From Membranes to Matrix Models

Denjoe O'Connor

School of Theoretical Physics
Dublin Institute for Advanced Studies
Dublin, Ireland

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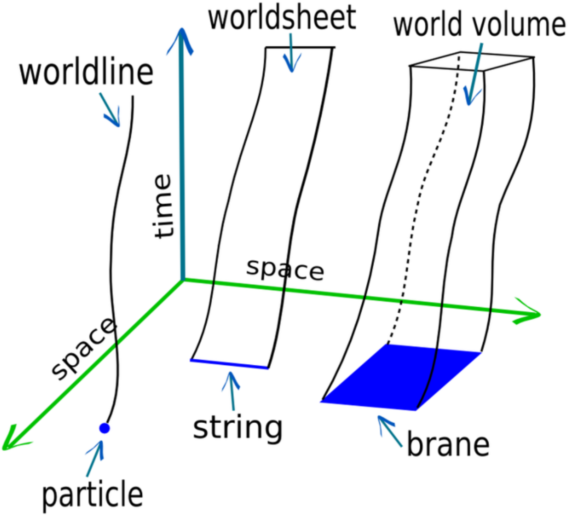
A particle, a string and a membrane



For a particle moving in the metric

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

the action functional is

$$S_{particle} = -mc^2 \int d\tau = -m \int \sqrt{g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt}} dt$$

A particle moving in flat Minkowski spacetime

$$c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2 = c^2 (1 - \frac{\vec{v}^2}{c^2}) dt^2$$

For small (non-relativistic) velocities this gives

$$S_{particle} = -mc^2 \int \sqrt{1 - \frac{\vec{v}^2}{c^2}} dt - = -mc^2 \int dt + \int \frac{m\vec{v}^2}{2} dt$$

In a Schwartschild background

$$c^2 d\tau^2 = (1 - \frac{r_s}{r})c^2 dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\Omega^2 \qquad r_s = \frac{2GM}{c^2}$$

$$S_{particle} = -mc^2 \int \sqrt{1 - \frac{r_s}{r} - \frac{\dot{r}^2}{c^2(1 - \frac{r_s}{r})} - \frac{r^2\vec{\omega}^2}{c^2}} dt$$

when $c \to \infty$

$$S_{particle} \simeq -mc^2 \int dt + \int \{\frac{m\dot{r}^2}{2} + mr^2\vec{\omega}^2 + \frac{GmM}{r} + \cdots\} dt$$

Coupling to an electromagnetic field

$$S_{charged-particle} = -m \int d\tau + q \int A_\mu dx^\mu$$

$$S_{charged-particle} = -m \int dt \sqrt{g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt}} + q \int A_\mu \frac{dX^\mu}{dt} dt$$

again for a particle moving in flat Minkowski spacetime this becomes

$$S_{charged-particle} = -m \int \sqrt{1 - \vec{v}^2} dt + \int (-q\phi + q\vec{v} \cdot \vec{A}) dt$$

The general Nambu Goto action

Nambu-Goto action

$$S_{NG} = -T \int d\sigma_0 d^p\sigma \sqrt{|\det G|}$$

$$G_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N g_{MN} \quad \alpha, \beta = 0, 1, \ldots, p$$

The point particle

$$S_{particle} = -m \int d\sigma_0 \sqrt{|G_{00}|}$$

A string has two coordinates σ_0 and σ

$$S_{NG-string} = -\frac{1}{2\pi\alpha'} \int d\sigma_0 d\sigma \sqrt{|detG|}, \qquad \alpha, \beta = 0, 1$$

A membrane has σ_0,σ_1 and σ_2

$$S_{NG-membrane} = -T \int d\sigma_0 d^2\sigma \sqrt{|detG|} \qquad \alpha, \beta = 0, 1, 2$$

The equations of motion

The discussion closely follows J. Hoppe (PhD thesis 1982)

$$\delta S_{NG} = T \int d\sigma_0 d^M\sigma \partial_\alpha(\sqrt{|G|}G^{\alpha\beta}\partial_\beta X^M \eta_{NM})\delta X^N$$

$$-\frac{1}{\sqrt{|G|}}\partial_\alpha(\sqrt{|G|}G^{\alpha\beta}\partial_\beta X^M) = 0$$

With time $t = \sigma_0$

$$G_{\alpha\beta} = \begin{pmatrix} \dot{X}^0\dot{X}^0 - \dot{\vec{X}}\cdot\dot{\vec{X}} & \dot{X}^0\partial_j X^0 - \dot{\vec{X}}\cdot\partial_j\vec{X} \\ \partial_i X^0\dot{X}^0 - \partial_i\vec{X}\cdot\dot{\vec{X}} & \partial_i X^0\partial_j X^0 - \partial_i\vec{X}\cdot\partial_j\vec{X} \end{pmatrix}$$

Lightcone coordinates

$$X^{\pm} = \frac{X^0 \pm X^D}{\sqrt{2}} \qquad d\tau^2 = 2dX^+ dX^- - d\vec{X} \cdot d\vec{X}$$

$$\begin{pmatrix} 2\dot{X}^+\dot{X}^- - \dot{\vec{X}}_\perp \cdot \dot{\vec{X}}_\perp & \dot{X}^+\partial_j X^- + \partial_j X^+ \dot{X}^- - \dot{\vec{X}}_\perp \cdot \partial_j \vec{X}_\perp \\ \partial_i X^+ \dot{X}^- + \dot{X}^+ \partial_i X^- - \partial_i \vec{X}_\perp \cdot \dot{\vec{X}}_\perp & \partial_i X^+ \partial_j X^- + \partial_i X^+ \partial_j X^- - \partial_i \vec{X}_\perp \cdot \partial_j \vec{X}_\perp \end{pmatrix}$$

Choose time to be $t = X^+$

So that $\dot{X}^+ = 1$ and $\partial_i X^+ = 0$, $\vec{X}_\perp$ is a $D-1$ vector and

$$G_{\alpha\beta} = \begin{pmatrix} 2\dot{X}^- - \dot{\vec{X}}_\perp \cdot \dot{\vec{X}}_\perp & \partial_j X^- - \dot{\vec{X}}_\perp \cdot \partial_j \vec{X}_\perp \\ \partial_i X^- - \partial_i \vec{X}_\perp \cdot \dot{\vec{X}}_\perp & -\partial_i \vec{X}_\perp \cdot \partial_j \vec{X}_\perp \end{pmatrix}$$

The Lagrangian

Assuming $g_{ij} = \partial_i \vec{X}_\perp \cdot \partial_j \vec{X}_\perp$ is not degenerate so that it is invertible then

$$L_{NG} = -\int d^2\sigma \sqrt{|G|} = -\int d^2\sigma \sqrt{g}\sqrt{\Gamma}$$

where $\Gamma = G_{00} - G_{0i} g^{ij} G_{j0}$

$$\Gamma = 2\dot{X}^- - \dot{\vec{X}}_\perp \cdot \dot{\vec{X}}_\perp - u_i g^{ij} u_j$$

with $u_i = \dot{\vec{X}}_\perp \cdot \partial_i \vec{X}_\perp - \partial_i X^-$

The Canonical Momenta and a Constraint

The Lagrangian density

$$\mathcal{L} = \sqrt{g}\sqrt{\Gamma}$$

The canonical momentum density

$$\vec{P} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{X}}_\perp} = \sqrt{\frac{g}{\Gamma}}(\dot{\vec{X}}_\perp - \partial_i \vec{X}_\perp g^{ij} u_j)$$

$$P^+ = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = \sqrt{\frac{g}{\Gamma}}$$

This is a consequence of diffeomorphism invariance

Note using $u_i = \dot{\vec{X}}_\perp \cdot \partial_i \vec{X}_\perp - \partial_i X^-$ that:

$$(\partial_i \vec{X}) \cdot \vec{P} + \partial_i X^- P^+ \equiv 0$$

The Hamiltonian Density

$$\mathcal{H} = \dot{\vec{X}}_\perp \vec{P} + \dot{X}^- P^+ - \mathcal{L} = \sqrt{\frac{g}{\Gamma}}(\dot{X}^- - \partial_i X^- u^i)$$

Eliminating in $\dot{X}^-$, $\partial_i X^-$ and $\dot{\vec{X}}_\perp$ to express the Hamiltonian in cannonical form and after some algebra one finds

the Hamiltonian and constraint

$$\mathcal{H} = \frac{P^2 + g}{2P^+} \quad \text{and} \quad \sum_{a=1}^{D-1} \{\dot{X}^a, X^a\} = 0.$$

$$g = det\{\partial_i \vec{X}_\perp \partial_j \vec{X}_\perp\} = \epsilon_{ij}\partial_{kl}(\partial_i \vec{X}_\perp \cdot \partial_k \vec{X}_\perp)(\partial_j \vec{X}_\perp \cdot \partial_l \vec{X}_\perp)$$

Reorganising and defining
$\{f, g\} = \rho\epsilon_{ij}\partial_i f \partial_j g \implies g = \frac{1}{2\rho^2}\sum_{a=1}^{D-1}\sum_{b=1}^{D-1}\{X^a, X^b\}\{X^a, X^b\}$

The Hamiltonian and simplified Lagrangian

$$H = \int d^2\sigma \frac{1}{2P^+}\left(\vec{P}^2 + \frac{1}{2\rho^2}\sum_{a=1}^{D-1}\sum_{b=1}^{D-1}\{X^a, X^b\}\{X^a, X^b\}\right)$$

with the constraint

$$\sum_{a=1}^{D-1}\{\dot{X}^a, X^a\} = 0$$

These can be repackaged as the reduced Lagrangian

$$L = \int d^2\sigma \left(\frac{1}{2}\sum_{a=1}^{D-1}(D_t X^a)^2 - \frac{1}{4}\sum_{a=1}^{D-1}\sum_{b=1}^{D-1}\{X^a, X^b\}\{X^a, X^b\}\right)$$

where $D_t X^a = \partial_t X^a + \omega, X^a$ and ω is a gauge field for diffeomorphisms.

Comments

The Lorentz invariance of the system is now hidden in the equations. To demonstrating the Lorentz invariance of the system one has to study the equations for X^- which have been eliminated.

The direct quantization of the system has never been achieved! It is argued that it is probably not a renormalizable system.

The role played by t and the membrane coordinates σ^1 and σ^2 is very different. If we treat the potential as a perturbation we only have ∂_t.

The system is a gauge theory with gauge group area preserving diffeomorphisms

Quantization via non-commutative regularization

The membrane has a 2-dimensional surface which we are thinking of as a phase space. If we regulate it by replacing the Poisson brackets with commutators so that

$$f(\sigma) \to F \qquad \text{functions go to matrices}$$

$$\{X^a, X^b\} \to i[X^a, P^b] \qquad \text{Poisson brackets go to commutators}$$

$$\int d^2 f \sigma \to \mathbf{Tr}(F) \qquad \text{integration goes to trace of matrices}$$

The Hamiltonian becomes

$$H = \mathbf{Tr}(\frac{1}{2} P^a P^a - \frac{1}{2}[X^a, X^b])$$

and the constraint that physical states are $U(N)$ invariant.

The regularizatioon is replacing functions by $N \times N$ matrices $f \to \mathrm{F}$, and $\int_\Sigma f \to \mathbf{Tr}\mathrm{F}$ then quantizing by $\dot{X}^a = \frac{1}{i}\frac{\partial}{\partial X^a}$
The Hamiltonian becomes

$$\mathrm{H} = -\frac{1}{2}\nabla^2 - \frac{1}{4}\sum_{i,j=1}^{d} \mathrm{Tr}[X^i, X^j]^2$$

and describes a "fuzzy" relativistic membrane in $d+1$ space and one time dimensions.
H appears to realize a proposed requirement of quantum gravity of Doplicher, Fredenhagen and Roberts,1995 arXiv:hep-th/0303037

Fuzzy Regularization

One can study matrix regularized field theories more generally. E.g. a scalar field on the fuzzy sphere.

$$S(\phi) = \int_{S^2} \{ \frac{1}{2} (\partial_a \phi)^2 + V(\phi) \}$$

regulated becomes

$$S(\Phi) = \mathbf{Tr} \left(\frac{1}{2} \Phi \mathcal{L}_a^2 \Phi + V(\Phi) \right)$$

The geometry is encoded in the Laplacian $\Delta = \mathcal{L}_a^2$. The theory with $V(\Phi) = g\Phi^4$ has received much attention. However it suffers from ultravoilet/infra red mixing. The model has not just two phases but a 3rd non-uniform phase.
Note: For the membrane the classical topology and geometry are lost in the fuzzy regularized theory.

Path Integral formulation

The Euclidean finite temperature action for the model is

$$S_b = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{tr} \left\{ \frac{1}{2} (\mathcal{D}_t X^i)^2 - \frac{1}{4} [X^i, X^j]^2 \right\} \ .$$

where $\mathcal{D}_t X^i = \partial_t X^i + [A, X^i]$ and β, the period of the S^1, is the inverse temperature.

S_b is also the zero volume limit of Yang-Mills on the torus T^d.

Higher dimensional membranes have a classical potential in terms of Nambu Brackets whose fuzzy version becomes

$$S_b = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{tr} \left\{ \frac{1}{2} (\mathcal{D}_t X^i)^2 - \frac{1}{p!} [X^{i_1}, X^{i_2}, \dots, X^{i_p}]^2 \right\} \ .$$

Path Integral Formulation

For a discussion of the two matrix case see Mathaba, Mulokwe, Rodrigues, arXiv:2306.00935

Partition Function

$$Z = \int [dX][dA] \mathrm{e}^{-N \int_0^\beta dt \, \mathrm{Tr}\left(\frac{1}{2}(\mathcal{D}_t X^i)^2\right) - \frac{N}{4} \lambda^{abcd} \int_0^\beta dt \, X_a^i X_b^i X_c^j X_d^j} \, .$$

The commutator square term can be written as:

$$\mathrm{Tr}[X^i, X^j]^2 = \mathrm{Tr}\left([t^a, t^c][t^b, t^d]\right) X_a^i X_b^i X_c^j X_d^j = \lambda^{abcd} X_a^i X_b^i X_c^j X_d^j \, , \tag{1}$$

where t^a are $SU(N)$ generators.

$$Z = \int [dX][dA][dk] \mathrm{e}^{-\frac{N}{2} \int_0^\beta dt \left\{ \mathrm{Tr}(\mathcal{D}_t X^i)^2 + k^{ab} X_a^i X_b^i \right\} + \frac{N}{4} \mu_{abcd} \int_0^\beta dt \, k^{ab} k^{cd}} \, .$$

The saddle point approximation for k^{ab} gives $k^{ab} = d^{2/3} \delta^{ab}$. A detailed $1/d$ analysis of the membrane model shows there are in fact two phase transitions for large enough d.

The effective dynamics of the Bosonic membrane is given by the action

$$S_{\text{eff}} \approx N \int_{-\infty}^{\infty} dt \, \mathrm{Tr} \left(\frac{1}{2} (\mathcal{D}_t X^i)^2 - \frac{1}{2} m^2 (X^i)^2 \right) .$$

One can derive this using a large $1/d$ expansion which to leading order in large d gives the Euclidean finite temperature action

$$S_b = N \int_0^{\beta} dt \, \mathrm{Tr} \left(\frac{1}{2} (\mathcal{D}_t X^i)^2 + \frac{d^{2/3}}{2} (X^i)^2 \right)$$

This model can of course be solved analytically.

Observations

- The eigenvalues of the matrices X^i have a Wigner semi-circle distribution.
- At zero temperature the gauge field A can be gauged away, while at high temperature A becomes an additional matrix in a pure matrix model.
- The entry of A as an additional matrix in the dynamics should signal a phase transition. This should be a Gross-Witten type transition. It occurs at

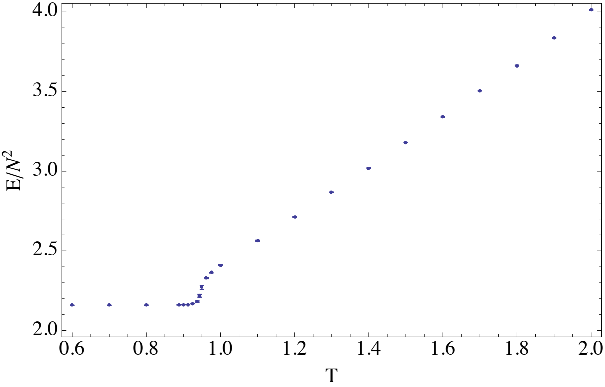
  $$T_c = \frac{m}{\ln d}$$

  The transition can be observed as centre symmetry breaking in the Polyakov loop.

All of these phenomena should be present in the membrane model.

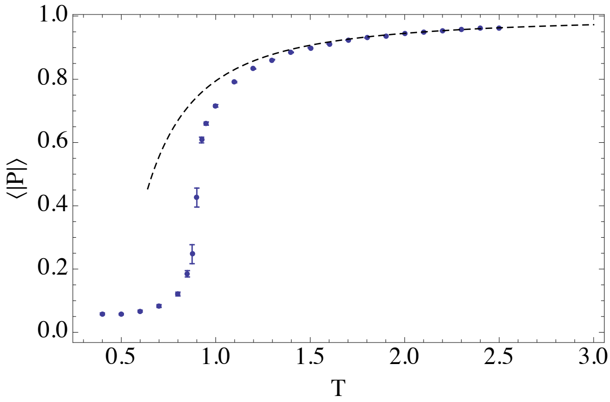
The gauged Gaussian model has a phase transition:

The energy for $N = 32$ and $a = 0.05$.

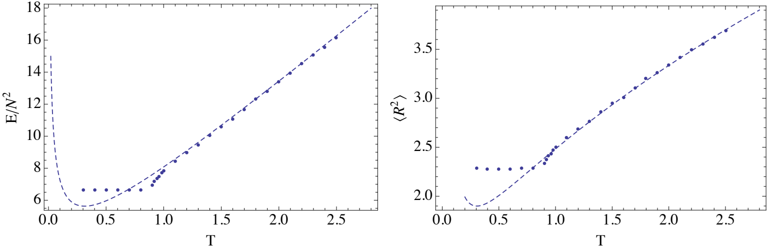


The gauge field is responsible for the phase transition. At low temperatures the eigenvalues of A are uniformly distributed but at high temperatures it becomes another matrix whose eigenvalues have a Wigner distribution.

Polyakov loop

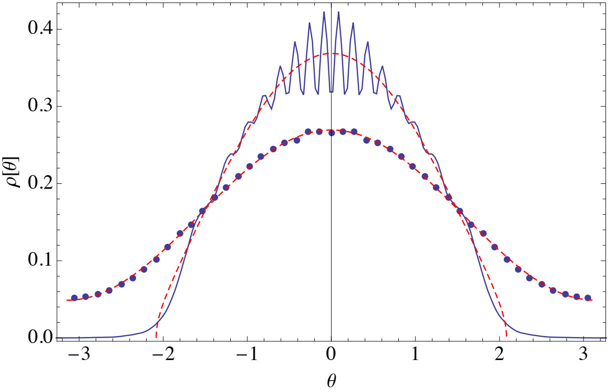


Dashed curves are the high temperature series expansions to 2nd order.



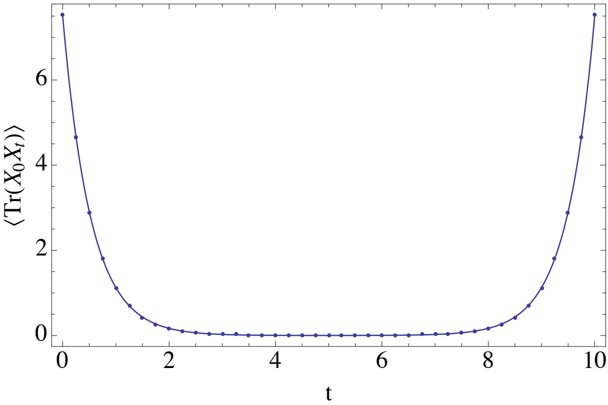
Plots of the scaled energy E/N^2 and $\langle R^2 \rangle = \frac{1}{N\beta} \int_0^\beta dt \langle \mathrm{Tr}(X^i)^2 \rangle$ as functions of the temperature. The dashed curves correspond to the high temperature expansion. One can see that near $T \approx 0.9$ the plots suggest the existence of a second order phase transition. The energy and temperature in the plots are in units of $\lambda^{1/3}$.

The eigenvalue distribution of the holonomy



Plots of the distribution of the holonomy P for temperatures $T = 0.900$ (the gapped phase) and $T = .9006$ (the ungapped phase). The plots are for size $N = 16$ and lattice spacing $a \approx 0.05$. The dashed curves correspond to fits to the gapped and ungapped distributions.

Correlation function



The correlator $\left\langle \mathrm{Tr}\left(X^1(0)X^1(t)\right)\right\rangle$ for $N = 30$, $\beta = 10$ and lattice spacing $a = 0.25$. Fitting to $A\left(e^{-m\,t} + e^{-m(\beta - t)}\right)$

$\implies m = E_1 - E_0 \approx (1.90 \pm .01)\,\lambda^{1/3}$

Comments

- The zero volume limit of Yang-Mills exhibits two phase transitions, very close in temperature.
- The bosonic relativistic membrane has a mass gap and at low temperatures is very well described by a system of oscillators.

In contrast to the bosonic string the relativistic bosonic membrane has only massive excitations!

The BFSS model

$$S_{SMembrane} = \int \sqrt{-G} - \int C + \quad \text{Fermionic terms}$$

The susy version only exists in $4, 5, 7$ and 11 spacetime dimensions.

BFFS Model — The supersymmetric membrane à la Hoppe

$$\mathrm{H} = \mathbf{Tr}(\tfrac{1}{2}\sum_{a=1}^{9} P^a P^a - \frac{1}{4}\sum_{a,b=1}^{9}[X^a, X^b][X^a, X^b] + \frac{1}{2}\Theta^T \gamma^a [X^a, \Theta])$$

It also describes a system of N interacting D0 branes.

Finite Temperature Model

The partition function and Energy of the model at finite temperature is

$$Z = Tr_{Phys}(\mathrm{e}^{-\beta\mathcal{H}}) \qquad \text{and} \qquad E = \frac{Tr_{Phys}(\mathcal{H}\mathrm{e}^{-\beta\mathcal{H}})}{Z} = \langle\mathcal{H}\rangle$$

The 16 fermionic matrices $\Theta_\alpha = \Theta_{\alpha A} t^A$ are quantised as

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}$$

The $\Theta_{\alpha A}$ are $2^{8(N^2-1)}$ and the Fermionic Hilbert space is

$$\mathcal{H}^F = \mathcal{H}_{256} \otimes \cdots \otimes \mathcal{H}_{256}$$

with $\mathcal{H}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ suggestive of
the graviton (**44**), anti-symmetric tensor (**84**) and gravitino (**128**) of $11 - d$ SUGRA.

For an attempt to find the ground state see: J. Hoppe et al arXiv:0809.5270

Lagrangian formulation

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The BFSS matrix model is also the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory down to one dimension:

$$
\begin{aligned}
S_M &= \frac{1}{g^2}\int dt\, \mathrm{Tr}\left\{\frac{1}{2}(\mathcal{D}_0 X^i)^2 + \frac{1}{4}[X^i, X^j]^2 \right. \\
&\left. -\frac{i}{2}\Psi^T C_{10}\,\Gamma^0 D_0\Psi + \frac{1}{2}\Psi^T C_{10}\,\Gamma^i[X^i,\Psi]\right\} ,
\end{aligned}
$$

where Ψ is a thirty two component Majorana–Weyl spinor, Γ^μ are ten dimensional gamma matrices and C_{10} is the charge conjugation matrix satisfying $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^{\mu\,T}$.

The gravity dual and its geometry

Gauge/gravity duality predicts that the strong coupling regime of the theory is described by II_A supergravity, which lifts to 11-dimensional supergravity.

The bosonic action for eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2} \int [\sqrt{-g} R - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{6} A_3 \wedge F_4 \wedge F_4]$$

where $2\kappa_{11}^2 = 16\pi G_N^{11} = \frac{(2\pi l_p)^9}{2\pi}$.

The relevant solution to eleven dimensional supergravity for the dual geometry to the BFSS model corresponds to N coincident $D0$ branes in the IIA theory. It is given by

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

with $A_3 = 0$
The one-form is given by $C = H^{-1} - 1$ and $H = 1 + \frac{\alpha_0 N}{r^7}$ where $\alpha_0 = (2\pi)^2 14\pi g_s l_s^7$.

A thermal bath and black hole geometry

$$ds_{11}^2 = -H^{-1} F dt^2 + F^{-1} dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - C dt)^2$$

Set $U = r/\alpha'$ and we are interested in $\alpha' \to \infty$
$H(U) = \frac{240\pi^5\lambda}{U^7}$ and the black hole time dilation factor
$F(U) = 1 - \frac{U_0^7}{U^7}$ with $U_0 = 240\pi^5\alpha'^5\lambda$. The temperature

$$\frac{T}{\lambda^{1/3}} = \frac{1}{4\pi\lambda^{1/3}} H^{-1/2} F'(U_0) = \frac{7}{2^4 15^{1/2} \pi^{7/2}} \left(\frac{U_0}{\lambda^{1/3}}\right)^{5/2}.$$

From black hole entropy we obtain the prediction for the Energy

$$S = \frac{A}{4G_N} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{9/2} \implies \frac{E}{\lambda N^2} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{14/5}$$

Checks of the predictions

We found excellent agreement with this prediction V. Filev and D.O'C. arXiv:1506.01366 and 1512.02536.
The best current results (Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki and Vranas arXiv 1606.04951) give

$$\frac{1}{N^2}\frac{E}{\lambda^{1/3}} = 7.41\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{14}{5}} - (10.0\pm 0.4)\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{23}{5}} + (5.8\pm 0.5)\,T^{\frac{29}{5}} + \ldots - \frac{5.77\,T^{\frac{2}{5}} + (3.5\pm 2.0)\,T^{\frac{11}{5}}}{N^2} + \ldots$$

The BMN or PWMM

The supermembrane on the maximally supersymmetric plane wave spacetime

$$ds^2 = -2dx^+dx^- + dx^a dx^a + dx^i dx^i - dx^+ dx^+ ((\frac{\mu}{6})^2 (x^i)^2 + (\frac{\mu}{3})^2 (x^a)^2)$$

with

$$dC = \mu dx^1 \wedge dx^2 \wedge dX^3 \wedge dx^+$$

so that $F_{123+} = \mu$. This leads to the additional contribution to the Hamiltonian

$$\Delta \mathrm{H}_\mu = \frac{N}{2}\mathbf{Tr}\Big((\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 + \frac{2\mu}{3} i\epsilon_{ijk} X^i X^j X^k + \frac{\mu}{4} \Theta^T \gamma^{123} \Theta \Big)$$

The BMN model

The BMN action

$$
\begin{aligned}
S_{BMN} \;=\; \int_0^\beta d\tau \,\mathrm{Tr}\Big\{ & \frac{1}{2}(\mathcal{D}_\tau X^i)^2 + \frac{1}{2}(\frac{\mu}{6})^2 (X^a)^2 + \frac{1}{2}(\frac{\mu}{3})^2 (X^i)^2 \\
& + \Psi^T D_\tau \Psi + \frac{\mu}{4} \Psi^T i\gamma^{123} \Psi \\
& - \frac{1}{4}[X^i, X^j]^2 + \frac{2\mu}{3} i\epsilon_{ijk} X^i X^j X^k + \frac{1}{2}\Psi^T \Gamma^i [X^i, \Psi] \Big\} \;,
\end{aligned}
$$

The X^i enter the potential as $\mathrm{Tr}(i[X^i, X^j] + \frac{\mu}{3}\epsilon^{ijk} X^k)^2$.
New non-trivial solutions $X^a = 0$, $X^i = -\frac{\mu}{3} J^i$, with J^i $su(2)$ generators.

$$\Delta S_\mu = -\frac{1}{2g^2}\int_0^\beta d\tau \mathbf{Tr}\Big((\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2 + \frac{2\mu}{3} i\epsilon_{ijk}X^iX^jX^k + \frac{\mu}{4}\Psi^T\gamma^{123}\Psi\Big)$$

The Bosonic model has been studied in
N.S. Dhindsa, A. Joseph, A. Samlodia, and D. Schaich, arXiv:2308.02538",
N.S. Dhindsa, R.G. Jha, A. Joseph, A. Samlodia, and D. Schaich, arXiv:2201.08791

Large mass expansion

For large μ the model becomes the supersymmetric Gaussian model

Finite temperature Euclidean Action

$$S_{BMN} = \frac{1}{2g^2}\int_0^\beta d\tau \, \mathrm{Tr}\Big\{(\mathcal{D}_\tau X^i)^2 + (\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2 \\ \Psi^T D_\tau \Psi + \frac{\mu}{4}\Psi^T\gamma^{123}\Psi\Big\}$$

This model has a phase transition at $T_c = \frac{\mu}{12\ln 3}$

Perturbative expansion in large μ.

Three loop result of *Hadizadeh, Ramadanovic, Semenoff and Young* [hep-th/0409318]

$$T_c = \frac{\mu}{12 \ln 3} \left\{ 1 + \frac{2^6 \times 5}{3^4} \frac{\lambda}{\mu^3} - \left(\frac{23 \times 19927}{2^2 \times 3^7} + \frac{1765769 \ln 3}{2^4 \times 3^8} \right) \frac{\lambda^2}{\mu^6} + \cdots \right\}$$

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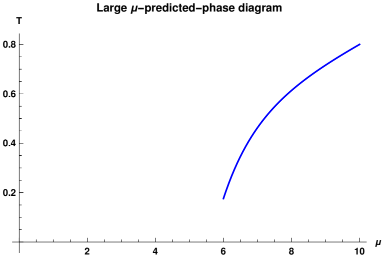
Three loop result of *Hadizadeh, Ramadanovic, Semenoff and Young* [hep-th/0409318]

$$T_c = \frac{\mu}{12 \ln 3}\left\{1 + \frac{2^6 \times 5}{3^4}\frac{\lambda}{\mu^3} - \left(\frac{23 \times 19927}{2^2 \times 3^7} + \frac{1765769 \ln 3}{2^4 \times 3^8}\right)\frac{\lambda^2}{\mu^6} + \cdots\right\}$$



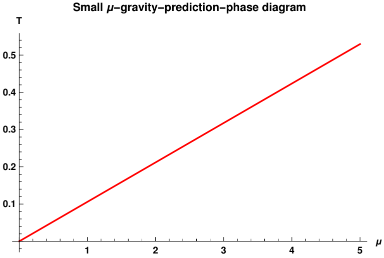
Passes through zero at $\mu = 5.65$.

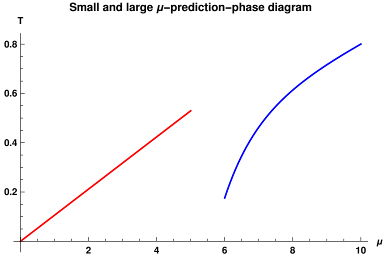
Gravity prediction at small μ

Costa, Greenspan, Penedones and Santos, [arXiv:1411.5541]

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{SUGRA}}}{\mu} = 0.105905(57)\,.$$

The prediction is for low temperatures and small μ the transition temperature approaches zero linearly in μ.





Padé approximant prediction of T_c

$$T_c = \frac{\mu}{12\ln 3}\left\{1 + r_1\frac{\lambda}{\mu^3} + r_2\frac{\lambda^2}{\mu^6} + \cdots\right\}$$

with $r_1 = \frac{2^6\times 5}{3}$ and $r_2 = -(\frac{23\times 19927}{2^2\times 3} + \frac{1765769\ln 3}{2^4\times 3^2})$

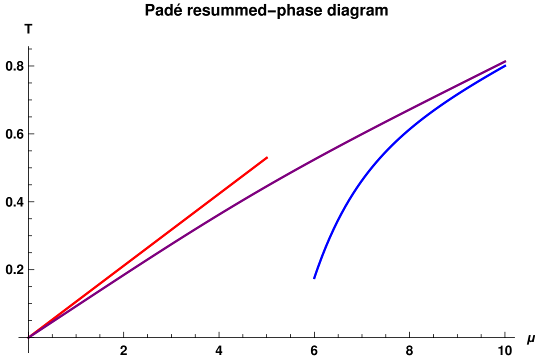
Using a Padé Approximant: $1 + r_1 g + r_2 g^2 + \cdots \to 1 + \frac{1+r_1 g}{1-\frac{r_2}{r_1}g}$

$$\Longrightarrow\ T_c^{\text{Padé}} = \frac{\mu}{12\ln 3}\left\{1 + \frac{r_1\frac{\lambda}{\mu^3}}{1 - \frac{r_2}{r_1}\frac{\lambda}{\mu^3}}\right\}$$

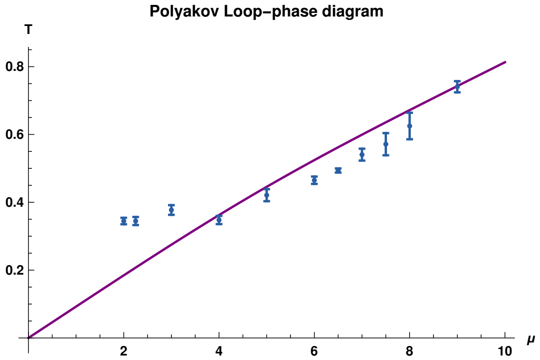
Now we can take the small μ limit

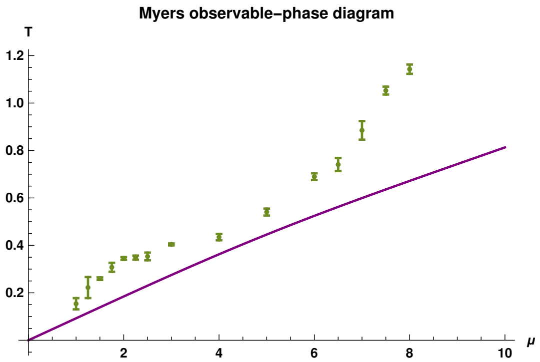
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{Padé}}}{\mu} \simeq \frac{1}{12\ln 3}\left(1-\frac{r_1^2}{r_2}\right) = 0.0925579$$

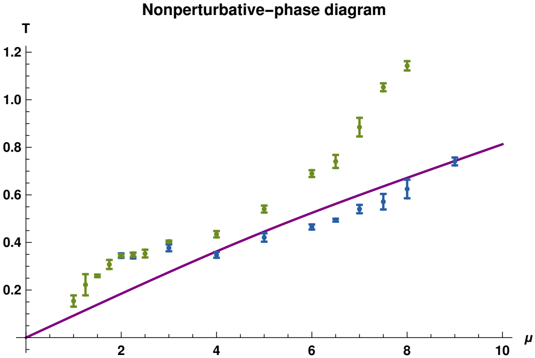
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{SUGRA}}}{\mu} = 0.105905(57)\,.$$



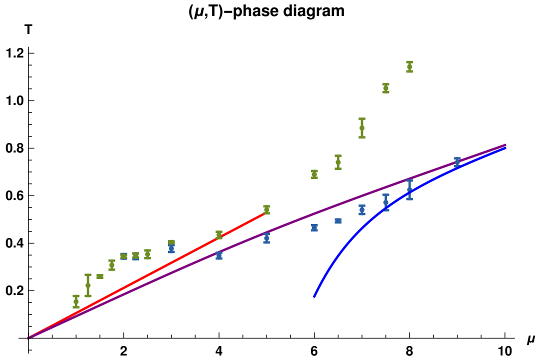
A non-perturbative phase diagram from the Polyalov Loop.




Myers observable–phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ



Green Myers transition
Blue Polyakov loop transition
Purple Padé prediction for the transition


(μ,T)–phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ

4-parameter Lattice discretisation

The bosonic lattice Laplacian

$$\Delta_{Bose} = \Delta + r_b a^2 \Delta^2 \,, \quad \text{where} \quad \Delta = \frac{2 - e^{aD_\tau} - e^{-aD_\tau}}{a^2} \,.$$

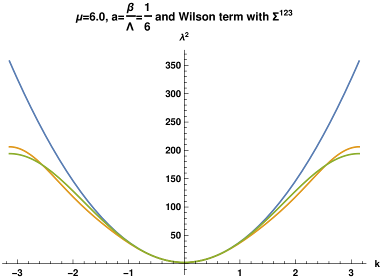
Lattice Dirac operator

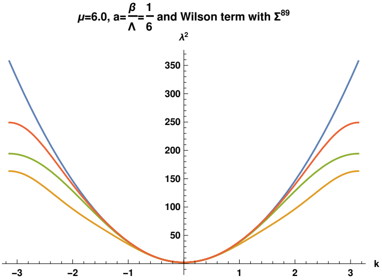
$$D_{Lat} = K_a \mathbf{1}_{16} - i\frac{\mu}{4}\gamma^{567} + \Sigma^{123} K_w \,, \quad \text{where} \quad \Sigma^{123} = i\gamma^{123} \,.$$

$$K_a = (1-r)\frac{e^{aD_\tau} - e^{-aD_\tau}}{2a} + r\frac{e^{2aD_\tau} - e^{-2aD_\tau}}{4a} \qquad \text{lattice derivative}$$

$$K_w = r_{1f}\, a\Delta + r_{2f}\, a^3 \Delta^2 \qquad \text{the Wilson term}$$

Lattice Dispersion relations

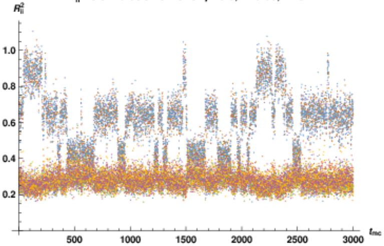




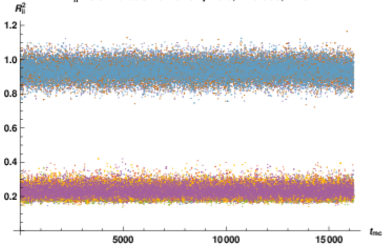
Eigenvalues $k^2 + \frac{\mu^2}{4}$ (blue parabola),
$\Delta_{Bose} + \frac{\mu^2}{4}$ light green,
Σ^{89} splitting red and orange curves.

Observables

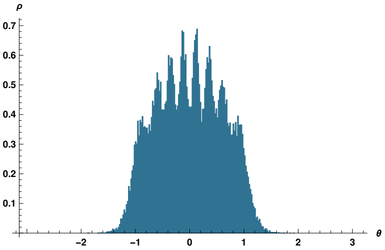
R^2_{ii} vs simulation time for μ=6.0, T=0.69, N=8



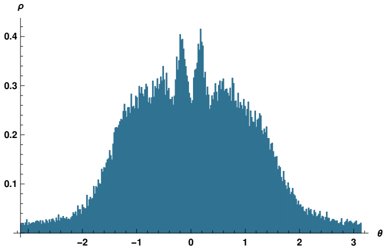
R^2_{ii} vs simulation time for μ=6.0, T=0.365, N=8

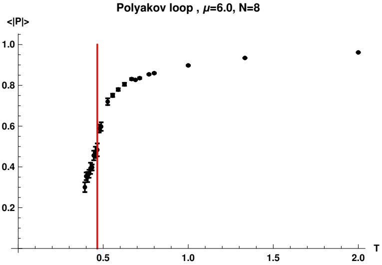


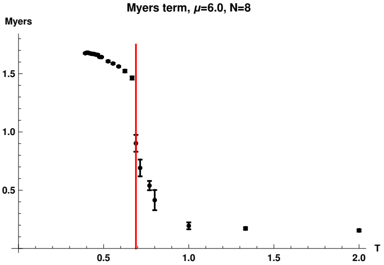
Eigenvalue density of A for μ=6.0, T=0.69, N=8, Λ=24

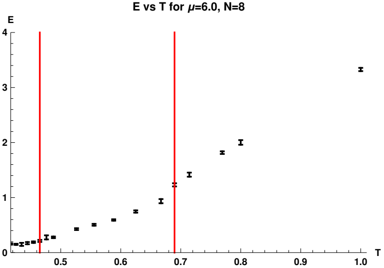


Eigenvalue density of A for μ=6.0, T=0.465, N=8, Λ=24

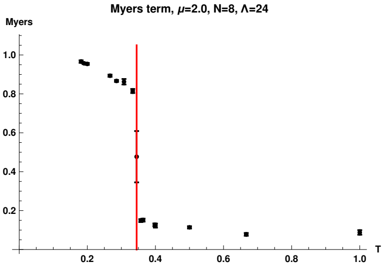


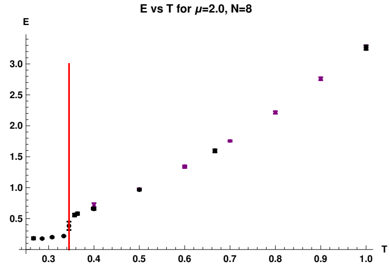

Polyakov loop , μ=6.0, N=8
<|P|>
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
T

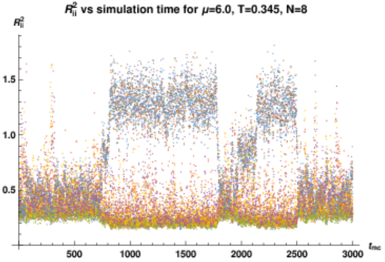

Myers term, μ=6.0, N=8
Myers
1.5
1.0
0.5
0.5
1.0
1.5
2.0
T

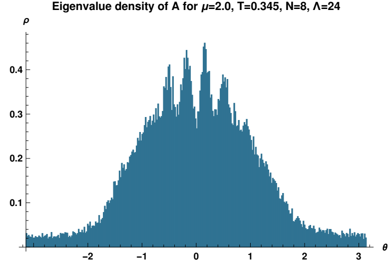

E vs T for μ=6.0, N=8
E
4
3
2
1
0
0.5
0.6
0.7
0.8
0.9
1.0
T

Small μ

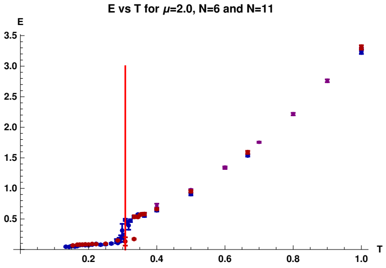

Myers term, μ=2.0, N=8, Λ=24
Myers
T
0.2
0.4
0.6
0.8
1.0
0.2
0.4
0.6
0.8
1.0

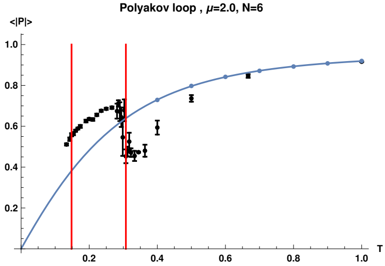

E vs T for μ=2.0, N=8
E
T
0.5
1.0
1.5
2.0
2.5
3.0
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0


R_{ii}^2 vs simulation time for μ=6.0, T=0.345, N=8
R_{ii}^2
t_{mc}
0.5
1.0
1.5
500
1000
1500
2000
2500
3000


Eigenvalue density of A for μ=2.0, T=0.345, N=8, Λ=24
ρ
θ
0.1
0.2
0.3
0.4
−2
−1
0
1
2
3

Non-monotonic Polyakov loop





Where do we go from here

- Use BProbe to visualise the fuzzy sphere phase.
- Study the bosonic BMN model—its phase diagram, theoretical predictions.
- Implications of $SU(4|2)$ symmetry.
- M2-branes.
- Probe BMN with D4-branes—already coded.
- $N = 1^*$ model — at coding stage.
- $N = 2$ models.
- Black dual geometries?
- M5-brane matrix models?
- Quantise (numerically) the diffeomorphism invariant model on the sphere.

The Berkooz Douglas Model

Adding fundamental degrees of freedom to the BFSS model yields the Berkooz–Douglas matrix model

$$\mathcal{L}=\mathcal{L}_{BFSS}+\mathbf{tr}\left(D_0\bar{\Phi}^\rho D_0\Phi_\rho+i\chi^\dagger D_0\chi\right)+\mathcal{L}_{\rm int}\ ,$$

where:

$$\begin{aligned}\mathcal{L}_{\rm int}=\mathbf{tr}&\left(\bar{\Phi}^\alpha[\bar{X}^{\beta\dot{\alpha}},X_{\alpha\dot{\alpha}}]\Phi_\beta+\frac{1}{2}\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\alpha-\bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\right)\\&+\mathbf{tr}\left(\sqrt{2}\,i\,\varepsilon_{\alpha\beta}\,\bar{\chi}\lambda_\alpha\Phi_\beta-\sqrt{2}\,i\,\varepsilon_{\alpha\beta}\,\bar{\Phi}^\alpha\bar{\lambda}_\beta\chi\right)\\&-\sum_{i=1}^{N_f}\left((\bar{\Phi}^\rho)^i(X^a-m_i^a\mathbf{1})(X^a-m_i^a\mathbf{1})(\Phi_\rho)_i+\bar{\chi}^i\gamma^a(X^a-m_i^a\mathbf{1})\chi_i\right)\end{aligned}$$

$a=1,\ldots,5$ are transverse to the D4-brane, m_i^a are the positions of the D4-branes, λ^ρ and $\theta^{\dot{\alpha}}$ are BFSS and χ the fundamental fermions.

The D4-brane as a probe of the geometry.

The dual adds N_f D4 probe branes. In the probe approximation $N_f \ll N_c$, their dynamics is governed by the Dirac-Born-Infeld action:

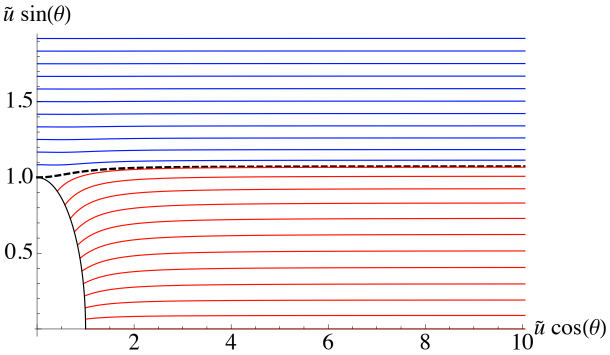
$$S_{\mathrm{DBI}} = -\frac{N_f}{(2\pi)^4\,\alpha'^{5/2}\,g_s}\int d^4\xi\, e^{-\Phi}\sqrt{-\mathrm{det}||G_{\alpha\beta} + (2\pi\alpha')F_{\alpha\beta}||}\ ,$$

where $G_{\alpha\beta}$ is the induced metric and $F_{\alpha\beta}$ is the $U(1)$ gauge field of the D4-brane. For us $F_{\alpha\beta} = 0$.

$$d\Omega_8^2 = d\theta^2 + \cos^2\theta\, d\Omega_3^2 + \sin^2\theta\, d\Omega_4^2$$

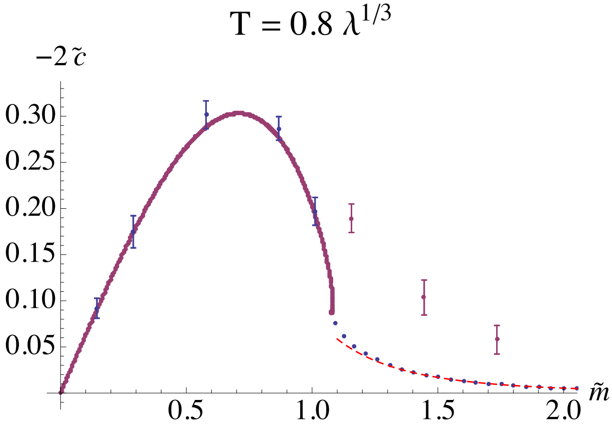
and taking a D4-brane embedding extended along: t, u, Ω_3 with a non-trivial profile $\theta(u)$.

Embeddings



$$\tilde{u}\sin\theta = m + \frac{\tilde{c}}{\tilde{u}^2} + \ldots .$$

The condensate and the dual prediction



V. Filev and D. O'C. arXiv 1512.02536.

The data overlaps surprisingly well with the gravity prediction in the region where the $D4$ brane ends in the black hole.

The Backreacted Problem

For the backreacted problem we need a solution to 11-dim sugra (Filev and D. O'C. arXiv:2203.02472) in an M5-brane background of the form

$$
\begin{aligned}
ds_{11}^2 = & -K_1(u,v)\,dt^2 + K_3(u,v)(dx_{11} + A_0(u,v)\,dt)^2 \\
& + K_2(u,v)(du^2 + u^2 d\Omega_3^2) + \\
& + K_4(u,v)(dv^2 + v^2 d\Omega_4^2)\ , \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{(4)} &= F'(v)\, v^4 \sin^3\psi \sin\tilde{\alpha} \cos\tilde{\alpha}\, d\psi \wedge d\tilde{\alpha} \wedge d\tilde{\beta} \wedge d\tilde{\gamma}\ , \qquad (3) \\
d\Omega_3^2 &= d\alpha^2 + \sin^2\alpha\, d\beta^2 + \cos^2\alpha\, d\gamma^2\ , \qquad (4) \\
d\Omega_4^2 &= d\psi^2 + \sin^2\psi\, d\tilde{\Omega}_3^2\ , \quad d\tilde{\Omega}_3^2 = d\tilde{\alpha}^2 + \sin^2\tilde{\alpha}\, d\tilde{\beta}^2 + \cos^2\tilde{\alpha}\, d\tilde{\gamma}^2\ .
\end{aligned}
$$

$$\int \mathcal{F}_{(4)} = \frac{8}{3}\pi^2 v^4 F'(v) = -Q_5 \qquad \text{the M5-brane charge.}$$

gives

$$F(v) = 1 + \frac{Q_5}{8\pi^2 v^3} \equiv 1 + \frac{v_5^3}{v^3} = 1 + \frac{N_f}{N_c}\frac{4\pi^3\alpha'^3\lambda}{v^3}\ ,$$

The solution preserving supersymmetry is given by:

$$ds_{11}^2 = \left(1 + \frac{v_5^3}{v^3}\right)^{-1/3} \left(-H(u,v)^{-1}\, dt^2 + \right.$$
$$+ H(u,v)\left(dx_{11} + (H(u,v)^{-1} - 1)\, dt\right)^2 +$$
$$\left. du^2 + u^2\, d\Omega_3^2\right) + \left(1 + \frac{v_5^3}{v^3}\right)^{2/3} \left(dv^2 + v^2\, d\Omega_4^2\right) \ .$$

Note: Supersymmetry does not restrict the shape of the function $H(u,v)$. The equation of motion for H can be obtained either by using the Einstein equations or by requiring that the angular momentum along x_{11} is conserved.

Equation for $H(u, v)$

The non-trivial equation requiring a solution is:

$$\partial_v^2 H(u,v) + \frac{4}{v}\partial_v H(u,v) + \left(1 + \frac{v_5^3}{v^3}\right)\left(\partial_u^2 H(u,v) + \frac{3}{u}\partial_u H(u,v)\right) = 0$$

Perturbation in v_5 recovers the probe limit.

In the $v \to \infty$ limit of equation (equivalent to the $v_5 \to 0$ limit) $SO(9)$ symmetry is recovered and

$$H_0(u, v) = 1 + \frac{r_0^7}{(u^2 + v^2)^{7/2}} \ ,$$

The parameter r_0^7 is proportional to the number of D0–branes, N_c:

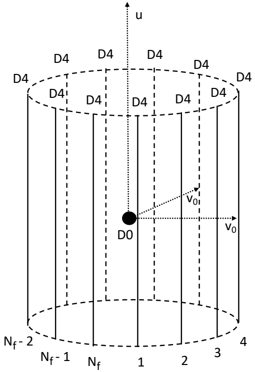
$$r_0^7 = N_c \, 60 \, \pi^3 \, g_s \, \alpha'^{7/2} \ .$$

$$1 + \frac{v_5^3}{v^3} = 1 + \frac{N_f}{N_c} \frac{4\pi^3 \lambda}{(v/\alpha'^3)} \ .$$

Perturbation in v_5 recovers the probe approximation.

Instability of overlap intersection

There is an instability in the system when the D0-branes lie in the D4-branes. We move them off into a shell



D0-branes at the origin surrounded by uniform density of D4-branes separated in the $\mathbb{R}^5$ transverse to the D4–branes and a distance $v = v_0$ from the D0-branes.

Just as in electrostatics, the solution interior to the shell is the same as that in the absence of the D4s, however the interior expression is modified from

$$H(u,v) = 1 + \frac{r_0^7}{r^7} \qquad \text{to} \qquad H(u,v) = 1 + \frac{\gamma^3 r_0^7}{r^7}$$

where $\gamma^2 = 1 + \frac{N_f}{N_c}\frac{\lambda}{2m_q^3}$ and $r^2 = u^2 + \gamma^2 v^2$ is the interior radial coordinate. The dependence on N_f/N_c is because the parameter r_0 is measured at infinity in u at fixed v.

The backreacted exterior solution takes the fom

$$H(u,v) \;\; = \;\; 1 + \frac{r_0^7}{(u^2+v^2)^{\frac{7}{2}}}\left[1 + \frac{v_5^3}{v_0^3} H_c\left(\frac{u}{v_0}, \frac{v}{v_0}, \frac{v_5}{v_0}\right)\right] \; .$$

It is similar to the leading perturbative solution $H_c \sim 1$. The principal effect of increasing v_5/v_0 is that the geometry outside of the shell approaches that of the D4-branes geometry in the absence of D0-branes.

We need a black hole solution

Work in progress

For useful comparisons with numerical simulations we need an 11-dim gravitational M5-brane solution in the presence of a black hole. This seems accessible only via numerics.