6th Mandelstam Theoretical Physics School and Workshop 2024

Complex Langevin Study of Spontaneous SO(10) Symmetry Breaking in Euclidean IKKT Matrix Model

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16th Jan, 12:00 - 12:30 SAST
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Ongoing work with
Anosh Joseph and Piyush Kumar

PLAN OF TALK

1. IKKT matrix model, SSB, and associated sign problem
2. Complex Langevin Method (CLM)
3. CLM to IKKT: Challenges and results
4. Summary and ongoing work

IKKT MATRIX MODEL

Ishibashi, Kawai, Kitazawa, Tsuchiya (1996)

- Matrix models are promising candidate for non-perturbative formulation of superstrings

IKKT: *0d matrix model in large-N limit* | IIB: *10d superstring theory*

- Inspired from Green-Schwarz action of type IIB superstring with Schild gauge

Matrix Regularization

Hoppe (1982)

$$\{A^a, A^b\} \to i\left[X^a, X^b\right]$$

phase-space vol $\to$ Tr

- Can be derived from dimensional reduction of *10d* $\mathcal{N} = 1$ SYM theory to a point

Jevicki, Yoneya (1997)

- Spacetime emerges dynamically from the eigenvalues of Hermitian matrices

- Our motivation: dynamical compactification (phenomenological admissibility)

Why we live in three spatial dimensions?

IKKT MATRIX MODEL

Ishibashi, Kawai, Kitazawa, Tsuchiya (1996)

Euclidean IKKT Model

Wick rotated: $X_0 \to iX_{10},\ \Gamma^0 \to -i\Gamma_{10}$

$$Z = \int \mathscr{D}X\, \mathscr{D}\psi\, e^{-(S_b+S_f)}$$

$$S_b = -\frac{1}{4}N\ \mathrm{tr}\left([X_\mu, X_\nu]^2\right)$$

$$S_f = -\frac{1}{2}N\ \mathrm{tr}\left(\psi_\alpha\, \Gamma^\mu_{\alpha\beta}\, [X_\mu, \psi_\beta]\right)$$

Vectors X_μ

Majorana-Weyl spinors ψ_α

X_μ, ψ_α : $N \times N$ *Hermitian traceless matrices*

Γ^μ : $2^4 \times 2^4$ *gamma matrices*

$\mu, \nu = 1,2\ldots,10$

$\alpha, \beta = 1,2,\ldots,16$

Z finite Krauth, Nicolai and Staudacher (1998)

Austing and Wheater (2001)

Euclidean IKKT: Symmetries

- Euclidean action is invariant under following symmetries

  A. SU(N) gauge symmetry
  B. $\mathcal{N} = 2$ supersymmetry
  C. SO(10) rotational symmetry

$$Z = \int \mathcal{D}X \, \mathcal{D}\psi \, e^{-(S_b + S_f)}$$

$$S_b = -\frac{1}{4} N \, \mathrm{tr} \left([X_\mu, X_\nu]^2 \right)$$

$$S_f = -\frac{1}{2} N \, \mathrm{tr} \left(\psi_\alpha \, \Gamma^\mu_{\alpha\beta} \, [X_\mu, \psi_\beta] \right)$$

A. Gauge Symmetry: SU(N)

$$X_\mu \to X'_\mu = U^\dagger X_\mu U$$

$$\psi_\alpha \to \psi'_\alpha = U^\dagger \psi_\alpha U$$

- Inherited from *10D* $\mathcal{N} = 1$ SYM, for $U \in \mathrm{SU}(N)$

Euclidean IKKT: Symmetries

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$$Z = \int \mathscr{D}X\, \mathscr{D}\psi\, e^{-(S_b + S_f)}$$

$$S_b = -\frac{1}{4} N \,\mathrm{tr}\left([X_\mu, X_\nu]^2 \right)$$

$$S_f = -\frac{1}{2} N \,\mathrm{tr}\left(\psi_\alpha\, \Gamma^\mu_{\alpha\beta}\, [X_\mu, \psi_\beta] \right)$$

A. SU(N) gauge symmetry
B. $\mathcal{N} = 2$ supersymmetry
C. SO(10) rotational symmetry

B. Supersymmetry: $\mathcal{N} = 2$

$$\delta^{(1)} X_\mu = i\bar{\epsilon}_1 \Gamma_\mu \psi \qquad \delta^{(2)} X_\mu = 0$$

$$\delta^{(1)} \psi = -\frac{i}{2} \left[X_\mu, X_\nu \right] \Gamma^{\mu\nu} \epsilon_1 \qquad \delta^{(2)} \psi = \epsilon_2$$

- Two set of supersymmetries: $\delta^{(1)} S = \delta^{(2)} S = 0$

Euclidean IKKT: Symmetries

- Euclidean action is invariant under following symmetries

A. SU(N) gauge symmetry
B. $\mathcal{N} = 2$ supersymmetry
C. SO(10) rotational symmetry

$$Z = \int \mathscr{D}X\, \mathscr{D}\psi\, e^{-(S_b + S_f)}$$
$$S_b = -\frac{1}{4} N \operatorname{tr}\left([X_\mu, X_\nu]^2 \right)$$
$$S_f = -\frac{1}{2} N \operatorname{tr}\left(\psi_\alpha\, \Gamma^\mu_{\alpha\beta}\, [X_\mu, \psi_\beta] \right)$$

C. Rotational Symmetry: SO(10)

$$X_\mu \to X'_\mu = \Lambda^\rho_\mu X_\rho$$

- Spontaneously broken:

SO(10) → SO(d)

matrix d.o.f. → gravitational d.o.f.

dynamical compactification

Numerical Sign Problem and SSB Investigation

$$Z = \int \mathscr{D}X \, \mathscr{D}\psi \, e^{-(S_b + S_f)}$$

$$Z = \int \mathscr{D}X \, \mathrm{Pf}\mathscr{M} \, e^{-S_b} = \int \mathscr{D}X \, e^{-S_{\mathrm{eff}}}$$

$$S_{\mathrm{eff}} = S_b - \ln(\mathrm{Pf}\mathscr{M})$$

$$S_b = -\frac{1}{4} N \, \mathrm{tr}\left([X_\mu, X_\nu]^2 \right)$$

- Integrating out fermions, we obtain Fermion matrix $\mathscr{M}$:

$$\mathscr{M}_{a\alpha, b\beta} = \frac{N}{2} \Gamma^\mu_{\alpha\beta} \mathrm{Tr}\left(X_\mu \left[t^a, t^b \right] \right)$$

anti-symmetric matrix: $16(N^2 - 1) \times 16(N^2 - 1)$

complex nature: $\mathrm{Pf}\mathscr{M} = |\mathrm{Pf}\mathscr{M}| \, e^{i\theta}$

~~$\theta \approx 0$~~ *fluctuates wildly*

Numerical Sign Problem and SSB Investigation

- MC and $1/D$ expansion to bosonic IKKT model - No SSB
  Hotta, Nishimura and Tsuchiya (1998)
- MC to phase-quenched IKKT model with $|\mathrm{Pf}\mathscr{M}|$ - No SSB
  Ambjorn, Anagnostopoulos, Bietenholz, Hotta and Nishimura (2000)
  Anagnostopoulos, Azuma and Nishimura (2013)

why phase quenched?

$e^{-S_{\mathrm{eff}}} \rightarrow$ ~~probability weight~~

sign problem in Monte Carlo!

- Phase $e^{i\theta}$ responsible for SSB!

how to incorporate complex phase?

complex Langevin

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Complex Langevin Method

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; \mathcal{O}(\phi) \; e^{-S_E}$$

→ *generate $\{\phi_i\}$ with probability $e^{-S[\phi_i]}$*

↓

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\phi_i)$$

Path integral Monte Carlo

For complex actions: $S = S_{re} + iS_{im}$

$e^{-(S_{re}+iS_{im})} \rightarrow$ ~~probability weight~~

sign problem in Monte Carlo!

↓

complex Langevin dynamics

Complex Langevin Method in a nutshell

Complex extension of Stochastic Quantization

expectation values ↔ equilibrium values

Parisi and Wu (1981)
Klauder (1983), Parisi (1983)

$$\partial_\tau \Phi(\tau) = v(\Phi, \tau) + \eta(\tau)$$

drift (friction)

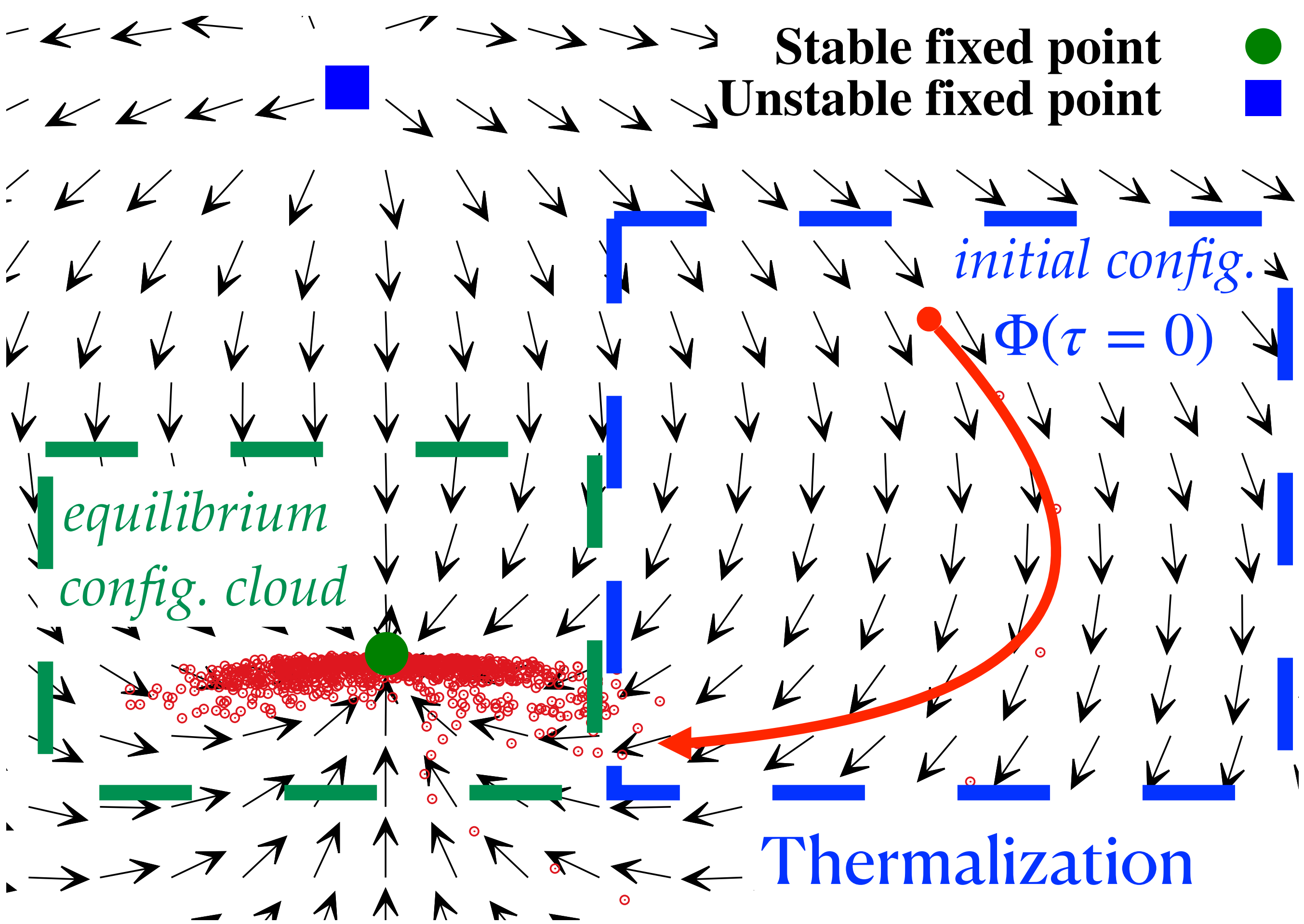
$$v(\Phi, \tau) = -\frac{\delta S[\Phi]}{\delta \Phi(\tau)}$$

noise (kick)

$$\langle \eta(\tau) \rangle = 0$$

$$\langle \eta(\tau)\eta(\tau') \rangle = 2\delta(\tau - \tau')$$

$\Phi = \phi_x + i\phi_y$

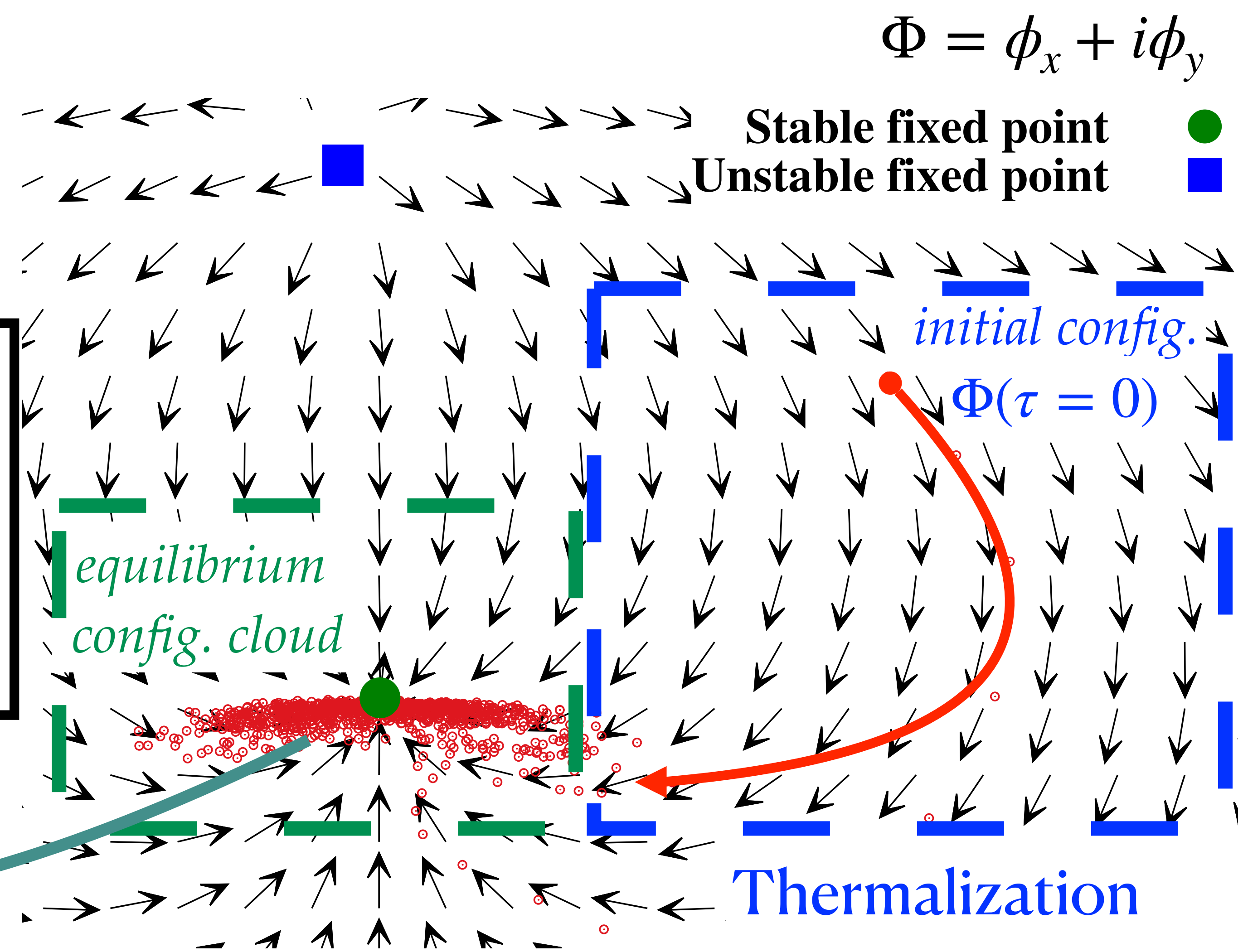


Complex Langevin Method in a nutshell

Klauder (1983), Parisi (1983)

$$\partial_\tau \Phi(\tau) = v(\Phi, \tau) + \eta(\tau)$$

$$\langle \mathcal{O}[\Phi(\tau)] \rangle_P = \int d\phi_x d\phi_y \times P[\phi_x, \phi_y; \tau]\, \mathcal{O}[\Phi(\tau)]$$



Complex Langevin Method
Correct Convergence

Klauder (1983), Parisi (1983)

- Expectation values of holomorphic observables $\mathcal{O}$ in real and complex configuration space

$$\langle \mathcal{O} \rangle_{P(\tau)} = \frac{\int d\phi_x d\phi_y \; P(\phi_x, \phi_y; \tau) \; \mathcal{O}(\phi_x + i\phi_y)}{\int d\phi_x d\phi_y \; P(\phi_x, \phi_y; \tau)}$$

$$\langle \mathcal{O} \rangle_{\rho(\tau)} = \frac{\int d\phi \; \rho(\phi; \tau) \; \mathcal{O}(\phi)}{\int d\phi \; \rho(\phi; \tau)}$$

- Complex Langevin is justified if

$$\lim_{\tau \to \infty} \langle \mathcal{O} \rangle_{P(\tau)} \overset{?}{\simeq} \langle \mathcal{O} \rangle_{\rho(\tau)}$$

- When action is complex, no such proof of convergence exists!

Complex Langevin Method Correctness Criteria

Aarts, Seiler, and Stamatescu (2009)

$$\langle \mathscr{L}^T \mathcal{O} \rangle = 0$$

$$\mathscr{L}^T = \frac{\partial}{\partial \phi_x}\left(\frac{\partial}{\partial \phi_x} + \mathrm{Re}\left[\frac{\partial S[\Phi]}{\partial \phi_x}\right]\right) + \frac{\partial}{\partial \phi_y}\left(\mathrm{Im}\left[\frac{\partial S[\Phi]}{\partial \phi_y}\right]\right)$$

$\mathscr{L}$: Langevin operator

Nagata, Nishimura, and Shimasaki (2016)

magnitude of drift : $u = \left|\frac{\partial S[\Phi]}{\partial \Phi}\right|$

Probability distribution $P(u)$ falls off exponentially or faster

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Applying Complex Langevin Method To IKKT Matrix Model

Investigate Spontaneous SO(10) Symmetry Breaking

Extent of space-time as an order parameter

$$\langle \lambda_\mu \rangle = \left\langle \frac{1}{N} \mathrm{tr} \left(X_\mu \right)^2 \right\rangle$$

Intact SO(10)

$\mu = 1, 2, \ldots, 10$:
all equivalent directions

SSB SO(10) → SO(d)

$\mu = 1, 2, \ldots, d$: extended directions
$\mu = d+1, d+2, \ldots, 10-d$: shrunken directions

Applying complex Langevin method to IKKT model

Computational side-note

- MPI-*based* X_μ *on* μ*-th core*
- *Broadcast/Gather to all cores*

Bosonic IKKT model

- Langevin evolution of X_μ at Langevin time τ

$$\frac{d(X_\mu)_{ji}}{d\tau} = -\frac{\partial S_b}{\partial (X_\mu)_{ji}} + (\eta_\mu)_{ij}(\tau)$$

$$\frac{\partial S_b}{\partial (X_\mu)_{ji}} = -N\left(\left[X_\nu, [X_\mu, X_\nu]\right]\right)_{ij}$$

Hermitian noise obeying

$$\propto \exp\left(-\frac{1}{4}\int \mathrm{Tr}\left(\eta_\mu^2(\tau)\right) d\tau\right)$$

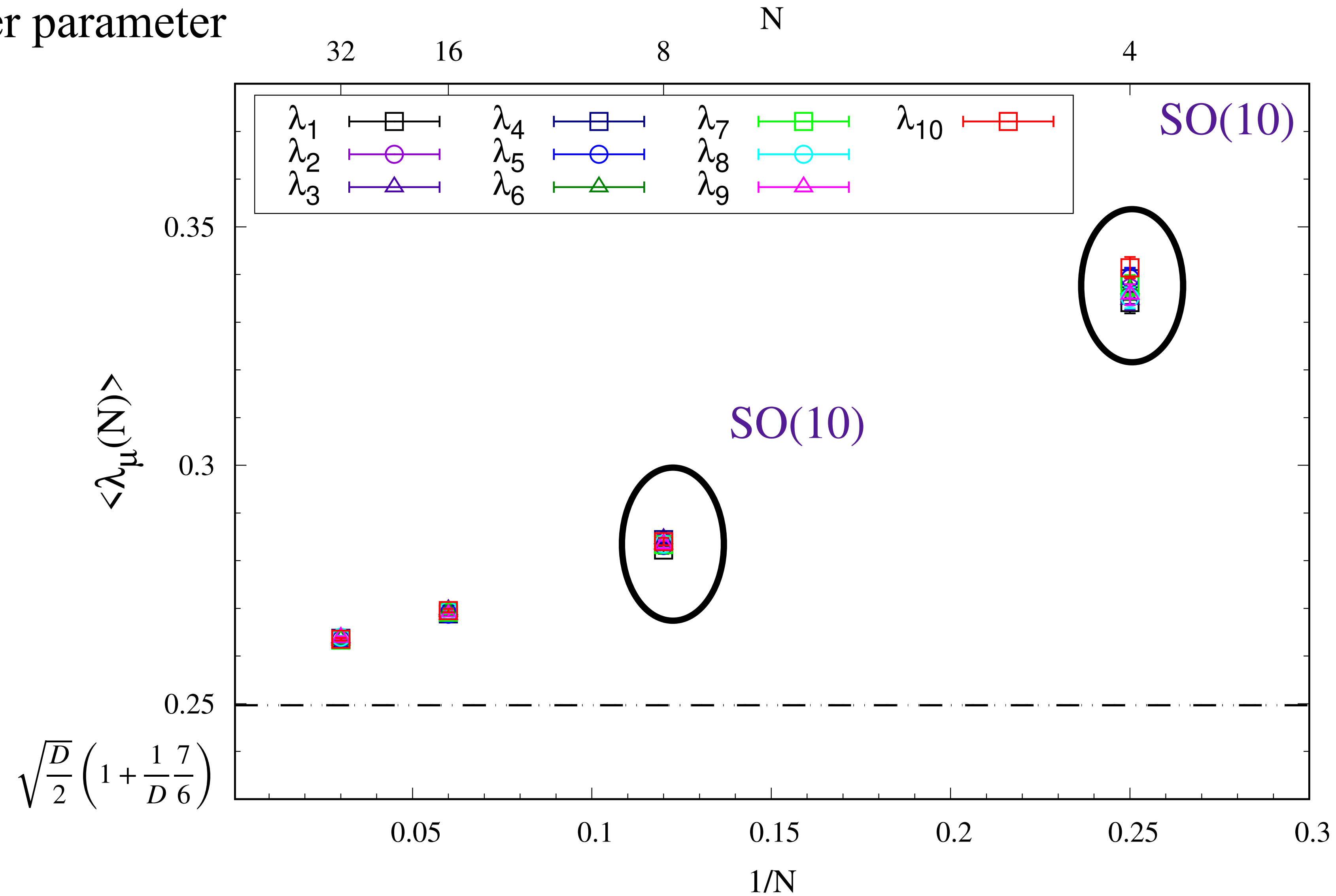
Applying complex Langevin method to IKKT model

Bosonic IKKT model

- Extent of space-time as an order parameter

$$\langle \lambda_\mu \rangle = \left\langle \frac{1}{N} \mathrm{tr} \left(X_\mu \right)^2 \right\rangle$$

- Results: No SSB
  SO(10) *at finite N*



Applying complex Langevin method to IKKT model

Computational side-note

- MPI-*based* X_μ *on* μ*-th core*
- *Broadcast/Gather to all cores*

IKKT model

- Langevin evolution of X_μ at Langevin time τ

$$\frac{d(X_\mu)_{ji}}{d\tau} = -\frac{\partial S_{\text{eff}}}{\partial (X_\mu)_{ji}} + (\eta_\mu)_{ij}(\tau)$$

$$\frac{\partial S_{\text{eff}}}{\partial (X_\mu)_{ji}} = \frac{\partial S_b}{\partial (X_\mu)_{ji}} - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}}\mathcal{M}^{-1}\right)$$

Hermitian noise obeying

$$\propto \exp\left(-\frac{1}{4}\int \mathrm{Tr}\left(\eta_\mu^2(\tau)\right) d\tau\right)$$

- Challenges: violation of correctness criteria

A. Excursion problem
B. Singular drift problem

Challenges: Excursion Problem

- Langevin evolution $X_\mu : \mathrm{SU}(N) \to \mathrm{SL}(N, \mathbb{C})$
- X_μ too far away from Hermitian: results are unreliable
- Hermiticity Norm

$$\mathcal{N}_H = -\frac{1}{10N} \sum_{\mu=1}^{10} \mathrm{Tr}\left[\left(X_\mu - X_\mu^\dagger\right)^2\right]$$

- Possible solutions: Gauge cooling
  Seiler, Sexty and Stamatescu (2012)

  Dynamical stabilization
  Attanasio and Jäger (2019)

Challenges: Singular Drift

- Fermion operator $\mathscr{M}$ has near-zero eigenvalues

$$\mathscr{M}_{a\alpha,b\beta} = \frac{N}{2}\Gamma^{\mu}_{\alpha\beta}\mathrm{Tr}\left(X_\mu\left[t^a, t^b\right]\right)$$

- The drift term diverges: results unreliable

$$\frac{\partial S_{\mathrm{f}}}{\partial (X_\mu)_{ji}} = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial \mathscr{M}}{\partial (X_\mu)_{ji}}\mathscr{M}^{-1}\right)$$

- Possible solutions: Mass deformations!

Challenges: Singular Drift

Mass Deformations:

$$S \to S = S_{\text{IKKT}} + \Delta S$$

Ito and Nishimura (2016)

- In general, $\Delta S = \Delta S_b + \Delta S_{\text{f}}$

$$\Delta S_b \propto \text{Tr}\left(M_{\mu\nu} X_\mu X_\nu\right)$$

$$\Delta S_{\text{f}} \propto \text{Tr}\left(\psi_\alpha \gamma_{\alpha\beta} \psi_\beta\right)$$

shifts eigenvalue distribution of $\mathscr{M}$

$$\mathscr{M}_{a\alpha,b\beta} \to \tilde{\mathscr{M}}_{a\alpha,b\beta} = \frac{N}{2}\Gamma^{\mu}_{\alpha\beta}\text{tr}\left(X_\mu\left[t^a, t^b\right]\right) + \gamma_{\alpha\beta}\delta_{ab}$$

- Explicitly break the SO(10) rotational symmetry *supersymmetry?*
- Recover original model: $\lim \Delta S \to 0$

Challenges: Singular Drift

SUSY-Breaking Mass Deformations:

$$S \to S = S_{\text{IKKT}} + \Delta S$$

Ito and Nishimura (2016)

- Recently, Anagnostopoulos, Azuma, Ito, Nishimura, Okubo, Papadoudis (2020)

breaks SO(10) symmetry: $\Delta S_{\text{b}} = \epsilon \frac{N}{2} \sum_{\mu} m_{\mu} \text{Tr} \left(X_{\mu} \right)^2$

shifts eigenvalue distribution of $\mathcal{M}$: $\Delta S_{\text{f}} = - m_{\text{f}} \frac{N}{2} \text{Tr} \left(\psi_{\alpha} \gamma_{\alpha\beta} \psi_{\beta} \right); \; \gamma = i \Gamma^{8} \Gamma^{9\dagger} \Gamma^{10}$

supersymmetry explicitly broken

- Probe SSB in original IKKT model by extrapolations:

A. $N \to \infty$
B. $\epsilon \to 0$
C. $m_{\text{f}} \to 0$

$\Longrightarrow$

$m_{\text{f}} = 3.0 \implies$ SO(7)
$m_{\text{f}} = 1.4 \implies$ SO(4)
$m_{\text{f}} = 1.0,\ 0.9,\ 0.7 \implies$ SO(3) consistent with GEM study

Nishimura, Okubo, Sugino (2011)

Challenges: Singular Drift

SUSY-Preserving Mass Deformations:

$$S \rightarrow S = S_{\text{IKKT}} + \Delta S$$

- Analogous to *1d* BMN matrix model

Berenstein, Maldacena, and Nastase (2002)
Bonelli and Natuurkunde (2002)

$$\Delta S = N\ \mathrm{tr}\left(-M^{\mu\nu} X_\mu X_\nu + \frac{i}{8} \overline{\psi} N_3 \psi + i N^{\mu\nu\rho} X_\mu [X_\nu, X_\rho] \right)$$

$M_{\mu\nu}$: bosonic mass matrix
N_3: fermion mass matrix
Myers term

Challenges: Singular Drift

SUSY-Preserving Mass Deformations:

$$S \rightarrow S = S_{\text{IKKT}} + \Delta S$$

$$\Delta S = N\,\mathrm{tr}\left(-M^{\mu\nu}X_\mu X_\nu + \frac{i}{8}\bar{\psi}N_3\psi + iN^{\mu\nu\rho}X_\mu[X_\nu, X_\rho]\right)$$

$M_{\mu\nu}$: bosonic mass matrix

N_3: fermion mass matrix

Myers term

- Simplest solution:

$$\delta X^\mu = -\frac{1}{2}\bar{\epsilon}\Gamma^\mu\psi$$

$$\delta\psi = \frac{1}{4}\left[X^\mu, X^\nu\right]\Gamma_{\mu\nu}\epsilon - \frac{i}{16}X^\mu\left(\Gamma_\mu N_3 + 2N_3\Gamma_\mu\right)\epsilon$$

provided flux constraint

$$\left[N_3(\Gamma^\mu N_3 + 2N_3\Gamma^\mu) + 4^3 M^{\mu\nu}\Gamma_\nu\right]\epsilon = 0 \quad \Longrightarrow$$

$$M = -\frac{\Omega^2}{4^3}\left(\mathbb{I}_7 \oplus 3\mathbb{I}_3\right)$$

$$N_3 = -\Omega\Gamma^8\Gamma^{9\dagger}\Gamma^{10}$$

$$N^{\mu\nu\rho} = \frac{\Omega}{3!}\sum_{\mu\nu\rho=8}^{10}\epsilon^{\mu\nu\rho}$$

Challenges: Singular Drift

SUSY-Preserving Mass Deformations:

$$S \to S = S_{\mathrm{IKKT}} + \Delta S$$

$$\Delta S = N\,\mathrm{tr}\left(\frac{\Omega^2}{4^3}\sum_{i=1}^{7} X_i^2 + \frac{3\Omega^2}{4^3}\sum_{a=8}^{10} X_a^2 + \frac{i\Omega}{3!}\epsilon^{abc}X_a[X_b, X_c] - \frac{N\Omega}{8}\psi_\alpha\gamma_{\alpha\beta}\psi_\beta\right)$$

bosonic mass term
fermion mass term
Myers term

$$\gamma_{\alpha\beta} = i(\Gamma^8\Gamma^{9\dagger}\Gamma^{10})_{\alpha\beta}$$

- Avoids singular drift problem?

shifts eigenvalue distribution of $\mathcal{M}$

$$\mathcal{M}_{a\alpha,b\beta} \to \mathcal{M}_{a\alpha,b\beta}(\Omega) = \frac{N}{2}\Gamma^{\mu}_{\alpha\beta}\mathrm{tr}\left(X_\mu\left[t^a, t^b\right]\right) - \frac{N\Omega}{8}\gamma_{\alpha\beta}\delta_{ab}$$

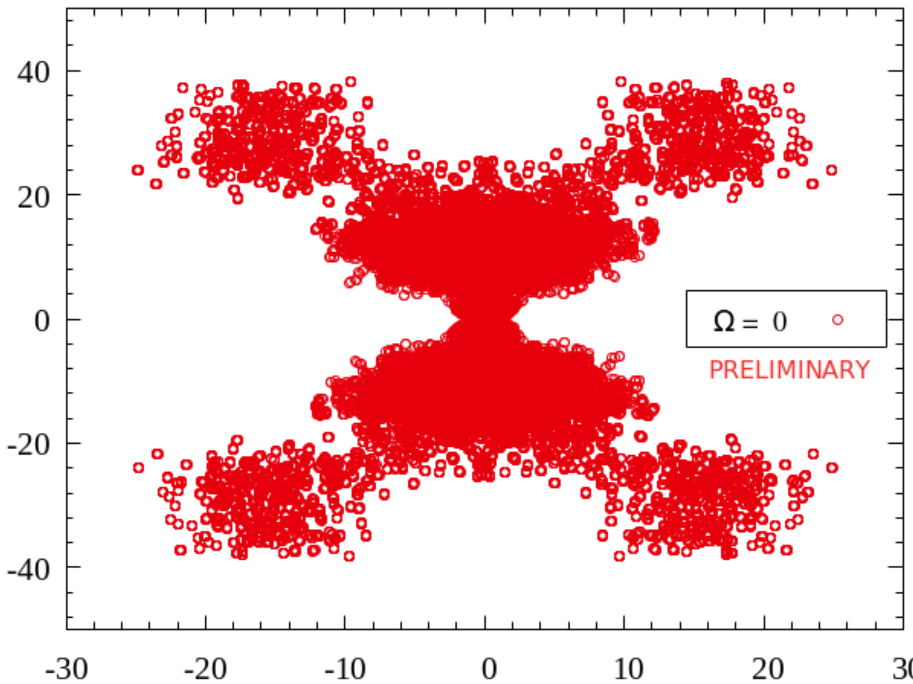
Challenges: Singular Drift

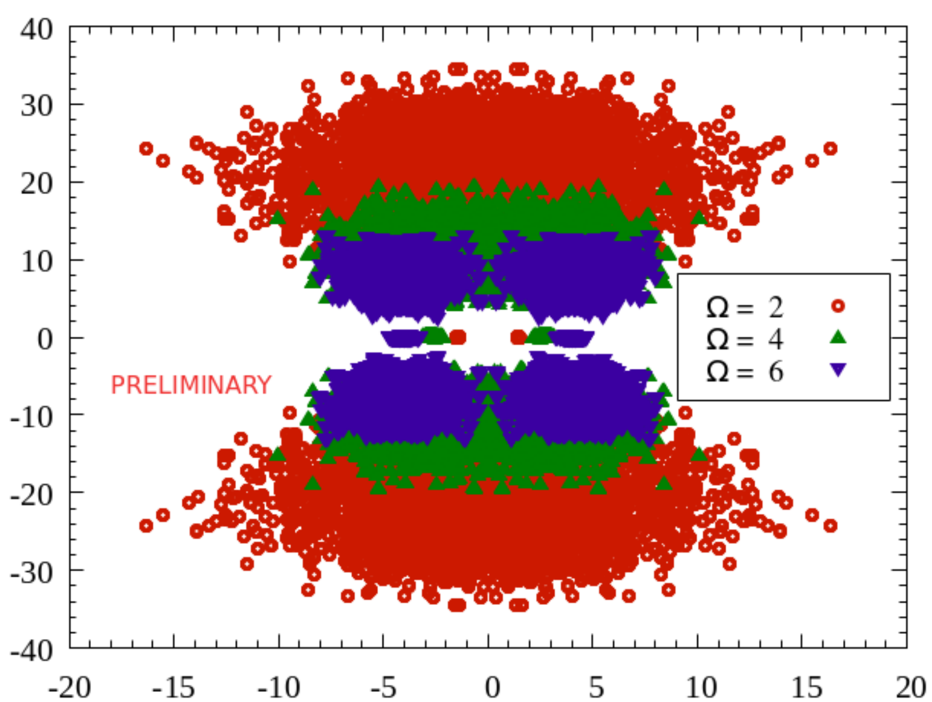
SUSY-Preserving Mass Deformations:

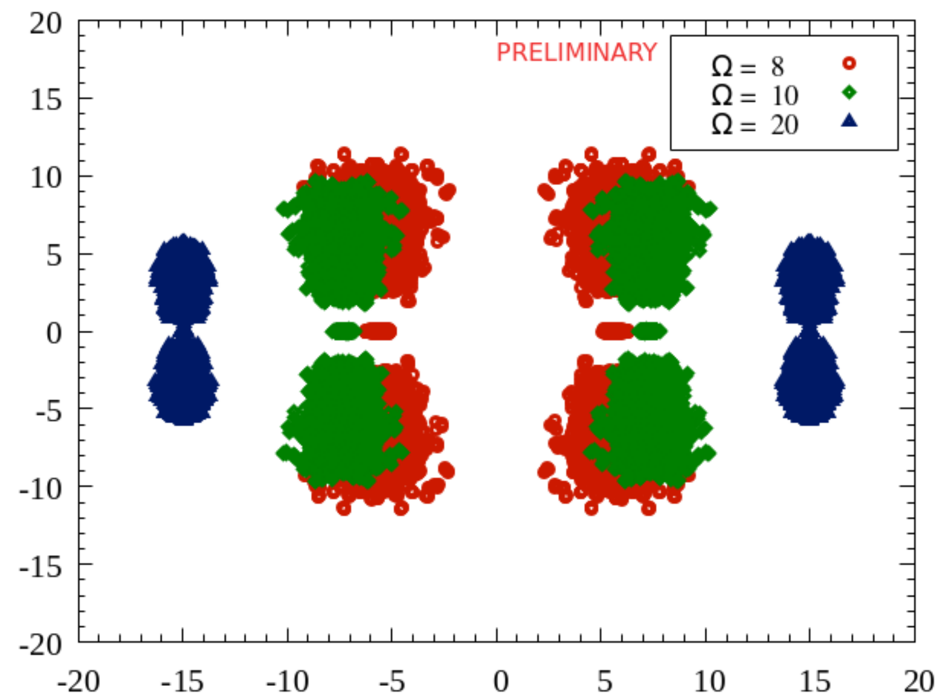
$$S \to S = S_{\rm IKKT} + \Delta S$$

$$\mathcal{M}_{a\alpha,b\beta} \to \mathcal{M}_{a\alpha,b\beta}(\Omega) = \frac{N}{2}\Gamma^{\mu}_{\alpha\beta}\mathrm{tr}\left(X_{\mu}\left[t^{a},t^{b}\right]\right) - \frac{N\Omega}{8}\gamma_{\alpha\beta}\delta_{ab}$$

SUSY preserving mass deformations evades singular drift problem!







$N = 6$

Probing SSB: SUSY preserving mass deformations

Bosonic IKKT model with Myers

$$S_{\mathrm{b}} = S_{\mathrm{bIKKT}} + N\ \mathrm{Tr}\left(\frac{\Omega^2}{4^3}\sum_{i=1}^{7} X_i^2 + \frac{3\Omega^2}{4^3}\sum_{a=8}^{10} X_a^2 + \frac{i\Omega}{3\,!}\epsilon^{abc}X_a[X_b, X_c] \right)$$

bosonic mass term

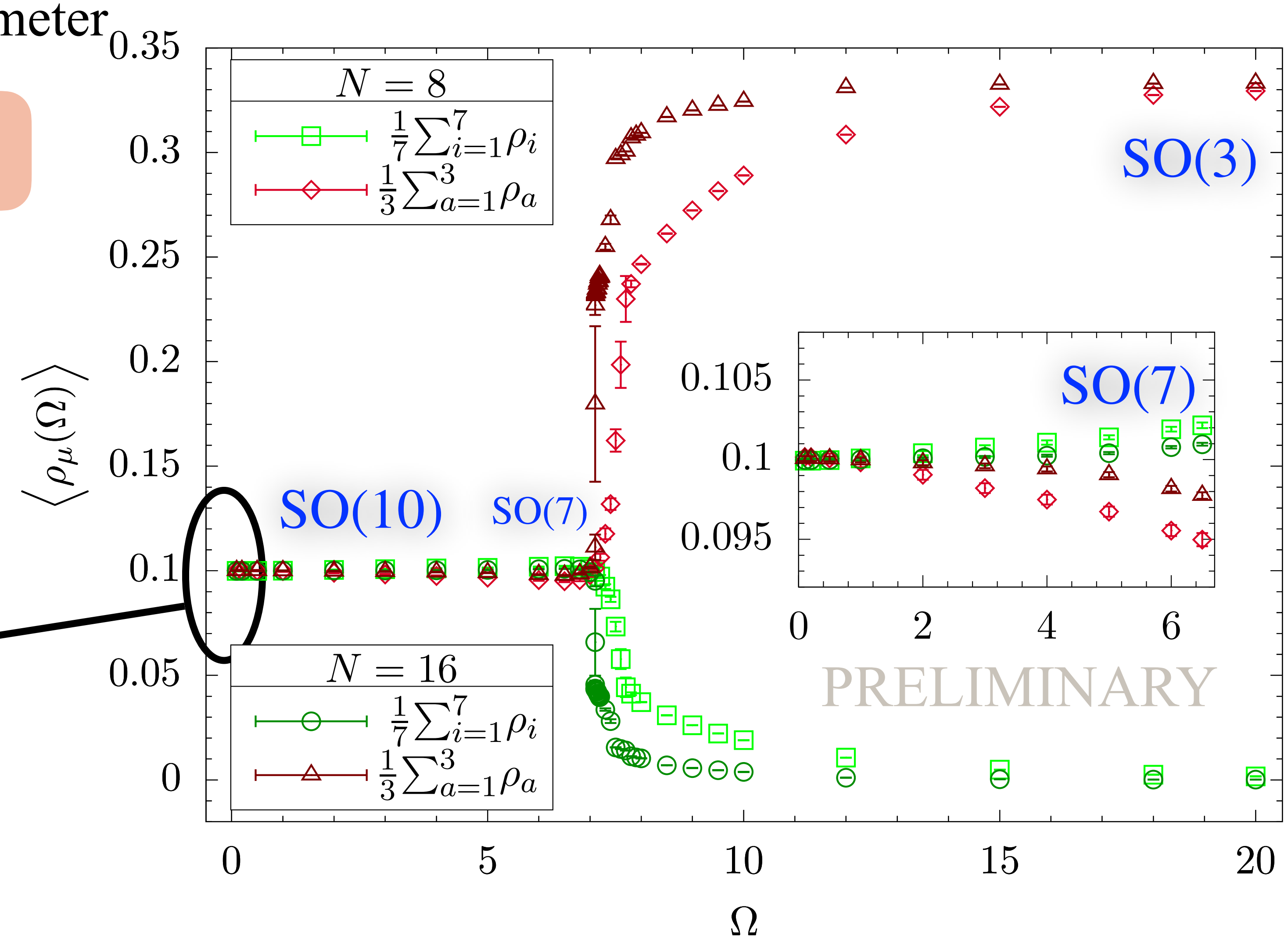
Myers term

- Extent of space-time as an order parameter

$$\lambda_\mu(\Omega) = \frac{1}{N}\mathrm{tr}\left(X_\mu\right)^2 \qquad \propto \quad \Omega^{-|p|}$$

$$\rho_\mu(\Omega) = \frac{\lambda_\mu(\Omega)}{\sum_\mu \lambda_\mu(\Omega)}$$

- Results:



SO(10) restored
$\Longrightarrow$ No SSB

Probing SSB: SUSY preserving mass deformations

IKKT deformed model with Myers

$$S = S_{\rm IKKT} + N\,{\rm tr}\left(\frac{\Omega^2}{4^3}\sum_{i=1}^{7} X_i^2 + \frac{3\Omega^2}{4^3}\sum_{a=8}^{10} X_a^2 + \frac{i\Omega}{3!}\epsilon^{abc}X_a[X_b, X_c] - \frac{N\Omega}{8}\psi_\alpha\gamma_{\alpha\beta}\psi_\beta\right)$$

bosonic mass term
fermion mass term
Myers term

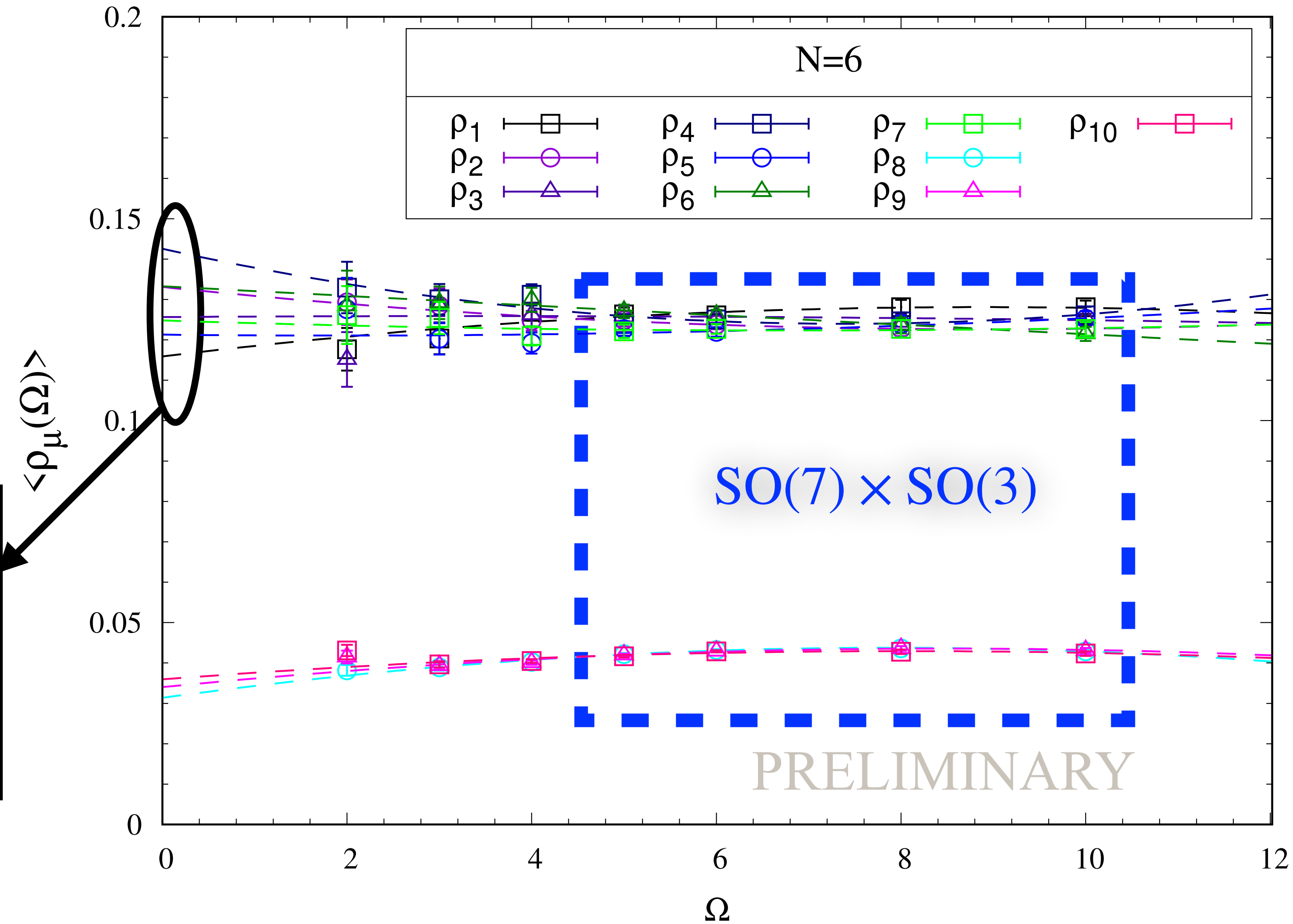
$$\gamma_{\alpha\beta} = i(\Gamma^8\Gamma^{9\dagger}\Gamma^{10})_{\alpha\beta}$$

- Extent of space-time as an order parameter

$$\rho_\mu(\Omega) = \frac{\lambda_\mu(\Omega)}{\sum_\mu \lambda_\mu(\Omega)}$$

- Results:

Even for $N = 6$ extents are not same
Spontaneously broken:
$SO(10) \rightarrow SO(d) \times SO(10-d)$
$\implies$ hints $SO(d) : d < 7$



Conclusions

- Complex Langevin method can detect spontaneous SO(10) symmetry breaking in IKKT matrix model

- SUSY-preserving mass deformations successfully evade singular drift problem

Ongoing work

- Large-N extrapolations to investigate exact nature of SO(d) symmetric vacuum

- Computation of $$\frac{\partial S_{\mathrm{f}}}{\partial (X_\mu)_{ji}} = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}}\mathcal{M}^{-1}\right)$$

  using *stochastic estimation (noisy estimator method)* with MPI-CUDA based parallel architecture.

Ongoing work

time complexity

- Computation of $\frac{\partial S_f}{\partial (X_\mu)_{ji}} = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}}\mathcal{M}^{-1}\right)$

$O(N^6) \to O(N^4)$

using *stochastic estimation (noisy estimator method)*
with MPI-CUDA based parallel architecture.

Stochastic Trace Estimation

$$\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}}\mathcal{M}^{-1}\right) \simeq \eta^\dagger \frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}} \underbrace{\mathcal{M}^{-1}\eta}_{\zeta}$$

$$\mathcal{M}^\dagger \mathcal{M} \zeta = \mathcal{M}^\dagger \eta$$

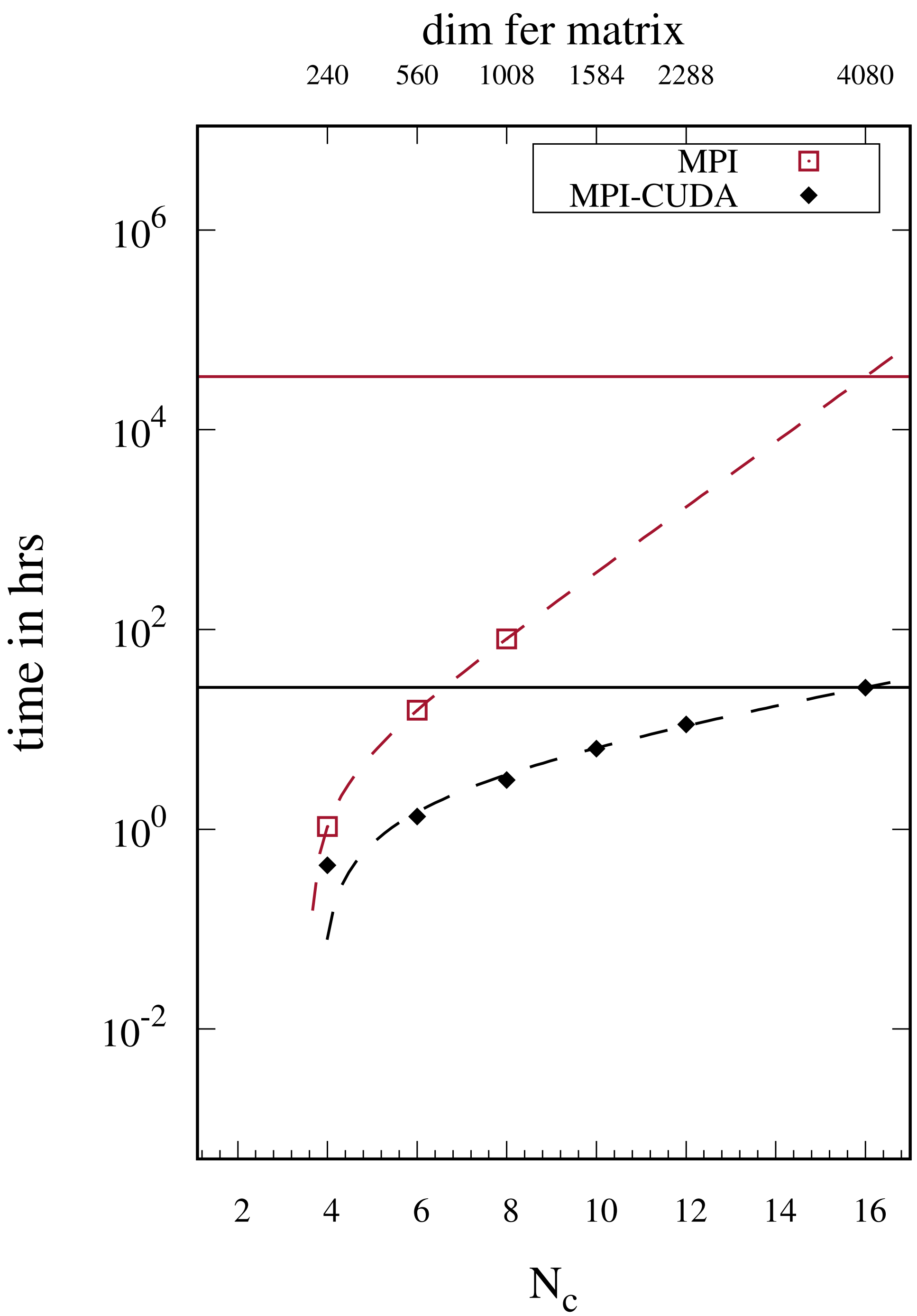
CG algorithm

- Linear transformation property of fermion matrix $\mathcal{M}$

$$\Psi_\alpha \to (\mathcal{M}\Psi)_\alpha \equiv (\Gamma_\mu)_{\alpha\beta}\left[X_\mu, \Psi_\beta\right]$$

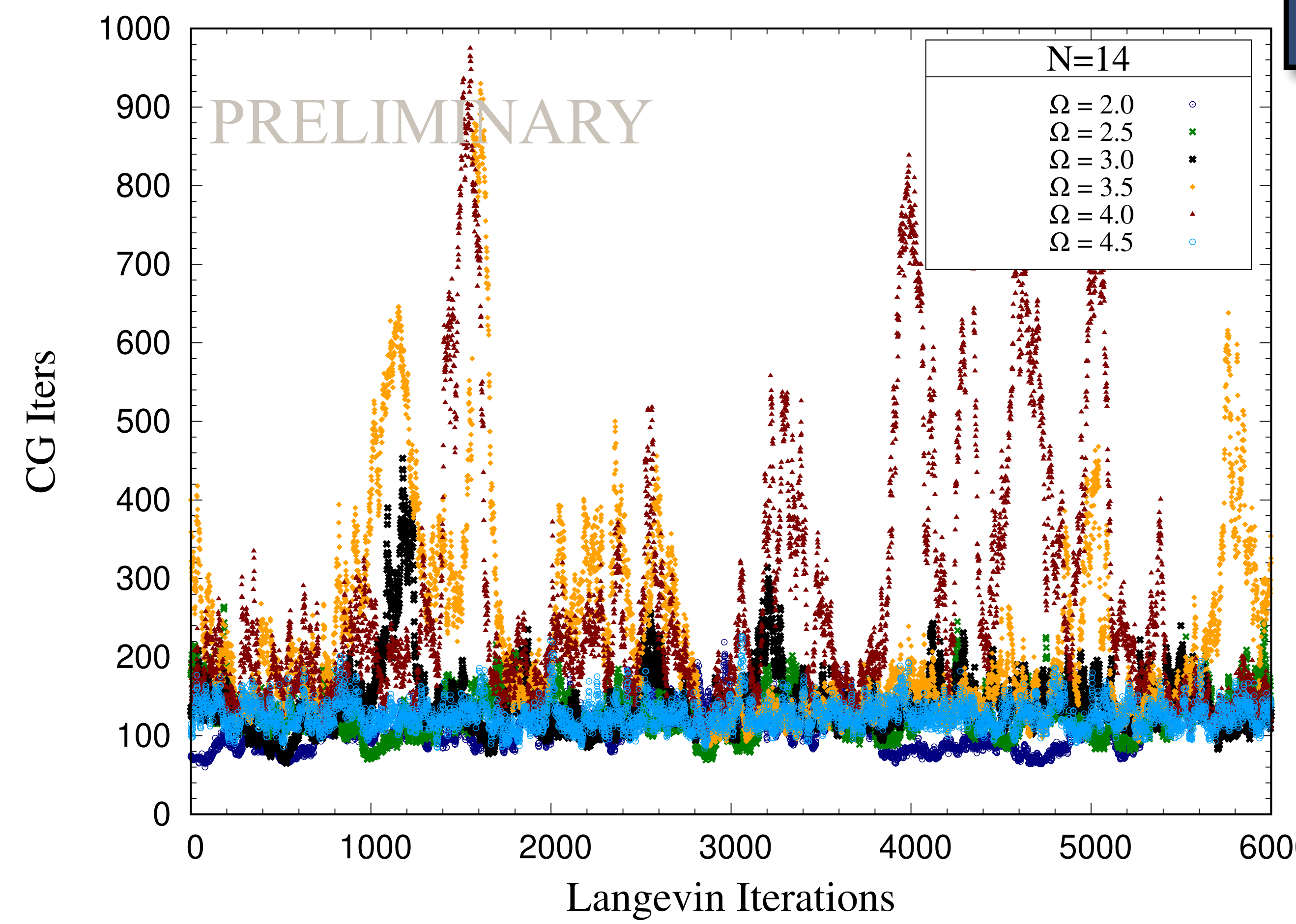
$O(N^4) \to O(N^3)$

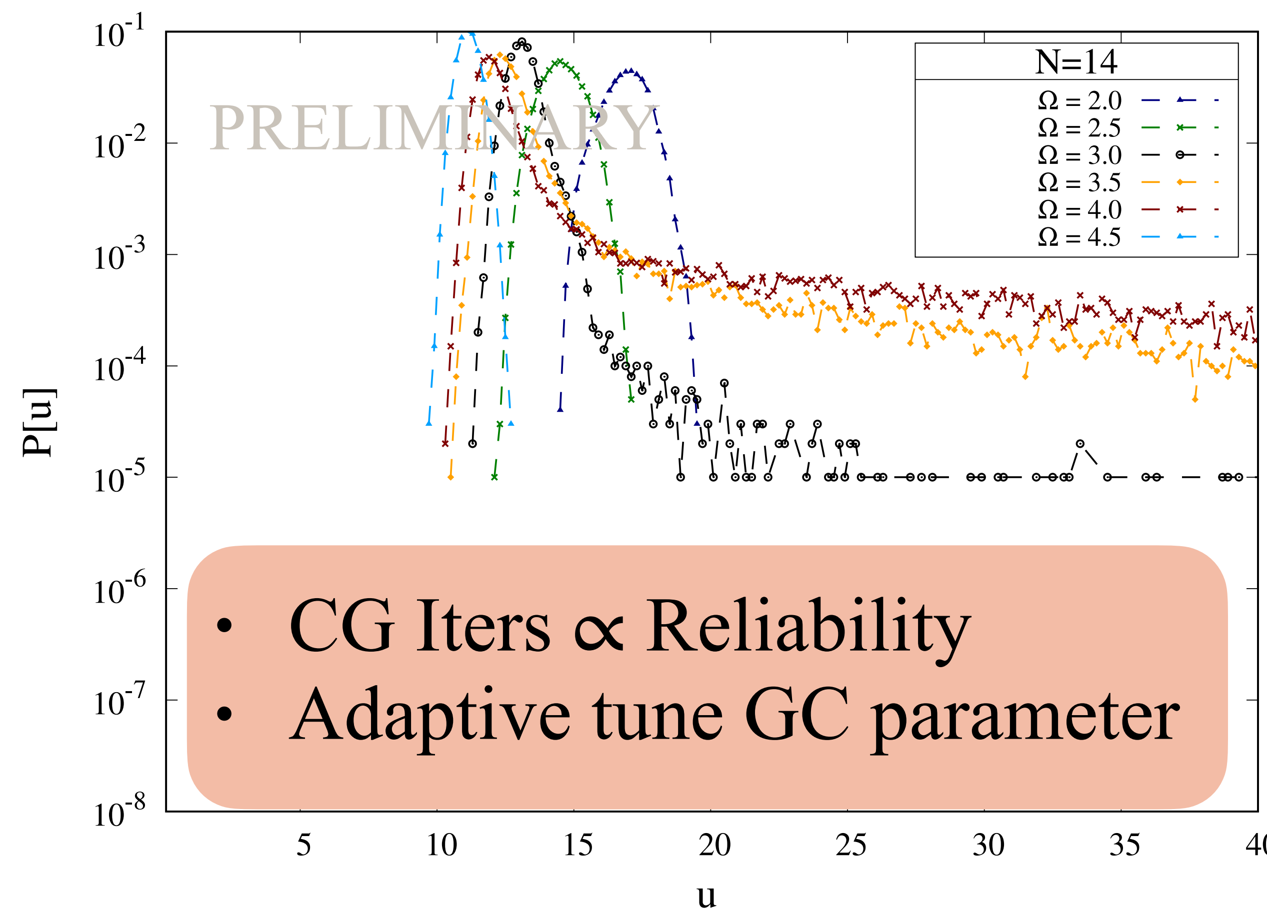
Ongoing work



Benchmarking:
10^5 *Langevin steps*

Deformation parameter:
$\Omega = 8$





- CG Iters ∝ Reliability
- Adaptive tune GC parameter

Thank You!