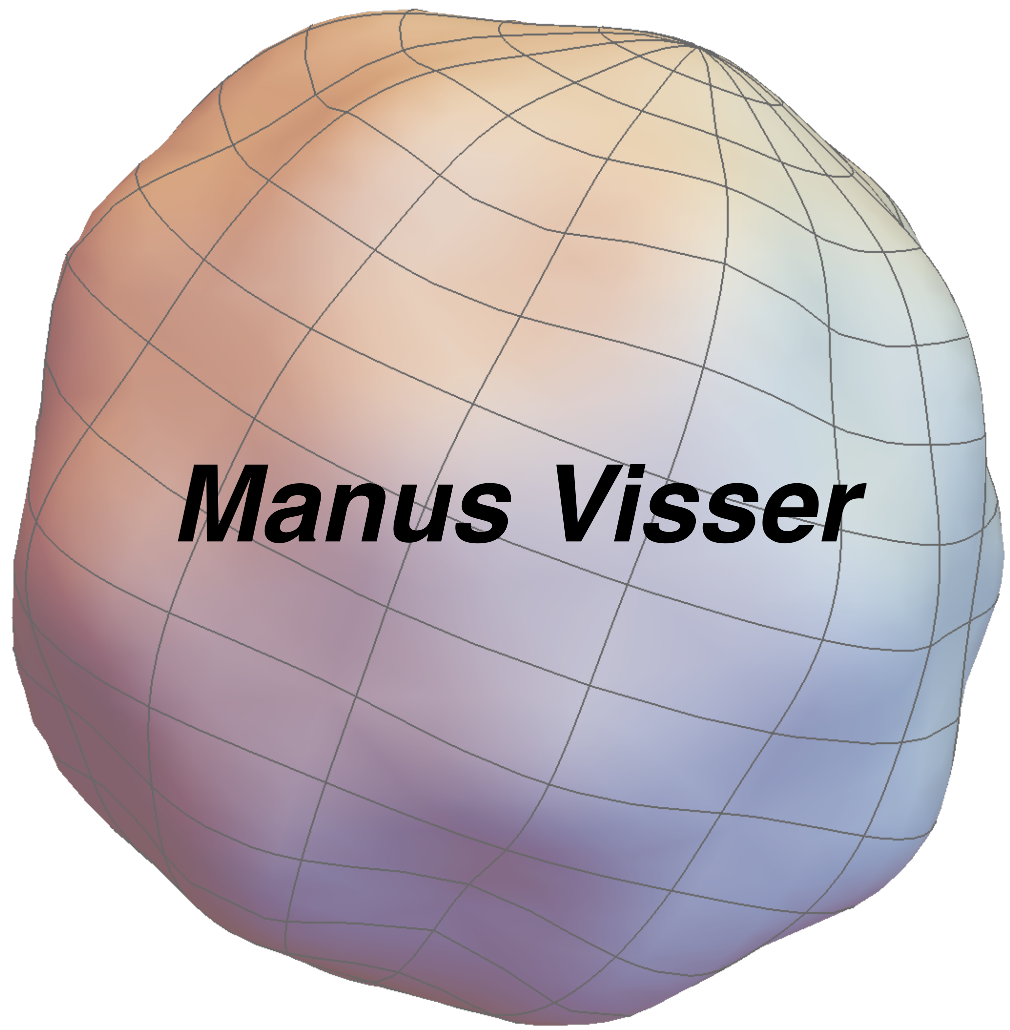
Partition function for a volume of space



Manus Visser

UNIVERSITY OF CAMBRIDGE

Swiss National Science Foundation

Sixth Mandelstam Theoretical Physics School and Workshop,
University of Witwatersrand, Johannesburg

12 January 2024

Horizon entropy

- Bekenstein-Hawking entropy provides a low-energy window into the realm of quantum gravity

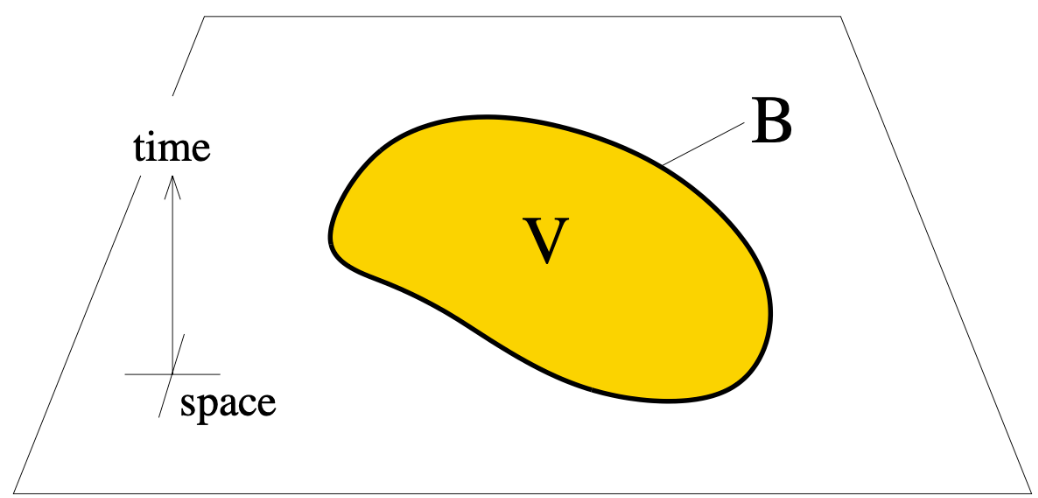
$$S = \frac{A}{4\hbar G}$$

- *Universal formula:* applies to black hole horizons, cosmological horizons, acceleration horizons, and entanglement wedges.

"Entropy = area" for any volume of space

- *Claim:* gravitational entropy is not only associated to the area of black hole or de Sitter horizon, but to the area of *any* boundary separating a region of space.

Bousso (1999), Banks-Fischler (2001), Jacobson-Parentani (2003), Bianchi-Myers (2014), …



How to justify this?

Gravitational path integral

Gibbons and Hawking (1977) derived the entropy of black hole and de Sitter horizons from a Euclidean saddle approximation of the quantum gravity partition function.

Can the entropy of any volume of space be derived from a quantum gravity partition function?

See also Banks-Draper-Farkas (2020)
and our statistical interpretation (Jacobson-MV 2022)

Gravitational path integral

Gibbons and Hawking (1977) derived the entropy of black hole and de Sitter horizons from a Euclidean saddle approximation of the quantum gravity partition function.

Can the entropy of any volume of space be derived from a quantum gravity partition function?

Yes! Using the method of constrained instantons

Outline

1. Gravitational partition function for de Sitter space

   - T. Jacobson, B. Banihashemi, Thermodynamic ensembles with cosmological horizons, JHEP, 2204.05324.

   - E. Morvan, J.P. Van der Schaar, MV, On the Euclidean action of de Sitter black holes and constrained instantons, SciPost, 2203.06155.

2. Constrained sphere partition function

   - T. Jacobson & MV, *Partition function of a volume of space*, PRL, 2212.10607.

Outline

1. **Gravitational partition function for de Sitter space**

   - T. Jacobson, B. Banihashemi, Thermodynamic ensembles with cosmological horizons, JHEP, 2204.05324.

   - E. Morvan, J.P. Van der Schaar, MV, On the Euclidean action of de Sitter black holes and constrained instantons, SciPost, 2203.06155.

2. Constrained sphere partition function

   - T. Jacobson & MV, *Partition function of a volume of space*, PRL, 2212.10607.

Euclidean gravity path integral

PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Action integrals and partition functions in quantum gravity

G. W. Gibbons* and S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England
(Received 4 October 1976)

One can evaluate the action for a gravitational field on a section of the complexified spacetime which avoids the singularities. In this manner we obtain finite, purely imaginary values for the actions of the Kerr-Newman solutions and de Sitter space. One interpretation of these values is that they give the probabilities for finding such metrics in the vacuum state. Another interpretation is that they give the contribution of that metric to the partition function for a grand canonical ensemble at a certain temperature, angular momentum, and charge. We use this approach to evaluate the entropy of these metrics and find that it is always equal to one quarter the area of the event horizon in fundamental units. This agrees with previous derivations by completely different methods. In the case of a stationary system such as a star with no event horizon, the gravitational field has no entropy.

GH represented the canonical partition function in gravity as a Euclidean path integral over metrics

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \longleftrightarrow \quad Z = \int \mathcal{D}g\, e^{-I_E[g]/\hbar}$$

Gibbons-Hawking partition function

- If the action is very large compared to Planck's constant, the path integral can be estimated as:

$$Z \sim \exp\left(-I_E^{\mathrm{saddle}}/\hbar\right)$$

- From the canonical partition $Z = \mathrm{Tr}\, e^{-\beta H}$ one usually gets thermodynamic quantities for the system

$$\ln Z = -\beta F$$

$$-\frac{d}{d\beta}\ln Z = \langle H \rangle_\beta$$

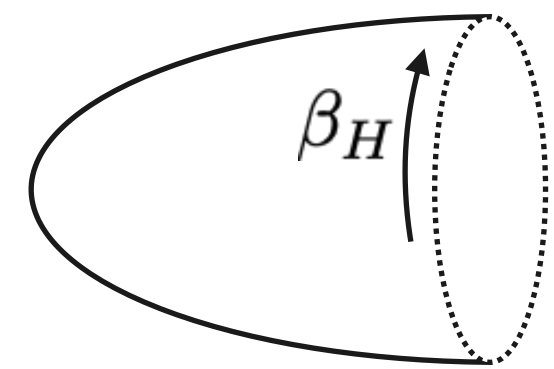
$$\left(1 - \beta\frac{d}{d\beta}\right)\ln Z = S$$

Entropy from the partition function

- If the saddle geometry is a Euclidean black hole spacetime, then

$$I_E/\hbar = \beta F = \beta M - S$$

$$S = \left(\beta\frac{d}{d\beta} - 1\right) I_E/\hbar = \frac{\text{horizon area}}{4\hbar G}$$

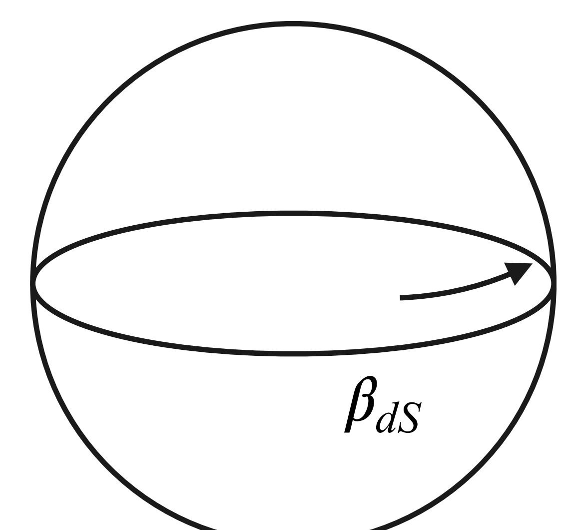


- If the saddle geometry is Euclidean de Sitter space (a round sphere whose radius is the dS curvature scale *L*), then

  - *M = 0* (since the saddle has no boundary)

  - the entropy is $S_{dS} = -I_E/\hbar = \dfrac{A(L)}{4\hbar G}$

NB: the action is independent of β

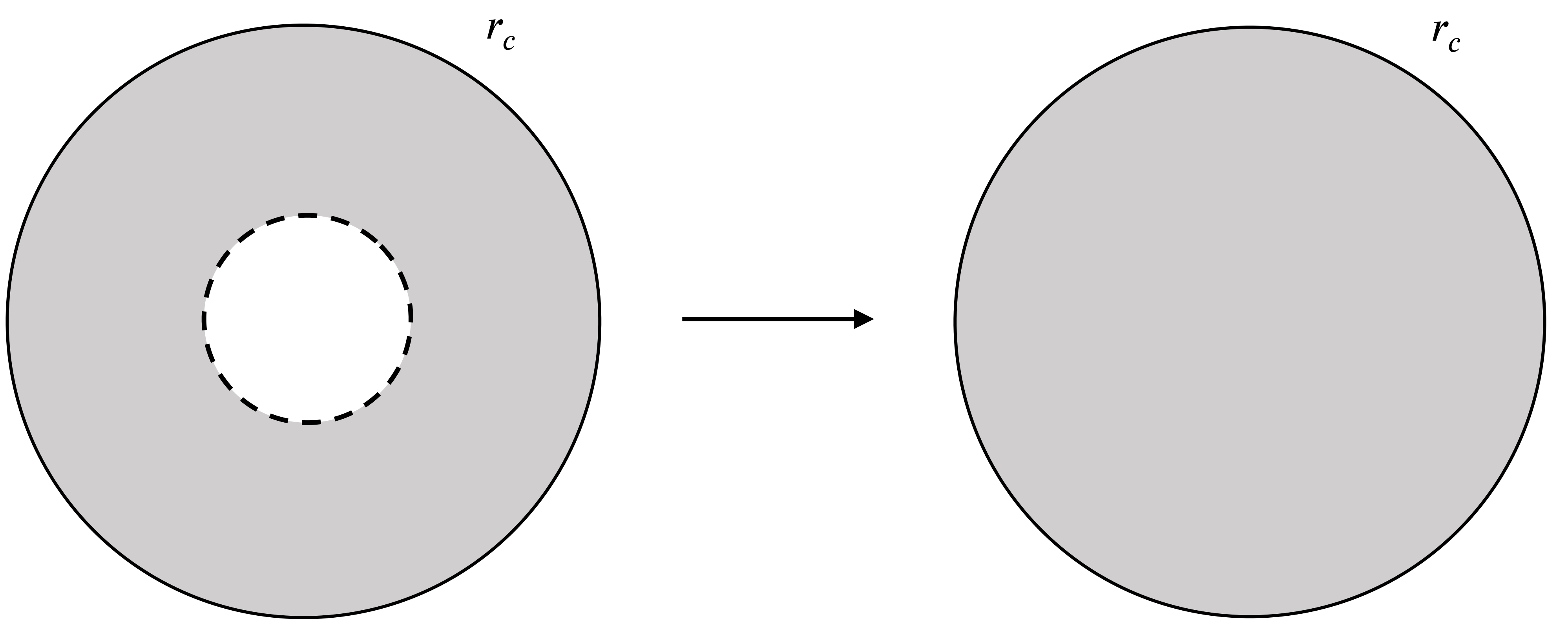


Sphere partition function

- What is the meaning of the sphere partition function?
  What is the thermodynamic ensemble?

Jacobson-Banihashemi (2022)

- Resolution: introduce an *artificial "York" boundary*, where H is defined, and examine the limit in which it disappears



topology: $D^2 \times S^{D-2}$ $\quad\quad$ S^D

Dimension of the Hilbert space

- Canonical partition function with a York boundary

$$H_{\rm BY} = -\frac{1}{8\pi G}\oint_S d^{D-2}x\sqrt{\sigma}k$$

$$Z = \mathrm{Tr}\, e^{-\beta H_{\rm BY}}$$

- In the vanishing boundary limit the Brown-York Hamiltonian vanishes $H_{\rm BY} = 0$, and the path integral is over all metric on the sphere S^D. Jacobson-Banihashemi (2022)

$\longrightarrow$

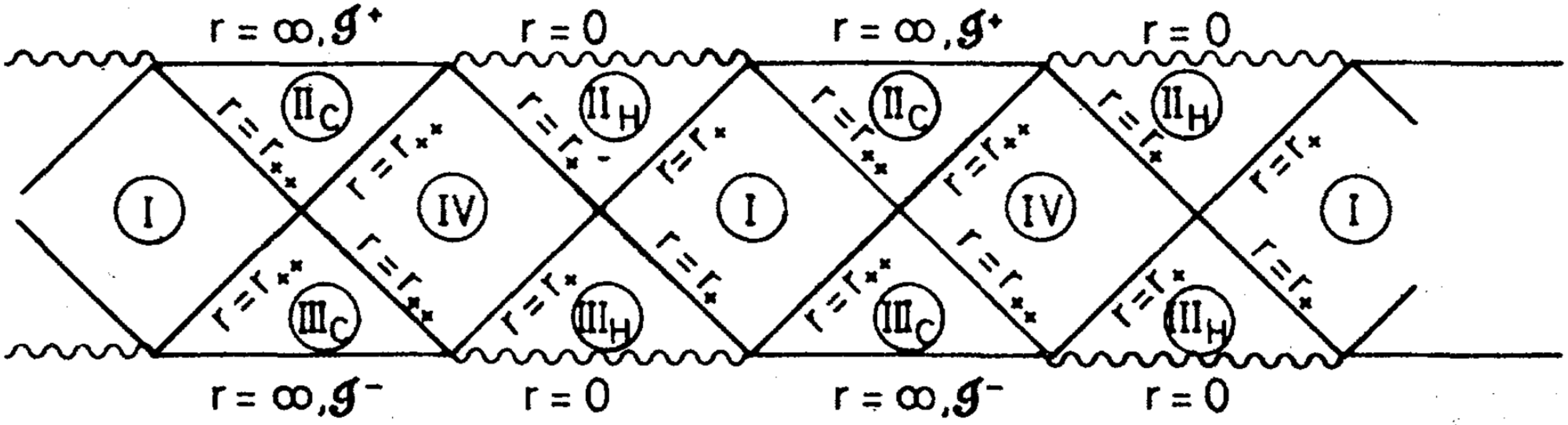
$$Z \to \mathrm{Tr}_{\mathcal{H}}\, 1 = e^{S_{dS}}$$

= dimension of Hilbert space
of states surrounded by a horizon,
i.e. states of a ball

de Sitter black holes

- What about black holes in de Sitter space? How do we take them into account in the Euclidean gravitational path integral?

- Let us focus on Schwarzschild-de Sitter space.

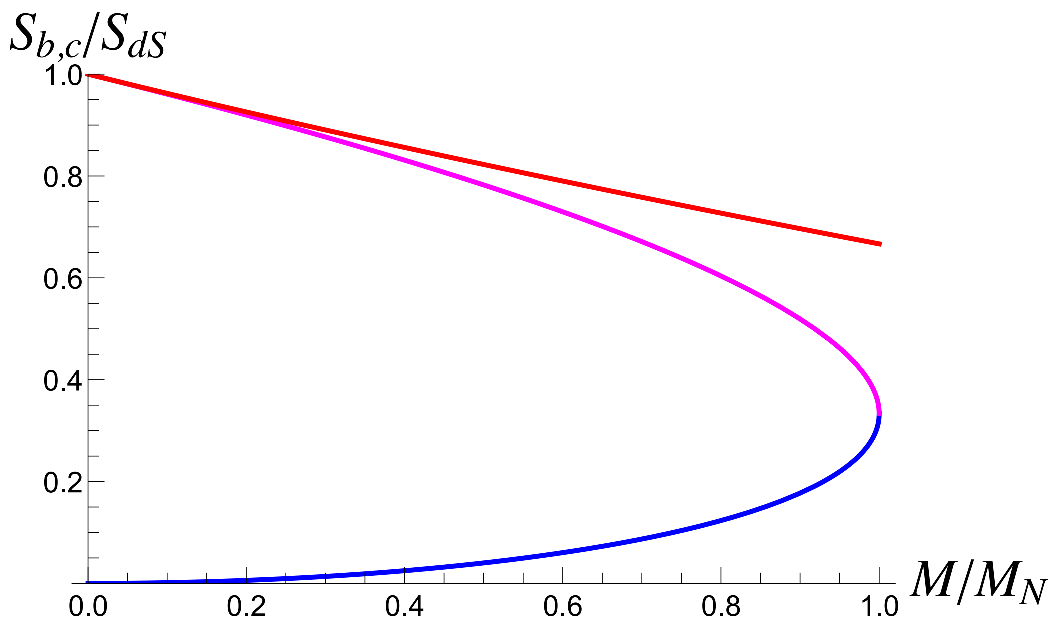


- SdS has both a black hole and a cosmological horizon, which have different temperatures. Only for the largest dS black hole (Nariai) are the temperatures the same.

de Sitter black holes

- The sum of the black hole entropy and cosmological horizon entropy is always lower than the de Sitter entropy, i.e. pure dS has maximum entropy.

$$S_{SdS} := S_b + S_c \leq S_{dS}$$



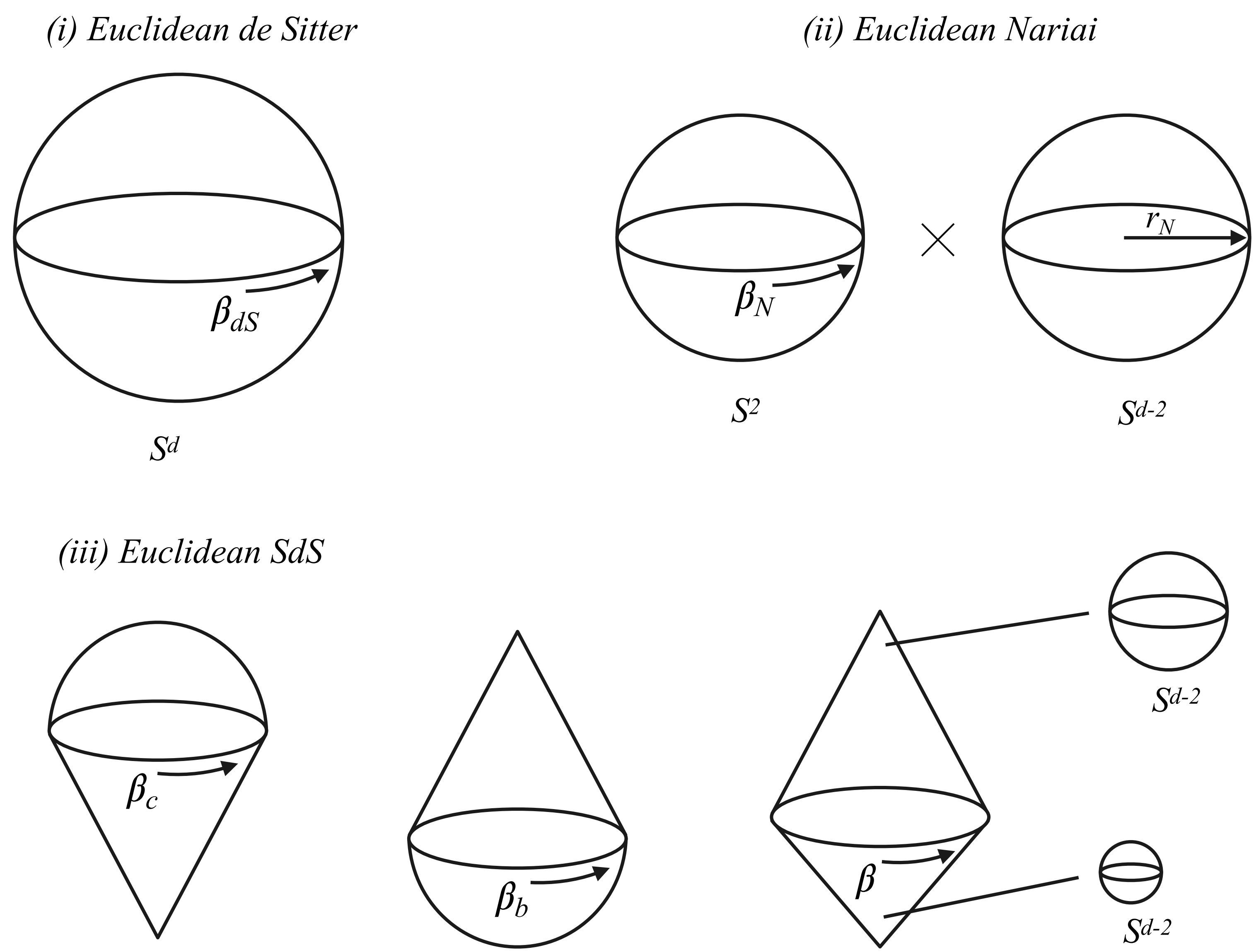
- This suggests:

  1. Euclidean de Sitter black holes are *subdominant* in the gravitational path integral.

  2. Empty dS corresponds to a maximally mixed state, Banks-Fisschler (2001) and the presence of a black hole constitutes a *constraint* on the state, reducing the entropy. Banks-Fiol-Morisse (2006)

Euclidean SdS geometries

- *Possible obstacle:* The Euclidean continuation of SdS is singular at (at least one of) the horizons, hence the Einstein equations are not satisfied there.



Constrained path integral

Draper-Farkas (2022), Morvan, Van der Schaar, MV (2022)

- Even though Euclidean SdS black holes are not regular gravitational instantons, they can be interpreted as stationary points of a constrained path integral at fixed $\mathcal{A}$

$$Z = \int \mathcal{D}g e^{-I_E[g]} = \int \mathcal{D}\mathcal{A}\mathcal{D}g\delta(\mathcal{C}[g] - \mathcal{A})e^{-I_E[g]}$$

Affleck (1981), Cotler-Jensen (2021)

- Introduce a Lagrange multiplier to impose the constraint

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}g\mathcal{D}\lambda e^{-I_E[g]+\lambda(\mathcal{C}[g]-\mathcal{A})} = \int D\mathcal{A}Z[\mathcal{A}]$$

- Perform stationary-point approximation at fixed $\mathcal{A}$

$$Z[\mathcal{A}] \approx e^{-I_E[\mathcal{A}]}$$

equations of motion

$$\delta I_E[g] + \lambda\,\delta\mathcal{C}[g] = 0\,,\ \mathcal{C}[g] = \mathcal{A}$$

Constrained path integral

- Now let's take the constraint to be the sum of the horizon areas (at a fixed horizon radius)

$$\mathcal{C}[g] = \left[\oint_{\mathcal{H}_b} + \oint_{\mathcal{H}_c}\right] d^{D-2}x\sqrt{\gamma}$$

- The partition function at fixed $\mathcal{A}$ is then

$$Z[\mathcal{A}] = \int \mathcal{D}g\mathcal{D}\lambda \exp\left(\frac{1}{16\pi G}\int d^D x\sqrt{g}(R-2\Lambda) - \frac{\lambda}{4G}\left(\left[\oint_{\mathcal{H}_b} + \oint_{\mathcal{H}_c}\right] d^{D-2}x\sqrt{\gamma} - \mathcal{A}\right)\right)$$

- The saddle point equations are

$$\mathcal{A} = A_b + A_c \qquad R = \frac{2D}{D-2}\Lambda + 4\pi\lambda\frac{\sqrt{\gamma}}{\sqrt{g}}\left(\delta(r=r_b) + \delta(r=r_c)\right)$$

- Euclidean SdS is a saddle (= constrained instanton):

if $\lambda = 1$ & $\sqrt{\gamma} = \sqrt{g}$ $\longrightarrow$ $R = R_{\text{bulk}} + R_{\text{con}}$

Euclidean action of SdS black hole

- Off-shell Euclidean action

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{g}(R-2\Lambda)$$

- For Euclidean SdS the Ricci scalar is constant everywhere, except at the conical singularities at the horizons $r = r_{b,c}$.

$$R = R_{\rm bulk} + R_{\rm con}$$

- The on-shell actions for both these terms are finite

$$I_{E,\rm bulk} = -\frac{\mathcal{V}\Lambda}{(D-2)4\pi G}$$

Euclidean spacetime volume

$$\mathcal{V} = \int_{\mathcal{M}} d^d x\sqrt{g} = \beta\Theta$$

$$I_{E,\rm con} = -\frac{A_b}{4G} - \frac{A_c}{4G} + \beta\left(\frac{A_b\kappa_b}{8\pi G} + \frac{A_c\kappa_c}{8\pi G}\right)$$

$$\tau \sim \tau + \beta$$

Euclidean action of SdS black hole

- The total on-shell Euclidean action is

$$I_E = I_{E,\text{bulk}} + I_{E,\text{con}} = \beta \underbrace{\left(\frac{A_b \kappa_b}{8\pi G} + \frac{A_c}{8\pi G} - \frac{\Theta \Lambda}{(D-2)4\pi G} \right)} - \frac{A_b}{4G} - \frac{A_c}{4G}$$

Smarr formula for SdS $= 0$

- Thus,

$$\boxed{I_{E,SdS} = -\frac{A_b + A_c}{4G}}$$

NB the result is independent of the inverse temperature β
NB this holds for any value of the mass *M*

- *Constrained partition function:* $Z[\mathcal{A}] \approx e^{-I_E[\mathcal{A}]} = e^{(A_b + A_c)/4G}$

Outline

1. Gravitational partition function for de Sitter space

   - T. Jacobson, B. Banihashemi, Thermodynamic ensembles with cosmological horizons, JHEP, 2204.05324.

   - E. Morvan, J.P. Van der Schaar, MV, On the Euclidean action of de Sitter black holes and constrained instantons, SciPost, 2203.06155.

2. Constrained sphere partition function

   - T. Jacobson & MV, *Partition function of a volume of space*, PRL, 2212.10607.

Partition function for a volume of space

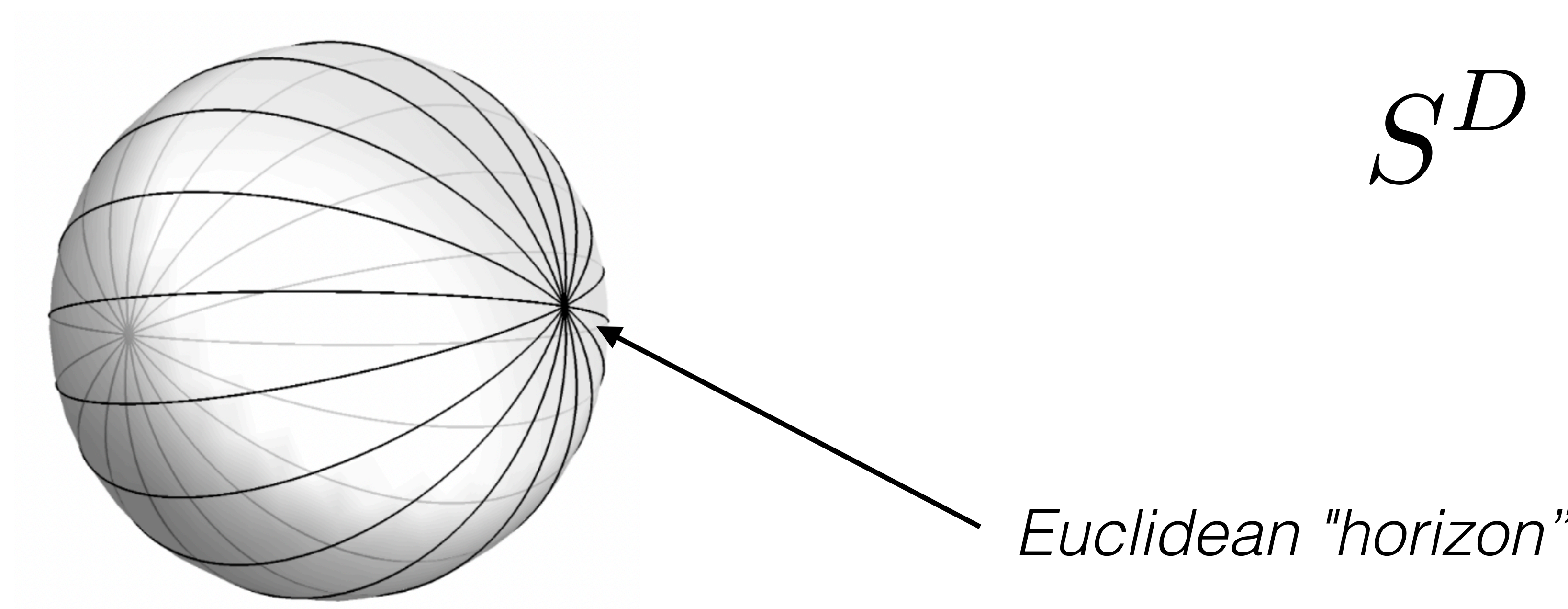
- Should not "area = entropy" apply to *any* volume of space (topological ball)?
- To specify a "region of space", one must somehow fix its size.
- We fix the spatial volume, by adding a constraint in the path integral that all spatial slices have volume V

$$Z[V] = \mathrm{Tr}_{\mathcal{H}} 1$$ = dimension of Hilbert space of geometries with spatial volume V

Euclidean sphere geometry

- What are the topologies that we integrate over in the path integral?
- Consider a spatial topological *(D-1)*-ball whose boundary has topology S^{D-2}.
- The Euclidean manifold generated by rotating the ball through a complete circle about the ball boundary is a <u>topological *D*-sphere</u>

e.g. *D=2* version:



Constrained sphere partition function

Method of constrained instantons
Affleck (1981), Stanford (2020), Cotler-Jensen (2021)

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\, \mathcal{D}g\, \exp\left[\frac{1}{16\pi\hbar G}\int d^D x\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\,\lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma} - V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Static, spherically symmetric saddle

- Saddle with minimum action presumably is the most symmetric one:

$$ds^2 = N^2(r)d\phi^2 + h(r)dr^2 + r^2 d\Omega^2_{D-2}$$

- *N(r)* is determined by the Tolman-Oppenheimer-Volkoff-equation, with boundary conditions:

1) $N = 0$ at $r = R_V$, the "horizon"

2) $N' = -\sqrt{h}$ at $r = R_V$, to prevent conical singularity

- $\Lambda = 0$ solution:

$$R_V = [(D-1)V/\Omega_{D-2}]^{1/(D-1)}$$

$$ds^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega^2_{D-2}$$

Euclidean action

- Euclidean constrained instanton has topology S^D, is conformally flat, and has a 1/(r-R_V) curvature singularity at the horizon.

- BUT the on-shell Euclidean action is finite:

for $\Lambda = 0$: $$I_{\text{saddle}} = -\frac{1}{16\pi G}\int d^D x \sqrt{g} R = -\frac{A_V}{4G}$$

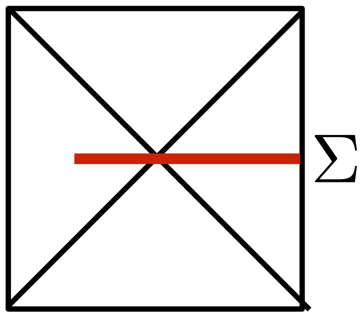
- Hence, in the zero-loop saddle-point approximation:

$$\boxed{Z[V] \approx \exp(A_V/4\hbar G)}$$

This shows that finite volumes of space have a BH “entropy” (log dim H).

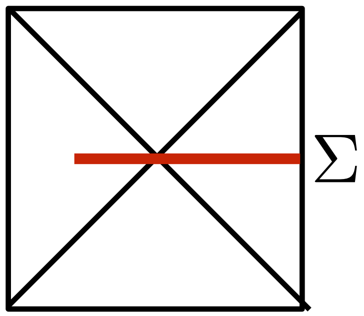
$\Lambda \neq 0$ **case**

For nonzero Λ a similar saddle exists, where spatial ball is embedded in H^{D-1} for $\Lambda < 0$ or S^{D-1} for $\Lambda > 0$



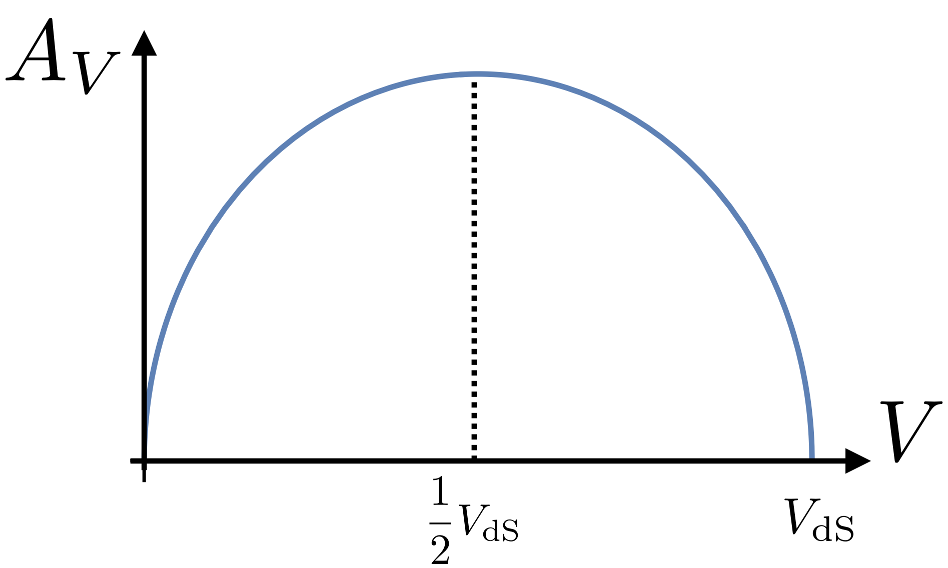
$\Lambda \neq 0$ case

For nonzero Λ a similar saddle exists, where spatial ball is embedded in
H^{D-1} for $\Lambda < 0$ or S^{D-1} for $\Lambda > 0$



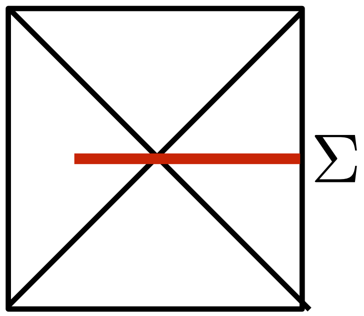
BUT for $\Lambda > 0$ there are significant differences:

- If V is larger than the dS spatial hemisphere, entropy *decreases* as V increases.
- There is no saddle if V is larger than the full spatial dS sphere.



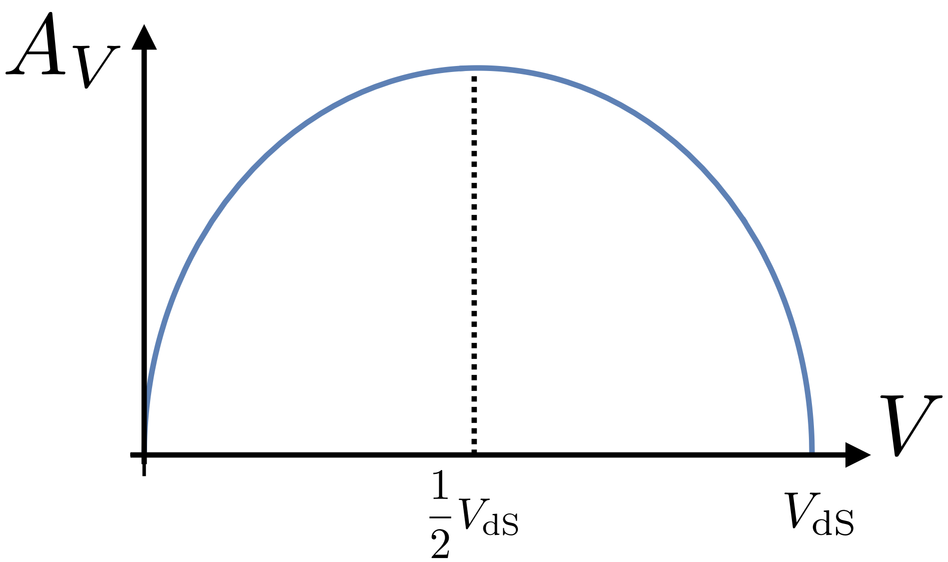
$\Lambda \neq 0$ **case**

For nonzero Λ a similar saddle exists, where spatial ball is embedded in H^{D-1} for $\Lambda < 0$ or S^{D-1} for $\Lambda > 0$



BUT for $\Lambda > 0$ there are significant differences:

- If V is larger than the dS spatial hemisphere, entropy *decreases* as V increases.
- There is no saddle if V is larger than the full spatial dS sphere.
- The integral over all V is indeed dominated by the de Sitter saddle:



$$Z = \int_0^{V_{\rm dS}} dV \exp(A_V/4\hbar G) \approx \exp(A_{\rm dS}/4\hbar G)$$

recovers Gibbons-Hawking entropy!

Regulation of the saddle singularity

- Bekenstein-Hawking result for action seems reasonable, but a curvature singularity exists at the horizon.

- Despite the singularity, the result could be reliable provided the corrections are small.

Regulation of the saddle singularity

- Bekenstein-Hawking result for action seems reasonable, but a curvature singularity exists at the horizon.

- Despite the singularity, the result could be reliable provided the corrections are small, EITHER because

  1. Curvature corrections might regularise the saddle without significantly changing the entropy, while remaining consistent with EFT

$$S_{\text{Wald}} \sim \frac{A}{\ell_P^2}(1 + \ell^2 R + \ell^4 R^2 + \ell^4 \Box R + \cdots)$$
$$\sim \frac{A}{\ell_P^2}\left(1 + \frac{\ell^2}{\ell_s R_V} + \frac{\ell^4}{\ell_s^2 R_V^2} + \frac{\ell^4}{\ell_s^3 R_V}\cdots\right)$$

Suppose curvature scalar saturates at $\rho = \ell_s$

$$R \sim \frac{1}{R_V \ell_s}$$

Regulation of the saddle singularity

- Bekenstein-Hawking result for action seems reasonable, but a curvature singularity exists at the horizon.

- Despite the singularity, the result could be reliable provided the corrections are small, EITHER because

1. Curvature corrections might regularise the saddle without significantly changing the entropy, while remaining consistent with EFT

$$S_{\text{Wald}} \sim \frac{A}{\ell_P^2}(1 + \ell^2 R + \ell^4 R^2 + \ell^4 \Box R + \cdots)$$
$$\sim \frac{A}{\ell_P^2}\left(1 + \frac{\ell^2}{\ell_s R_V} + \frac{\ell^4}{\ell_s^2 R_V^2} + \frac{\ell^4}{\ell_s^3 R_V} \cdots\right)$$

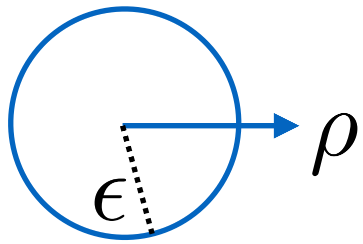
Suppose curvature scalar saturates at $\rho = \ell_s$

$$R \sim \frac{1}{R_V \ell_s}$$

OR

2. non-EFT UV completion exists but does not significantly change the entropy, because It is only relevant in a small region surrounding the horizon.

$$I \sim \int d^D x \sqrt{g} R \sim \int_0^\epsilon \frac{1}{\rho R_V} \rho d\rho \sim \epsilon / R_V \ll 1$$



Conclusions

- Partition function of a volume of space = dimension of the quantum gravity Hilbert space of a topological ball with fixed proper volume.

- To leading order in the coupling the Hilbert space dimension is given by the semiclassical entropy corresponding to the boundary of the saddle ball

$$Z[V] = \dim \mathcal{H} \approx \exp(A_V / 4\hbar G)$$

- This reflects the holographic nature of nonperturbative quantum gravity in a generic finite volume of space.

Future directions

- *Higher curvature corrections* deserve further work: do they regularise the curvature singularity at the horizon?

- Constrained partition function might give a nonperturbative rationale for Jacobson's maximal vacuum entanglement hypothesis,
  i.e. that entanglement entropy in small balls *at fixed volume* is maximized in the semiclassical, gravitationally dressed vacuum state.

- Different constraints? E.g. fixing spacetime volume gives a smooth saddle.