2 Dimensional Maximally Supersymmetric Yang-Mills Theory on the Lattice

Bana Singh

6^{th} **Mandelstam Theoretical Physics School & Workshop**
11^{th} **January 2024**

Outline

- $\mathcal{N} = 4, d = 4$ SYM in continuum
- Dimensional reduction method
- $\mathcal{N} = (8, 8), d = 2$ SYM in continuum
- Prescription for lattice
- $\mathcal{N} = (8, 8), d = 2$ SYM on the Lattice
- Observable using RHMC
- SUSY LATTICE code
- Future work

Twisted $\mathcal{N} = 4, d = 4$ SYM theory in the continuum

$\mathcal{S} = \mathcal{S}_{\mathcal{Q}-exact} + \mathcal{S}_{\mathcal{Q}-closed}$

$\{\mathcal{Q}, \mathcal{Q}\} = 0$, $\implies$ $\mathcal{Q}^2 = 0$, Preserved in discrete space-time

$$\mathcal{S}_{\mathcal{Q}-exact} = \frac{N}{2\lambda}\mathcal{Q}\int d^4x \mathrm{Tr}[(\chi_{ab}\mathcal{F}_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d)]$$

$$\mathcal{S}_{\mathcal{Q}-closed} = -\frac{N}{8\lambda}\int d^4x \mathrm{Tr}[\epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de}]$$

$$\mathcal{D}_a\phi = \partial_a\phi + [A_a, \phi], \quad a = 1, 2..5, \quad \mathcal{F}_{ab} = [\mathcal{D}_a, \mathcal{D}_b], \quad d = [\overline{\mathcal{D}}_a, \mathcal{D}_b]$$

The nilpotent transformations

$$\mathcal{Q}\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\psi_a = 0$$

$$\mathcal{Q}\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\eta = d \qquad \mathcal{Q}d = 0$$

Performing the $\mathcal{Q}$-variation and integrating out the auxiliary field d

$$\mathcal{S}_{\mathcal{Q}-exact} = \mathcal{S}_B + \mathcal{S}_F$$

$$\mathcal{S}_B = \frac{N}{2\lambda} \int d^4x \mathrm{Tr}[-\mathcal{F}_{ab}\mathcal{F}_{ab} + \frac{1}{2}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2]$$

$$\mathcal{S}_F = \frac{N}{2\lambda} \int d^5x \mathrm{Tr}[-\chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a]$$

$$\mathcal{S} = \frac{N}{2\lambda} \int d^4x \mathrm{Tr}\Bigg[- \mathcal{F}_{ab}\mathcal{F}_{ab} + \frac{1}{2}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a - \frac{1}{4}\epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \Bigg]$$

Dimensional reduction method

Compactifying the k spatial dimension on circles of radii R

x^m of $\mathbb{R}^{1,(d-1)} \rightarrow x^\mu$ of $\mathbb{R}^{1,(d-1-k)}$ & $y^1, y^2 ... y^k$ of S^1

$\mathbb{R}^{1,(d-1)} \rightarrow \mathbb{R}^{1,(d-1-k)} \times S^1 \times S^1 \cdots S^1$ k times

$\mathbb{R}^{1,(d-1)} \rightarrow \mathbb{R}^{1,(d-1-k)} \times \mathbb{T}^K$

$R \rightarrow 0$, compactified dimension y^i of k circle S^1 vanish to zero

This process is called dimensional reduction

Field decomposes into a direct sum $\bigoplus$ of representations

Similarly component of gauge field in reduced direction A_r for $r = 1, 2, 3..k$ becomes scalar fields

$\mathcal{N}=1, d=10$	$\mathcal{N}=1, d=6$	$\mathcal{N}=1, d=4$
$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{N}=2, d=6$	$\mathcal{N}=2, d=4$	$\mathcal{N}=2, d=3$
$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{N}=4, d=4$	$\mathcal{N}=4, d=3$	$\mathcal{N}=(2,2), d=2$
$\downarrow$	$\downarrow$	
$\mathcal{N}=8, d=3$	$\mathcal{N}=(4,4)d=2$	
$\downarrow$		
$\mathcal{N}=(8,8), d=2$		

TABLE: Dimensional reduction of SYM theories

$\mathcal{N}$ = (8,8), d = 2 SYM theory
Bosonic action in continuum

Reduction to 2d from $\mathcal{N}=4, d=4$ *SYM*
By reducing 4th, 5th direction as $\mathcal{A}_4=\varphi, \mathcal{A}_5=\phi$

$$\mathcal{A}_a \to \mathcal{A}_i \bigoplus \mathcal{A}_4 \bigoplus \mathcal{A}_5 \equiv \mathcal{A}_i \bigoplus \varphi \bigoplus \phi$$

$$\mathcal{F}_{ab}=[\mathcal{D}_a,\mathcal{D}_b] \to [\mathcal{D}_i,\mathcal{D}_j] \bigoplus [\mathcal{D}_i,\mathcal{D}_4] \bigoplus [\mathcal{D}_i,\mathcal{D}_5] \bigoplus [\mathcal{D}_4,\mathcal{D}_5]$$

$$\mathcal{F}_{ab} \equiv \mathcal{F}_{ij} \bigoplus \mathcal{D}_i\mathcal{A}_4 \bigoplus \mathcal{D}_i\mathcal{A}_5 \bigoplus [\mathcal{A}_4,\mathcal{A}_5]$$

$$S_B^{d=2}=\frac{N}{2\lambda}\int d^2x \mathrm{Tr}\Big[-|\mathcal{F}_{ij}|^2-2|\mathcal{D}_i\varphi|^2-2|\mathcal{D}_i\phi|^2-2|[\varphi,\phi]|^2$$

$$+\frac{1}{2}[\overline{\mathcal{D}}_i,\mathcal{D}_i]^2+\frac{1}{2}[\overline{\phi},\phi]^2+\frac{1}{2}[\overline{\varphi},\varphi]^2+[\overline{\mathcal{D}}_i,\mathcal{D}_i][\overline{\phi},\phi]+[\overline{\mathcal{D}}_i,\mathcal{D}_i][\overline{\varphi},\varphi]$$

$$+[\overline{\phi},\phi][\overline{\varphi},\varphi]\Big]$$

$\mathcal{N} = (8,8)$, d = 2 SYM theory
Fermionic action in continuum

$$\psi_a \to \psi_i \bigoplus \psi_4 \bigoplus \psi_5 = \psi_i \bigoplus \overline{\zeta} \bigoplus \overline{\eta}$$

$$\chi_{ab} \to \chi_{ij} \bigoplus \chi_{i4} \bigoplus \chi_{i5} \bigoplus \chi_{45} \bigoplus \chi_{44} \bigoplus \chi_{55} \bigoplus \chi_{4j} \bigoplus \chi_{5j} \bigoplus \chi_{54} \text{ where } \chi_{ab} = -\chi_{ba} \text{ (Antisymmetric)}$$

$$\chi_{ab} \to \chi_{ij} \bigoplus \overline{\psi}_i \bigoplus \overline{\theta}_i \bigoplus \zeta \bigoplus -\overline{\psi}_j \bigoplus -\overline{\theta}_j \bigoplus -\zeta$$

$$S_F = -\frac{N}{2\lambda} \int d^2x \mathrm{Tr} \Big[\chi_{ij} \mathcal{D}_{[i} \psi_{j]} + 2\overline{\psi}_i \big(\mathcal{D}_i \overline{\zeta} - [\varphi, \psi_i] \big) + 2\overline{\theta}_i \big(\mathcal{D}_i \overline{\eta} - [\phi, \psi_i] \big) + 2\zeta \big([\varphi, \overline{\eta}] - [\phi, \overline{\zeta}] \big) + \eta \overline{\mathcal{D}}_i \psi_i + \eta [\overline{\varphi}, \overline{\zeta}] + \eta [\overline{\phi}, \overline{\eta}] \Big]$$

$\mathcal{N} = (8{,}8)$, d = 2 SYM theory
Q-Closed Action in Continuum

$$\mathcal{S}^{\mathcal{N}=4,d=4}_{\mathcal{Q}-closed} = -\frac{N}{4\lambda}\int d^4x \mathrm{Tr}\left[\epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de}\right]$$

Reducing 4th, 5th direction

$$\epsilon_{abcde} \rightarrow \quad \epsilon_{ijk45} \bigoplus \epsilon_{i4k5m} \bigoplus \epsilon_{4jk5m} \bigoplus \epsilon_{4jkl5} \ldots \text{ 20 Permutations}$$

$$\chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \rightarrow \chi_{ij}\overline{\mathcal{D}}_k\chi_{45} \bigoplus \chi_{i4}\overline{\mathcal{D}}_k\chi_{5m} \bigoplus \chi_{4j}\overline{\mathcal{D}}_k\chi_{5m} \bigoplus \chi_{4j}\overline{\mathcal{D}}_k\chi_{l5} \bigoplus \ldots$$

$$\mathcal{S}^{2d}_{\mathcal{Q}-closed} =$$

$$\int d^2x \mathrm{Tr}\Big[\frac{1}{2}\epsilon_{ijk}\overline{\psi}_i[\overline{\phi},\chi_{jk}] - \frac{1}{2}\epsilon_{ijk}\overline{\theta}_i[\overline{\varphi},\chi_{jk}] + \frac{1}{2}\epsilon_{ijk}\zeta\overline{\mathcal{D}}_i\chi_{jk} + \epsilon_{ijk}\overline{\psi}_i\overline{\mathcal{D}}_j\overline{\theta}_k$$
$$+\frac{1}{2}\epsilon_{ijk}\chi_{ij}[\overline{\phi},\overline{\psi}_k] - \frac{1}{2}\epsilon_{ijk}\chi_{ij}[\overline{\varphi},\overline{\theta}_k] + \frac{1}{2}\epsilon_{ijk}\chi_{ij}\overline{\mathcal{D}}_k\zeta - \epsilon_{ijk}\overline{\theta}_i\overline{\mathcal{D}}_j\overline{\psi}_k\Big]$$

where $\epsilon_{ijk} = \epsilon_{54ijk}$

Prescription for Lattice

$$x \to n_x a, \quad t \to n_t a, \quad \int d^2x \to \sum_{n=(n_x,n_t)}$$

$$\mathcal{A}_a(x) \to e^{\mathcal{A}_a(n)} \equiv \mathcal{U}_a(n)$$

$$\chi_{ab}(x) \to \chi_{ab}(n), \quad \psi_a(x) \to \psi_a(n), \quad \eta(x) \to \eta(n)$$

$$\mathcal{D}_a \to \mathcal{D}_a^{(+)}, \quad \overline{\mathcal{D}}_a \to \overline{\mathcal{D}}_a^{(-)}$$

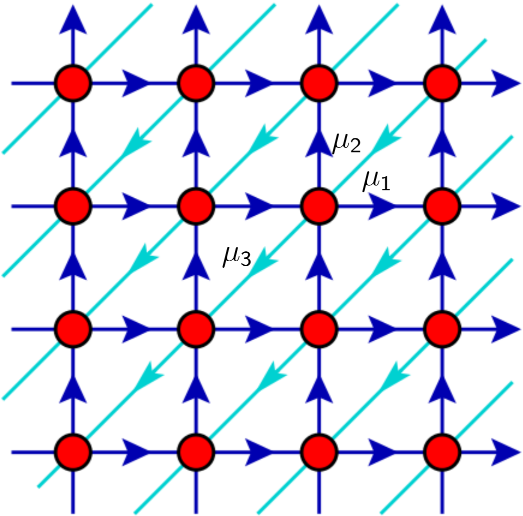
$$\mathcal{D}_c^{(+)} f(n) \equiv \mathcal{U}_c(n) f(n+\mu_c) - f(n)\mathcal{U}_c(n)$$

$$\mathcal{D}_c^{(+)} f_a(n) \equiv \mathcal{U}_c(n) f_a(n+\mu_c) - f_a(n)\mathcal{U}_c(n+\mu_a)$$

$$\overline{\mathcal{D}}_c^{(-)} f_c(n) \equiv f_c(n)\overline{\mathcal{U}}_c(n) - \overline{\mathcal{U}}_c(n-\mu_c) f_c(n-\mu_c)$$

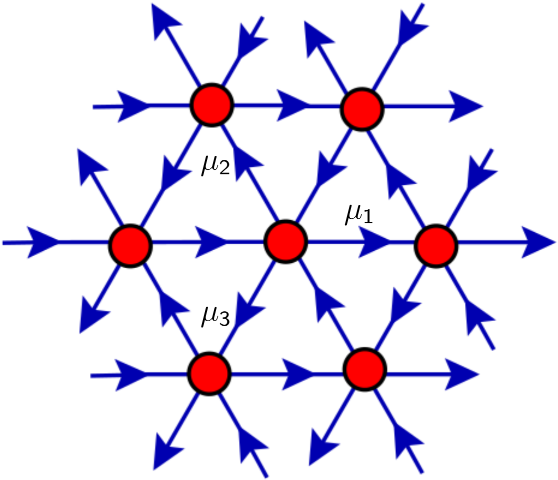
$$\overline{\mathcal{D}}_c^{(-)} f_{ab}(n) \equiv f_{ab}(n)\overline{\mathcal{U}}_c(n-\mu_c) - \overline{\mathcal{U}}_c(n+\mu_a+\mu_b-\mu_c) f_{ab}(n-\mu_c)$$

Square d = 2 Lattice



Square Lattice

Triangular d = 2 Lattice



Triangular lattice

Bosonic action on the lattice

$$
\begin{aligned}
S_B^{d=2,latt} = \tfrac{N}{2\lambda}\sum_n \mathrm{Tr}\Big[&\sum_{i=1,2,3} \frac{1}{2}\big(\mathcal{U}_i(n)\overline{\mathcal{U}}_i(n) - \overline{\mathcal{U}}_i(n-\mu_i)\mathcal{U}_i(n-\mu_i)\big)^2 \\
&+\tfrac{1}{2}([\overline{\varphi}(n),\varphi(n)])^2 + \tfrac{1}{2}([\overline{\phi}(n),\phi(n)])^2 \\
&+\big(\mathcal{U}_i(n)\overline{\mathcal{U}}_i(n) - \overline{\mathcal{U}}_i(n-\mu_i)\mathcal{U}_i(n-\mu_i)\big)[\overline{\varphi}(n),\varphi(n)] \\
&+\big(\mathcal{U}_i(n)\overline{\mathcal{U}}_i(n) - \overline{\mathcal{U}}_i(n-\mu_i)\mathcal{U}_i(n-\mu_i)\big) \\
&+[\overline{\phi}(n),\phi(n)] + [\overline{\varphi}(n),\varphi(n)][\overline{\phi}(n),\phi(n)] \\
&+\sum_{i,j=1,2,3} |\mathcal{U}_i(n)\mathcal{U}_j(n+\mu_i) - \mathcal{U}_j(n)\mathcal{U}_i(n+\mu_j)|^2 \\
&+2\sum_{i=1,2,3} |\mathcal{U}_i(n)\varphi(n+\mu_i) - \varphi(n)\mathcal{U}_i(n)|^2 \\
&+2\sum_{i=1,2,3} |\mathcal{U}_i(n)\phi(n+\mu_i) - \phi(n)\mathcal{U}_i(n)|^2 \\
&-2|[\varphi(n),\phi(n)]|^2\Big]
\end{aligned}
$$

Fermionic action on the lattice

$$
\begin{aligned}
S_F^{d=2latt.} = -\frac{N}{2\lambda} \sum_{n,i,j,k} \mathrm{Tr}\Bigg[& \frac{1}{2}\chi_{ij}(n)\big(\mathcal{U}_i(n)\psi_j(n+\mu_i) - \psi_j(n)\mathcal{U}_i(n+\mu_j)\big) \\
& -\tfrac{1}{2}\chi_{ij}(n)\big(\mathcal{U}_j(n)\psi_i(n+\mu_j) - \psi_i(n)\mathcal{U}_j(n+\mu_i)\big) \\
& +2\overline{\psi}_i(n)\big(\mathcal{U}_i(n)\overline{\zeta}(n+\mu_i) - \overline{\zeta}(n)\mathcal{U}_i(n) - [\varphi(n),\psi_i(n)]\big) \\
& +2\overline{\theta}_i(n)\big(\mathcal{U}_i(n)\overline{\eta}(n+\mu_i) - \overline{\eta}(n)\mathcal{U}_i(n) - [\phi(n),\psi_i(n)]\big) \\
& +2\zeta(n)\big([\varphi(n),\overline{\eta}(n)] - [\phi(n),\overline{\zeta}(n)]\big) \\
& +\eta(n)\big(\psi_i(n)\overline{\mathcal{U}}_i(n) - \overline{\mathcal{U}}_i(n-\mu_i)\psi_i(n-\mu_i)\big) \\
& +\eta(n)[\overline{\varphi}(n),\overline{\zeta}(n)] + \eta(n)[\overline{\phi}(n),\overline{\eta}(n)]\Bigg]
\end{aligned}
$$

Q-Closed Action on the Lattice

$$= \frac{1}{8} \sum_n \mathrm{Tr} \Big[$$

$$+ \epsilon_{ijk} \overline{\psi}_i(n) \overline{\phi}(n) \chi_{jk}(n+\mu_i) - \epsilon_{ijk} \overline{\psi}_i(n) \chi_{jk}(n+\mu_i) \overline{\phi}(n+\mu_i)$$

$$- \epsilon_{ijk} \overline{\theta}_i(n) \overline{\varphi}(n) \chi_{jk}(n+\mu_i) - \epsilon_{ijk} \overline{\theta}_i(n) \chi_{jk}(n+\mu_i) \overline{\varphi}(n+\mu_i)$$

$$+ \epsilon_{ijk} \chi_{ij}(n+\mu_k) \overline{\phi}(n+\mu_k) \overline{\psi}_k(n) - \epsilon_{ijk} \chi_{ij}(n+\mu_k) \overline{\psi}_k(n+\mu_k) \overline{\phi}(n)$$

$$- \epsilon_{ijk} \chi_{ij}(n+\mu_k) \overline{\varphi}(n+\mu_k) \overline{\theta}_k(n) - \epsilon_{ijk} \chi_{ij}(n+\mu_k) \overline{\theta}_k(n+\mu_k) \overline{\varphi}(n)$$

$$+ \epsilon_{ijk} \zeta(n) \chi_{jk}(n+\mu_i) \overline{\mathcal{U}}_i(n) - \epsilon_{ijk} \zeta(n) \overline{\mathcal{U}}_i(n+\mu_j+\mu_k) \chi_{jk}(n)$$

$$+ 2\epsilon_{ijk} \overline{\psi}_i(n+\mu_j+\mu_k) \overline{\theta}_k(n+\mu_j) \overline{\mathcal{U}}_j(n) - 2\epsilon_{ijk} \overline{\psi}_i(n+\mu_j+\mu_k) \overline{\mathcal{U}}_j(n+\mu$$

$$+ \epsilon_{ijk} \chi_{ij}(n+\mu_k) \zeta(n+\mu_k) \overline{\mathcal{U}}_k(n) - \epsilon_{ijk} \chi_{ij}(n+\mu_k) \overline{\mathcal{U}}_k(n) \zeta(n)$$

$$- 2\epsilon_{ijk} \overline{\theta}_i(n+\mu_j+\mu_k) \overline{\psi}_k(n+\mu_j) \overline{\mathcal{U}}_j(n)$$

$$- 2\epsilon_{ijk} \overline{\theta}_i(n+\mu_j+\mu_k) \overline{\mathcal{U}}_j(n+\mu_k) \overline{\psi}_k(n) \Big]$$

Observable & RHMC

$$S(\mathcal{U},\overline{\mathcal{U}},k,\overline{k},\Psi) = S_B(\mathcal{U},\overline{\mathcal{U}},k,\overline{k}) + \Psi^T\mathcal{D}(\mathcal{U},\overline{\mathcal{U}},k,\overline{k})\Psi$$

Kähler–Dirac fermion field Ψ

$$\Psi^T = \left(\eta^T, \overline{\eta}^T, \zeta^T, \overline{\zeta}^T, \psi_i^T, \overline{\psi}_i^T, \overline{\theta}_i^T, \chi_{ij}^T\right)$$

$\mathcal{D}(\mathcal{U},\overline{\mathcal{U}},k,\overline{k}) =$ Fermion Operator

$$< \mathcal{O} >= \frac{\int d\mathcal{U}d\overline{\mathcal{U}}dkd\overline{k}d\Psi\mathcal{O}e^{-S(\mathcal{U},\overline{\mathcal{U}},k,\overline{k},\Psi)}}{\int d\mathcal{U}d\overline{\mathcal{U}}dkd\overline{k}d\Psi e^{-S(\mathcal{U},\overline{\mathcal{U}},k,\overline{k},\Psi)}}$$

$$< \mathcal{O} >= \frac{\int d\mathcal{U}d\overline{\mathcal{U}}dkd\overline{k}\mathcal{O}e^{-S_B(\mathcal{U},\overline{\mathcal{U}},k,\overline{k})}pf\mathcal{D}(\mathcal{U},\overline{\mathcal{U}},k,\overline{k})}{\int d\mathcal{U}d\overline{\mathcal{U}}dkd\overline{k}e^{-S_B(\mathcal{U},\overline{\mathcal{U}},k,\overline{k})}pf\mathcal{D}(\mathcal{U},\overline{\mathcal{U}},k,\overline{k})}$$

$$|pf\mathcal{D}| = \left(det[\mathcal{D}^\dagger\mathcal{D}]\right)^{1/4} \propto \int[d\Phi^\dagger][d\Phi]exp\left[-\Phi^\dagger\left(\mathcal{D}^\dagger\mathcal{D}\right)^{-1/4}\Phi\right]$$

SUSY LATTICE Code

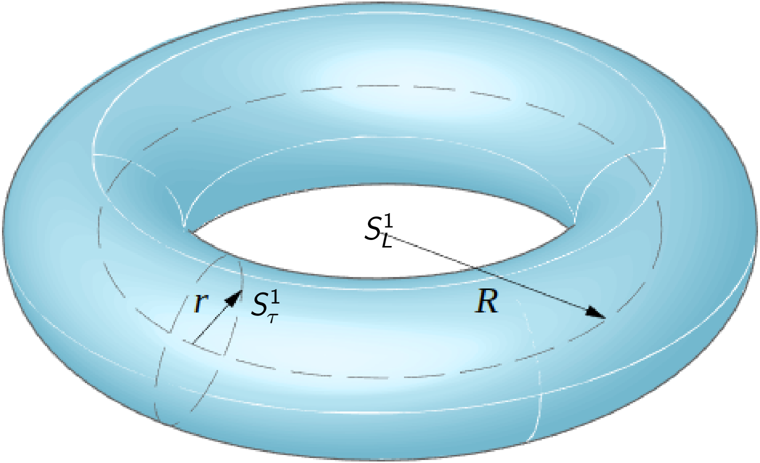
- SUSY LATTICE is parallel code for RHMC simulations of extended-supersymmetric Yang–Mills theories in various dimensions.

- SUSY LATTICE is based on the MILC code for lattice QCD.

- SUSY LATTICE is mostly based on C - Language.

- SUSY LATTICE is developed in a publicly accessible version control repository https://github.com/daschaich/susy.git

2-Torus



$$T^2 = S^1 \times S^1$$

Phase diagram

JHEP 01 (2006) 140

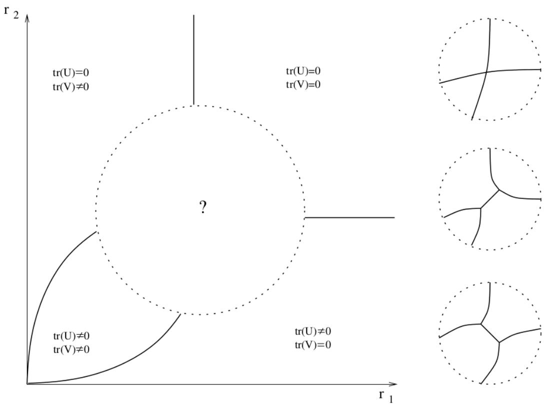


FIGURE: Phase diagram of massless $SU(N)$ bosonic gauge theory

Future work

- To study the phase diagram of two-dimensional massless $SU(N)$ bosonic gauge theory and look for three possible completions.
- To study a complete system with the differently skewed torus.
- To study certain black hole solutions in Type IIA and IIB supergravity. Phys.Rev.D 97 (2018) 8, 086020

Thank You !