Large but Finite N gauged matrix models and Discrete Gauge Groups.

Denjoe O'Connor

School of Theoretical Physics
Dublin Institute for Advanced Studies
Dublin, Ireland

Sixth Mandelstam Theoretical Physics School and Workshop 2024
University of Witwatersrand
January 16th 2024

Dimensional Reduction of Yang-Mills

Consider $SU(N)$ Yang-Mills compactified on a 3-torus.

The Yang-Mills action for $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{T}^3$:

$$S_{YM} = \frac{1}{4g^2} \int dt d^3x \mathbf{tr} F_{\mu\nu} F^{\mu\nu} \xrightarrow[V_{\mathbb{T}^3} \to 0]{} \frac{V_{\mathbb{T}^3}}{4g^2} \int \mathbf{tr} F_{\mu\nu}(t) F^{\mu\nu}(t)$$

Dimensional reduction on $\mathbb{T}^3$ gives a matrix model:
The spatial gauge fields become $N \times N$ matrices $A_a \to X_a$ and only $A_0 = A$ remains as a gauge field.

Reduced Hamiltonian

Lagrangian

$$L = \int_{\mathbb{T}^3} d^3x \frac{1}{2}\mathbf{tr}(\vec{E}^2 - \vec{B}^2) = \frac{V_{\mathbb{T}^3}}{4g^2}\mathbf{tr}(\frac{1}{2}[D_t, X_a]^2 + \frac{1}{4}[X_a, X_b][X^a, X^b])$$

Hamiltonian

$$H = \int_{\mathbb{T}^3} d^3x \frac{1}{2}\mathbf{tr}(\vec{E}^2 + \vec{B}^2) = \frac{V_{\mathbb{T}^3}}{4g^2}\mathbf{tr}(\frac{1}{2}[D_t, X_a]^2 - \frac{1}{4}[X_a, X_b][X^a, X^b])$$

This is now a quantum mechanical system of matrices. The gauge invariance is $X_a \to U^{-1}X_aU$, $A \to U^{-1}AU + iU^{-1}\partial_t U$.

Quantization in a Thermal Bath

- The gauge field, A, is non-dynamical—**the Lagrangian has no $\partial_t A$ dependence**.
- A is a Lagrange multiplier for a constraint—the Gauss law constraint.
- The constraint requires that the only physical degrees of freedom are gauge invariant observables.

Canonical Quantization

$$Z = \mathbf{Tr}_{\mathbf{Inv}}(\mathrm{e}^{-\beta \mathbf{H}})$$

The physical degrees of freedom are the invariants of the matrices X_a and $\Pi^a = E^a$, Note $[X_a, X_b] \neq 0$.

Path Integral Quantization

Since this is a quantum mechanical system we can follow the usual Feynman route to a path integral treatment and perform a Wick rotation to Euclidean (imaginary) time.

Path Integral Quantization in a Thermal Bath

$$Z = \int [dX][dA] \mathrm{e}^{-N \int_0^\beta d\tau \, \mathrm{Tr}(\frac{1}{2}(D_\tau X^a)^2 - \frac{1}{4}[X^a, X^b]^2)}$$

One can the evaluate observables with the path integral by standard techniques.

Hamiltonian Quantization

The residual gauge field A is not dynamical and appears only in

$$D_\tau X^a = \partial_\tau X^a - i[A, X^a].$$

It leads to a constraint on the dynamics.

Gauss law constraint

The Lagrange multiplier field, A, multiplies the Gauss law constraint and forces $SU(N)$ invariant physical states.

From the action we can obtain the Hamiltonian and once we have the Hamiltonian H we can equally consider thermal ensembles whose partition function is given by

$$Z = \mathbf{Tr}_{\mathbf{Inv}}(\mathrm{e}^{-\beta H}) = \sum_E \mathbf{\Omega(E)}\mathrm{e}^{-\beta E}.$$

Inv means $SU(N)$ singlets and $\mathbf{\Omega(E)}$ the energy degeneracy.

Reduction to a Gauge Gaussian Model

In leading order in a $1/d$ expansion the model becomes

A Gauge Gaussian Model

$$S_{GG}[X,A] = \frac{1}{2}\int_0^\beta d\tau\,\mathrm{Tr}\left\{(D_\tau X^a)^2 + m^2 X^a X^a\right\}$$

with $m \simeq d^{1/3}$ (V. Filev and D.O'C. arXiv:1506.01366).

Identifying and Counting Invariants

Consider counting the number of invariants of a system of $N \times N$ matrices, i.e. for $g \in U(N)$ invariant under conjugation:

$$X_i \to g X_i g^{-1}$$

$X \in \mathrm{Mat}(N)$ has N^2 degrees of freedom

But there are only N invariants—the N eigenvalues of X.

Eigenvalues are roots or the characteristic polynomial

$$P_N(\lambda) = Det[X - \lambda \mathbf{1}_N]$$

Hamilton-Cayley

The Hamilton-Cayley Theorem

Every finite rank square matrix, X, over a commutative ring satisfies its own characteristic equation

$$P_N(X) = 0$$

where $P_N(\lambda)$ is the characteristic polynomial of X.

$P_N(X)$ recursively

$$P_N(X) = P_{N-1}(X)X - \frac{1}{N}\mathbf{tr}(P_{N-1}(X)X).$$

with $P_1(X) = X - \mathbf{tr}(X)$.

$$\mathbf{tr}(P_N(X)) = 0 \quad \text{gives } \det(X) \text{ in terms of traces.}$$

Similarly $\mathbf{tr}(X^{N+1})$ becomes products of traces of lower powers.

2×2 matrices and 3×3 traceless matrices

For X, a generic 2×2 matrix,

$$P_2(x) = P_1(X)X - \frac{1}{N}\mathbf{tr}(P_1(X)X)\mathbf{1}_2 \qquad P_1(X) = X - \mathbf{tr}(X)$$

$$\implies P_2(X) = X^2 - X\mathbf{tr}(X) - \frac{1}{2}\left(\mathbf{tr}(X^2) - \mathbf{tr}^2(X)\right)\mathbf{1}_2$$

$$\mathbf{tr}(X^3) - \frac{3}{2}\mathbf{tr}(X)\mathbf{tr}(X^2) + \frac{1}{2}\mathbf{tr}^3(X) = 0\,.$$

For Y a generic traceless 3×3 traceless matrix

$$P_3(Y) = Y^3 - \frac{1}{2}\mathbf{tr}(Y^2)Y - \frac{1}{3}\mathbf{tr}(Y^3)$$

$$\implies \mathbf{tr}(Y^4) - \frac{1}{2}\left(\mathbf{tr}(Y^2)\right)^2 = 0\,.$$

More generally for an $N \times N$ matrix $\mathbf{tr}(X^{N+1})$ is expressible in terms of products of lower traces.

All matrix invariants are expressible in terms of the generating set $\{\mathbf{tr}(X^k)\}$ with $k \leq N$.

The algebra of GL_N invariants

The algebra of invariants of a single generic matrix X is generated by the N traces $\mathbf{tr}(X^k)$, $k = 1, \ldots, N$.

The invariants of X are, of course, the eigenvalues.

The number of invariants for a given power of the matrix is captured by a generating function (Hilbert-Poincaré series)

$$Z_N(t) = \sum_n^\infty \mathrm{dim}_n(N) t^n = \sum_{n=0}^\infty p_N(n) t^n$$

where dim_n is the number of invariants formed from n X's.
$\mathrm{dim}_n(N) = p_N(n) = \#$ partitions of n into N or less parts.

$$Z_N(t) = \prod_{m=1}^N \frac{1}{1 - t^m} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + \cdots .$$

Fock Space Realisation

For a single matrix the low lying states are:

$$
\begin{array}{ccccc}
|0\rangle, & & & & \\
\mathbf{tr}(a^\dagger)|0\rangle, & & & & \\
\mathbf{tr}^2(a^\dagger)|0\rangle, & \mathbf{tr}((a^\dagger)^2)|0\rangle, & & & \\
\mathbf{tr}^3(a^\dagger)|0\rangle, & \mathbf{tr}(a^\dagger)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}((a^\dagger)^3)|0\rangle, & & \\
\mathbf{tr}^4(a^\dagger)|0\rangle, & \mathbf{tr}^2(a^\dagger)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}((a^\dagger)^2)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}(a^\dagger)\mathbf{tr}((a^\dagger)^3)|0\rangle, & \mathbf{tr}((a^\dagger)^4)|0\rangle, \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

The partition function (Hilbert Poincaré series).

$$
Z_N(t) = \mathbf{Tr}_{\mathbf{Inv}}(\mathrm{e}^{-\beta(\mathbf{tr}(a^\dagger a))}) = \mathbf{Tr}_{\mathbf{Inv}}(t^{\hat{N}}) = \prod_{m=1}^{N} \frac{1}{1-t^m}..
$$

Where $t = \mathrm{e}^{-\beta}$, and **Inv** refers to $U(N)$—gauge invariant states.

$$
Z_\infty(t) = \frac{1}{\phi(t)} \qquad \phi(t) = \prod_{n=1}^{\infty}(1-t^n) \qquad \text{is the Euler function.}
$$

Two or more Matrices

What happens if we consider a pair of matrices X and Y?

For more than one matrix the invariants are no longer eigenvalues.

What can we say about the invariants of this system?

The invariants of 2×2 matrices

Two matrices X and Y

$$Z_2(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)(1-t_1^2)(1-t_1 t_2)(1-t_2^2)}$$

The invariants are built from $\mathbf{tr}(X)$, $\mathbf{tr}(X^2)$, $\mathbf{tr}(Y)$, $\mathbf{tr}(Y^2)$ and $\mathbf{tr}(X.Y)$.

Three matrices X, Y and Z

$$Z_2(t_1, t_2, t_3) = \frac{1 + t_1 t_2 t_3}{\prod_{a=1}^{3}(1-t_a)\prod_{b\leq c=1}^{3}(1-t_b t_c)}$$

The term $t_1 t_2 t_3$ indicates that we need $\mathbf{tr}(X.Y.Z)$ but not higher powers—it satisfies a quadratic relation. It captures a $\mathbb{Z}_2$ invariant.

Schur Polynomials

The low lying states and Schur Polynomials

$$Z_N(\rho t_1, \rho t_2, \rho t_3) = 1 + s_{(1,0,0)}\rho + 2s_{(2,0,0)}\rho^2 \\ +(2s_{(3,0,0)} + s_{(2,1,0)} + s_{(1,1,1)})\rho^3 + \cdots$$

where

$$s_{(1,0,0)} = t_1 + t_2 + t_3,\ s_{(2,0,0)} = t_1^2 + t_1 t_2 + t_2^2 + t_2 t_3 + t_3^2 + t_3 t_1$$
$$s_{(3,0,0)} = t_1^3 + t_1^2 t_2 + \cdots,\ s_{(2,1,0)} = t_1^2 t^2 + t_2 t_1^2 + \cdots$$
$$s_{(1,1,1}) = t_1 t_2 t_3$$

Traceless matrices

$$\prod_{a=1}^{3}(1 - t_a) Z_N(\rho t_1, \rho t_2, \rho t_3) = 1 + s_{(2,0,0)}\rho^2 + s_{(1,1,1)}\rho^3 + \cdots$$

The Molien-Weyl formula from Path Integrals

Lattice Gauge Gaussian Model

$$S[X,A] = \frac{1}{2}\int_0^\beta d\tau\, \mathrm{Tr}\left\{(D_\tau X)^2 + m^2 X^2\right\} \quad D_\tau = \partial_\tau + i[A,\cdot]\,.$$

$$D_\tau X \xrightarrow{lat} \frac{g_{n,n+1}X_{n+1}g_{n+1,n} - X_n}{a}\,, g_{n,n+1} = \mathcal{P}\mathrm{e}^{i\int_{na}^{(n+1)a} d\tau\, A(\tau)}\,,$$

with $\mathcal{P}$ a path ordered product, $g_{n+1,n} = g_{n,n+1}^{-1}$.

$$S_{\Lambda,g} = \sum_{n=0}^{\Lambda-1} \mathbf{tr}\left\{\frac{1}{a}\left(X_n^2 - X_n g_{n,n+1} X_{n+1} g_{n+1,n}\right) + \frac{a}{2}X_n^2\right\}\,,$$

$$Z_{N,\Lambda} = \int_{U(N)^\Lambda}\int_{\mathbb{R}^{N^2\Lambda}} \prod_{k=1}^{\Lambda} \mu(g_{k,k+1}) \frac{d^{N^2}X_k}{(2\pi a)^{N^2}} \mathrm{e}^{-S_{\Lambda,g}}$$

Collecting the gauge fields on the final link

Change of variables

$X_1' = X_1', \quad X_2' = g_{1,2} X_2 g_{2,1}, \cdots$ so that $X_1 g_{1,2} X_2 g_{2,1}$ becomes $X_1' X_2'$

$$S_{\Lambda,g} = -\frac{1}{a}\mathbf{tr}\left\{\sum_{n=1}^{\Lambda-1} X'_n X'_{n+1} + X'_\Lambda g\, X'_1 g^{-1}\right\} + \frac{1}{a}\sum_{n=1}^{\Lambda-1}\mathbf{tr}\left\{(1+\frac{a^2\beta^2 m^2}{2})\right.$$

$$g = g_{1,2}\dots g_{\Lambda,1} = \prod_{k=1}^{\Lambda} g_{k,k+1}$$

The action

$$S_{\Lambda,g} = \sum_{n,n'=1}^{\Lambda} \frac{1}{2} \mathbf{tr} X_{n'} a(\Delta_{\Lambda,g} + m^2 \mathbf{1})_{n',n} X_n$$

The matrix $(a^2\Delta_{\Lambda,g} + \frac{\beta^2 m^2}{\Lambda^2}\mathbf{1})_{n',n}$ is a ΛN^2 dimensional tri-diagonal matrix with $g \otimes g^{-1}$ in the right upper corner and its inverse in the lower corner. The diagonal elements are all $(2 + \frac{\beta^2 m^2}{\Lambda^2})\mathbf{1}$ and the off diagonals are $-\mathbf{1}$.

The partition function

$$\begin{aligned} Z_{N,\Lambda} &= \int \mu(g) \mathbf{Det}^{-1/2}(a^2 \Delta_{\Lambda,g} + \frac{\beta^2 m^2}{\Lambda^2} \mathbf{1})) \\ &= \int \mu(g) \frac{z_-^{\frac{N^2 \Lambda}{2}}}{\mathbf{det}[\mathbf{1} - z_-^{\Lambda} g \otimes g^{-1}]}. \end{aligned}$$

where $z_- = 1 + \frac{\mu^2}{2} - \sqrt{\mu^2(1 + \frac{\mu^2}{4})}$, with $\mu = \frac{m\beta}{\Lambda}$.

The continuum limit $\Lambda \to \infty$

$$\lim_{\Lambda\to\infty} z_-^\Lambda \to \mathrm{e}^{-\beta m}$$

$$\lim_{\Lambda\to\infty} Z_{N,\Lambda} = Z_N = \int \mu(g) \frac{\mathrm{e}^{-\frac{\beta m N^2}{2}}}{\mathbf{det}[\mathbf{1} - \mathrm{e}^{-\beta m} g \otimes g^{-1}]}.$$

We can replace the continuum gauge group with a discrete gauge group and all steps go through

Finite Group

$$Z_N = \frac{1}{|G|} \sum_{g\in G} \frac{\mathrm{e}^{-\frac{\beta m N^2}{2}}}{\mathbf{det}[\mathbf{1} - \mathrm{e}^{-\beta m} g \otimes g^{-1}]}.$$

Without the zero-point energy contributions these are the Molien-Weyl formulae.

The gauged d-matrix $U(N)$ gaussian model

The resulting normal ordered partition function with $t_a = \mathrm{e}^{-\beta m_a}$, is

$$Z_N(t_1, \cdots, t_d) = \frac{1}{N!} \oint \prod_{i=1}^{N} \frac{dz_i}{2\pi i} \Delta(z)\Delta(z^{-1}) \prod_{a=1}^{d} \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - t_a z_i z_j^{-1}}$$

where

$$\Delta(z) = \prod_{1 \le i < j \le N} (z_i - z_j)$$

The case of $d = 2$, with $t_1 = t_2 = t$ has received much attention due to its relevance to the counting of $\frac{1}{4}$-BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills.

writing $z_i = \mathrm{e}^{i\theta_i}$

$Z_N(t) = \frac{1}{N!}\int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_N}{(2\pi)^N} \mathrm{e}^{-S_N(\theta)}$

$$S_N(\theta) = \frac{d}{2}\sum_{i,j=1}^{N} \ln\left|1 - t\mathrm{e}^{i(\theta_i-\theta_j)}\right|^2 - \frac{1}{2}\sum_{i\neq j=1}^{N} \ln\left|1 - \mathrm{e}^{i(\theta_i-\theta_j)}\right|^2$$

The last sum is from the Vandermonde due to diagonalisation of g.

A large N analysis

Low temperature, $\beta \to \infty \implies t \ll 1$

Expanding the t dependent logarithms one finds

$$S_N(\theta) = -N^2 \sum_{n=1}^{\infty} \frac{dt^n}{n} |u_n|^2 - \frac{1}{2} \sum_{i \neq j = 1}^{N} \ln |1 - \mathrm{e}^{i(\theta_i - \theta_j)}|^2 .$$

where $u_n = \frac{1}{N} \sum_{i=1}^{N} \mathrm{e}^{in\theta_i}$.

An instability

From

$$S_N(\theta, d) \simeq N^2 \sum_{n=1}^{\infty} \frac{1 - dt^n}{n} |u_n|^2 ,$$

we see that the coefficient of $|u_1|^2$ changes sign at $dt = 1$ ($T_H = \frac{m}{\ln d}$). A transition occurs!

For small t one can find the exact expression in the limit of infinite N

$$Z_\infty = \prod_{n=1}^{\infty} \frac{1}{1 - dt^n}$$

see F. Dolan arXiv:0704.1038 . This expression counts low energy states and is exact up to t^N and terms grow as d^n.

The Polyakov Loop

Internal energy

$$E = \frac{1}{mN^2} t \frac{d}{dt} \ln Z = \frac{dt}{m} \langle |P_1^2| \rangle + \sum_{n=2}^{\infty} \frac{dt^n}{m} \langle |P_n|^2 \rangle$$

The Polyakov loop as Order Parameter

$\langle P_1 \rangle = 0$	the confined phase
$\langle P_1 \rangle \neq 0$	the deconfined phase

A proxy for the Polyakov loop at large N

$$\langle |P_1| \rangle \simeq \sqrt{\langle |P_1^2| \rangle} \simeq \sqrt{\frac{mE}{td}} \, .$$

The a_1 model in detail

$$Z(a_1) = \int [dg] \mathrm{e}^{a_1 \mathbf{tr}(g) \mathbf{tr}(g^{-1})}$$

Expanding directly in a_1 gives

$$Z(a_1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_R [d_R(S_k)]^2 a_1^k$$

where $d_R(S_k)$ is the dimension of the representation R of the permutation group S_k.

$$d_{n_1,\cdots,n_N}(S_k) = (n_1 + \cdots n_N)! \frac{\prod_{i=1}^{N-1} \prod_{j=i+1}^{N} (n_i - i - (n_j - j))}{\prod_{i=1}^{N} (n_i + N - i)!}$$

We have

$$\frac{1}{k!} \sum_R [d_R(S_k)]^2 = 1 \quad k \leq N$$

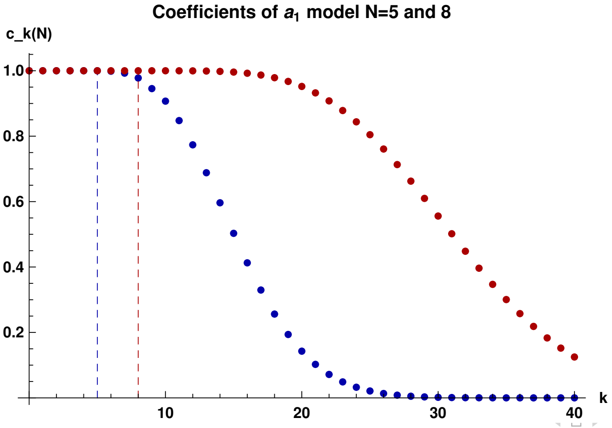
and decreases slowly above N (at least initially).

Coefficients of a_1 model

The N dependence of the a_1 model

The coefficients

$$c_k(N) = \frac{1}{k!}\sum_R [d_R(S_k)]^2$$



Solving the a_1 model for $a_1 > 1$

For large N we wish to solve

$$S_N(\theta) = \frac{d}{2}\sum_{i,j=1}^{N} \ln|1 - t\mathrm{e}^{i(\theta_i-\theta_j)}|^2 - \frac{1}{2}\sum_{i\neq j=1}^{N} \ln|1 - \mathrm{e}^{i(\theta_i-\theta_j)}|^2 .$$

for $\theta_n \to \theta(n)$ with $\frac{dn}{d\theta} = \rho(\theta)$ and

$$\frac{S(\rho)}{N^2} = \frac{d}{2}\int \rho(\alpha)\int \rho(\beta)\ln|1 - t\mathrm{e}^{i(\alpha-\beta)}|^2 d\alpha d\beta - \frac{1}{2}P\int \rho(\alpha)\rho(\beta)\ln|1 - \mathrm{e}^{i(\alpha-\beta)}|^2 d\alpha d\beta .$$

The large N a_1 model is solvable

$$\frac{S_{a_1}}{N^2} = -a_1|u_1|^2 - \frac{1}{2}P\int \rho(\alpha)\rho(\beta)\ln|1-\mathrm{e}^{i(\alpha-\beta)}|d\alpha d\beta\,.$$

Has the solution

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for} \quad a_1 < 1 \\ \frac{1}{\pi\sin^2(\frac{\theta_0}{2})}\sqrt{\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})}\cos(\frac{\theta}{2}) & \text{for} \quad a_1 > 1 \end{cases} \tag{1}$$

and θ_0 is specified by

$$s^2 \equiv \sin^2(\frac{\theta_0}{2}) = 1 - \sqrt{1-\frac{1}{a_1}}. \tag{2}$$

The large N free energy as

$$-\frac{1}{N^2}\ln Z = \beta F = \begin{cases} 0 & a_1 < 1 \\ \frac{1}{2} - \frac{1}{2s^2} - \frac{1}{2}\ln s^2 & a_1 > 1 \end{cases}$$

$$\simeq \quad \frac{1-a_1}{4} - \frac{1}{3}(a_1 - 1)^{3/2} + \qquad a_1 > 1$$

Adding one-loop corrections from the gauge field or from other fields *including fermions* only changes a_1

Note the charactiristic leading $\frac{1}{4}$ and exponent $\frac{3}{2}$.

The Gauge Gaussian Case

Taking $a_1 = d\mathrm{e}^{-\beta}$ and expanding in the vicinity of the Hagadorn temperature we find with $\beta_H = \ln d$

$$-\frac{1}{N^2}\ln Z = \beta F = \begin{cases} 0 & \beta > \beta_H \\ \frac{\beta-\beta_H}{4} - \frac{1}{3}(\beta_H - \beta)^{3/2} + \cdots & \beta < \beta_H \end{cases}$$

The energy

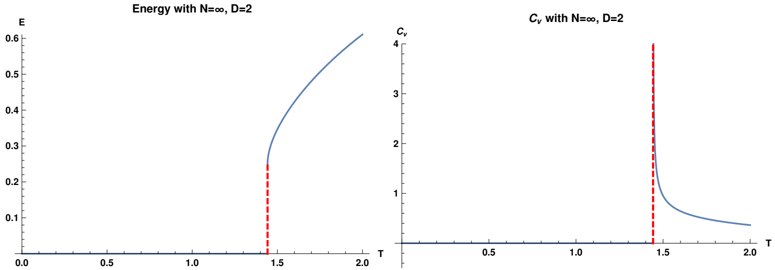
$$E = \frac{\partial(\beta F)}{\partial \beta} = \begin{cases} 0 & \beta > \beta_H \\ \frac{1}{4} + \frac{1}{2}\sqrt{\beta_H - \beta} + \cdots & \beta < \beta_H\,. \end{cases}$$

The transition occurs is $E = \frac{1}{4}$ or $n = \frac{N^2}{4}$.

The Phase Transition

The transition is NOT simply 1st order (it is 3/2 order).

The transition has a divergent specific heat on either side of the transition. The stronger divergence appears to be on the low temperature side, but this is coming from subdominant contributions as the limit is approached.



Entropy from the Free Energy

Thermodynamics

$$TdS = dE$$

$$\frac{dS}{dE} = \beta(E)$$

$$E = \frac{1}{4} + \frac{1}{2}\sqrt{\ln d - \beta} + \cdots \qquad \Longrightarrow \beta(E) = \ln d - 4(E - \frac{1}{4})^2 + \cdots$$

Inverting the expression for $E(\beta)$ and integrating gives and matching at $E = \frac{1}{4}$ gives

$$S(E) = E \ln d - \frac{4}{3}(E - \frac{1}{4})^3 + \cdots$$

Matching across the transition

The transition occurs at $E = \frac{1}{4}$.
At low temperatures

$$Z_\infty(t) = \prod_{n=1}^{\infty} \frac{1}{1 - dt^n} = \sum_{n=1}^{\infty} d^n t^n = e^{N^2 \ln(d) E}$$

using $E_n = \frac{n}{N^2}$

Entropy at the transition

$$S(\frac{1}{4}) = \frac{\ln d}{4}$$

So at the transition

$$n_c = \frac{N^2}{4}$$

The transition is one from where trace relations can be ignored to where they become significant.

Implications for Matrix Traces

- The number of states grows with energy as $\dim_n(N,d) \sim d^n = e^{N^2 \ln(d) E}$ (with $E = \frac{n}{N^2}$) below the transition.
- $\dim_n(N,d) \sim e^{N^2\{\ln(d)E - \frac{4}{3}(E-\frac{1}{4})^3 + \cdots\}}$ above.
- The low n and large n entropy match at $E = \frac{1}{4}$.

Trace relations switch on at $n = \frac{N^2}{4}$

Rephrasing:

Some Lessons for confining/deconfining transitions

- The phase transition is in the density of states—the entropy!
- In both the confined and deconfined phases all the observables are gauge invariant singlets.
- In the matrix model at large N it is trace relations that "switch on" at the transition and this occurs for traces of length $n = \frac{N^2}{4}$.
  This is true for bosons or fermions or a mixture of these and of the number of matrices.

There is an overlap of some of these results with D. Berenstein and Kai Yan arXiv:2307.06122.

- Summary:

$$\dim_n(N,d) = \begin{cases} cd^n & 0 \ll n \leq \frac{N^2}{4} \\ cd^n \mathrm{e}^{-\frac{4N^2}{3}(\frac{n}{N^2}-\frac{1}{4})^3 \cdots} & n \geq \frac{N^2}{4} \end{cases}.$$

- For large N trace relations are ignorable up to matrix words of length $\frac{N^2}{4}$.
  Trace relations are vital when words of longer than $\frac{N^2}{4}$ are excited.

The entropy, $S(n,d) = \frac{1}{N^2} \ln \dim_n(N,d)$ has **universal** large N transition at $n = \frac{N^2}{4}$

Thanks for Your Attention!