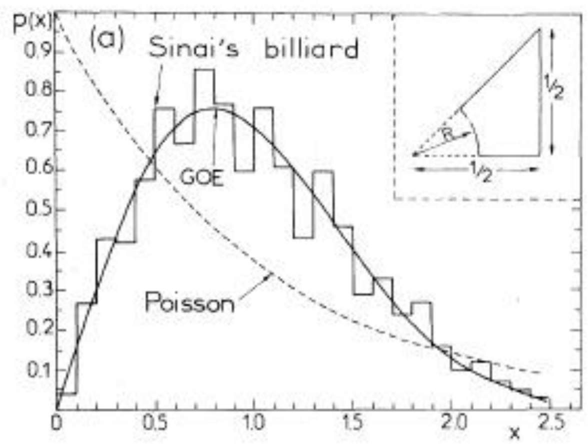
2112.12128: Rabinovici, Sanchez-Garrido, Shir, Sonner

Summary

Summary

- Chaos: beyond RMT, better resolution, dynamics. Krylov-complexity





Complex (nuclear spectrum) ~ Random matrix theory

- Level repulsion
- Spectral rigidity

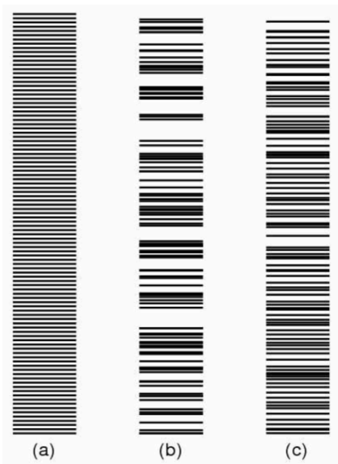
- No time ~ Long time limit (time-energy uncertainty)
- No operator or state dependence?

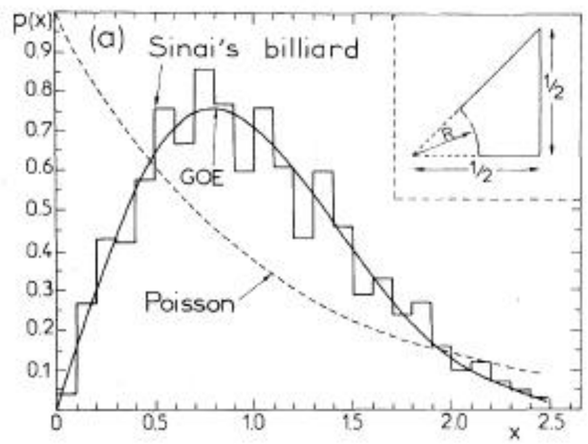
$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

- Krylov complexity has a better "resolution" than level statistics
- Krylov (spread) complexity has time dynamics
- Krylov basis has info for the operator or state
- Krylov(spread) complexity has no ambiguity by minimization over basis

Summary

- Chaos: beyond RMT, better resolution, dynamics.   Krylov-complexity





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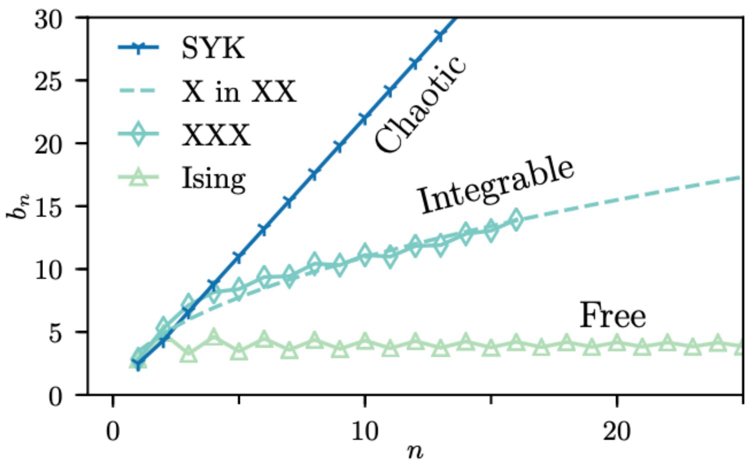
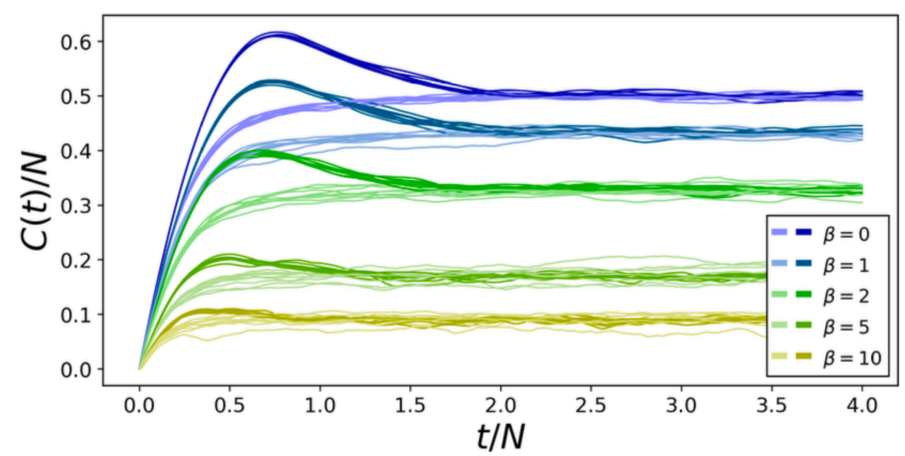
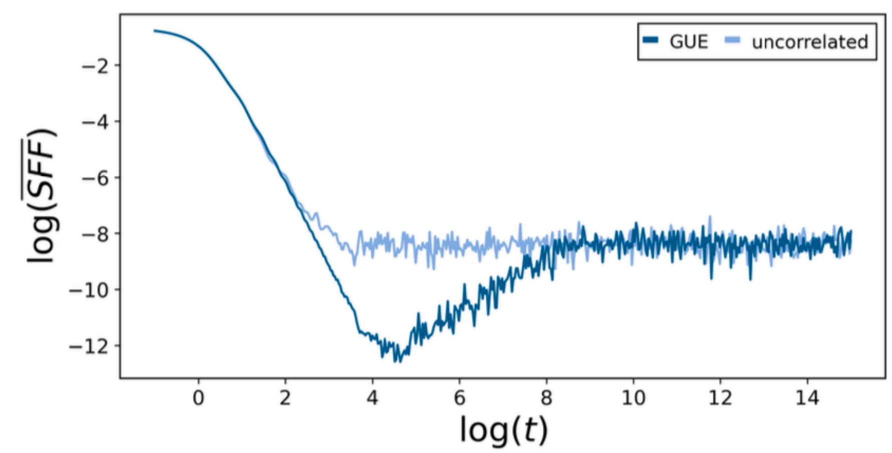
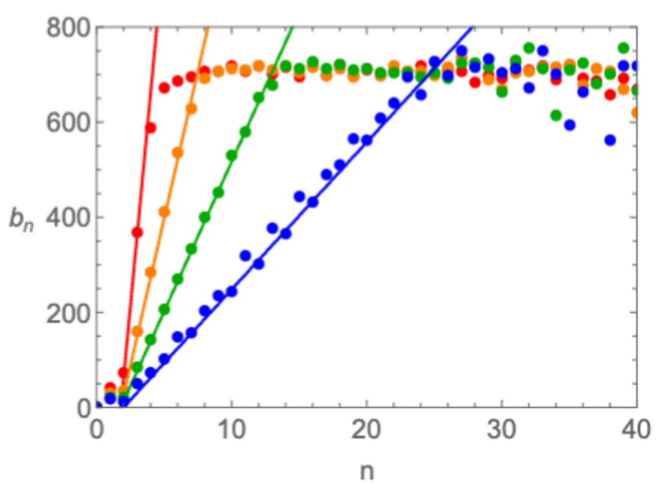
OTOC

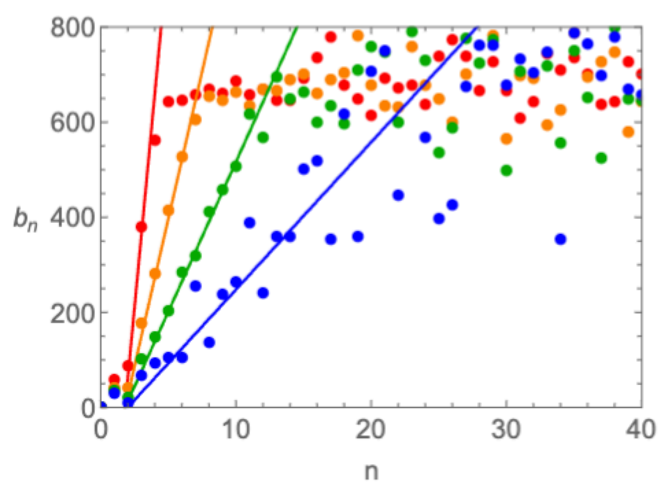








(a) Stadium billiard ($a/R = 1$)



(b) Circle billiard ($a/R = 0$)

- Observations, conjectures, justification?
- Holographic counterpart?
- Black hole interior?