Black holes in AdS/CFT 2

Seok Kim
(Seoul National University)

Mandelstam School & Workshop
Jan 11, 2024

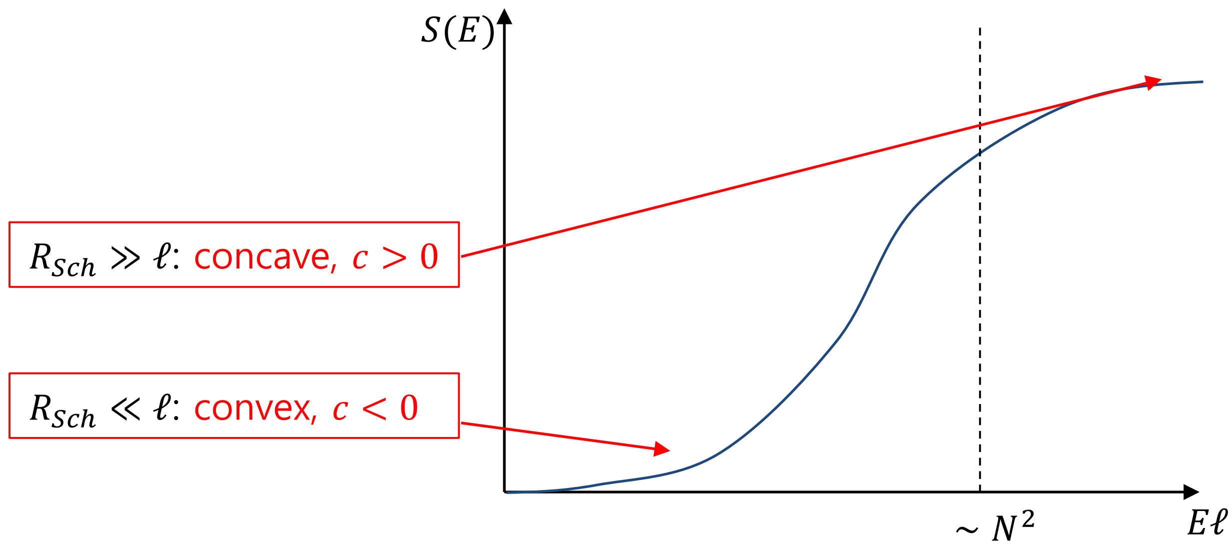
Recap of lecture 1

We discussed the followings:

- Asymptotically flat black holes exhibit exotic thermal behaviors: many high E d.o.f.

$$S_{BH} \propto M^{(D-2)/(D-3)}$$

- Quantum gravity in AdS: more exotic high T phase of gravity from black holes.



This lecture:

- Microscopic studies from AdS/CFT
- Strongly-coupled QFT problem & supersymmetric models

AdS/CFT & a toy model

For holographic semiclassical gravity to emerge, demand:

- Large N: To realize string theory w/ large d.o.f.
- Strong coupling: Otherwise, just Yang-Mills gauge theory of non-gravitational fields

Start from easier weak coupling limit, $g_{YM} \to 0$, hoping to get qualitative lessons

- U(N) Yang-Mills theory QFT on S^3(r=1) x R: gauge fields & possibly matters as well
- Having CFT & simple AdS/CFT models in mind later,

  "Canonical" couplings to background curvature, like scalars having conformal mass:

$$\mathcal{L} \sim \sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \# R\phi^2 + \cdots\right]$$

  $\rightarrow$ ∃mass gap for all fields & discrete spectrum (compact space)

  All fields in adjoint representations ($N \times N$ matrices)

Want to study the thermal partition function: $Z(r\beta) = Tr[e^{-\beta H}]$

- In free limit, each spherical harmonics mode is a "creation operator" for "free particles"

Weakly-interacting partition function

$Z(\beta)$ in free limit: [Sundborg] [Aharony, Marsano, Minwalla, Papadodimas, Raamsdonk]

- A mode carrying energy E and $U(1)^N \subset U(N)$ charges $(+1_a, -1_b)$ contribute to the trace as $(a, b = 1, \cdots, N)$

$$\frac{1}{1-e^{-\beta E}e^{i(\alpha_a-\alpha_b)}} = \exp\left[\sum_n \frac{1}{n} e^{-n\beta E} e^{in\alpha_{ab}}\right] \quad \text{for bosons}$$

$$1 + \mathrm{e}^{-\beta E}\mathrm{e}^{\mathrm{i}\alpha_{ab}} = \exp\left[\sum_n \frac{(-1)^{n-1}}{n} e^{-n\beta E} e^{in\alpha_{ab}}\right] \quad \text{for fermions} \qquad (\alpha_{ab} \equiv \alpha_a - \alpha_b)$$

- Collect ∞-ly many modes: $\exp[\sum_n \frac{1}{n}(f_B(n\beta) + (-1)^{n-1} f_F(n\beta)) \sum_{a,b} e^{in\alpha_{ab}}]$

  $f_B(\beta) = \sum_{bosons} e^{-\beta E}$ & $f_F(\beta) = \sum_{fermions} e^{-\beta E}$ is the "1-particle" or "letter" partition function

- We temporarily bookkept the charges to project to gauge invariant states.

$$Z(\beta) = \int_0^{2\pi} \frac{d^N\alpha}{(2\pi)^N} \boxed{\prod_{a\neq b}(1 - e^{i\alpha_{ab}})} \exp[\sum_n \frac{1}{n}\left(f_B(n\beta) + (-1)^{n-1} f_F(n\beta)\right) \sum_{a,b} e^{i\alpha_{ab}}]$$

"Haar measure"

Result: integral over a U(N) "unitary" matrix

Deconfinement from matrix model

Large N saddle points:

- “Fluctuations” of the integral suppressed away from the saddle points: $N^2 \sim 1/\hbar$
- Characterized by the distribution of eigenvalues, solving 0d equation of motion.

2-body potential: $\prod_{a\neq b}(1-e^{i\alpha_{ab}}) \exp\left[\sum_n \frac{1}{n}(f_B(n\beta)+(-1)^{n-1}f_F(n\beta))\sum_{a,b} e^{in\alpha_{ab}}\right] \equiv \exp[-\sum_{a,b} V(\alpha_{ab})]$

- Repulsive (vanishes at $\alpha_a = \alpha_b$) vs. Attractive (enhances at $\alpha_a = \alpha_b$).

The repulsive/attractive potentials compete

- Low T: Repulsion dominates. Uniform distribution **(figure)**
- High T: Attraction dominates. Inhomogeneous distribution **(figure)**

There is a phase transition at certain T.

- Confinement-deconfinement transition, proposed as the dual of Hawking-Page transition.
- Why deconfining? : E.g. at infinite T, all α_a’s same since attraction dominates.
- There are further quantitative studies of this transition, which I don’t elaborate on.

Supersymmetric black holes

The results so far qualitatively agree quite well with AdS black hole physics.

- But it is never going to be precise, due to the weak/strong coupling discrepancies.

BPS black holes: We generally expect that certain quantum corrections due to strong coupling are under control.

- Maximal SYM on S^3 x R ↔ IIB strings on AdS_5 x S^5: BPS black holes [Gutowski, Reall] , …
- Carry 5 distinct charges: spins J_1, J_2 in AdS_5, charges Q_1, Q_2, Q_3 in S^5.

mass~energy $E = (Q_1 + Q_2 + Q_3 + J_1 + J_2)/\ell$

- The known solutions carry 4 independent parameters: "charge relation" on solution space

$$Q_1Q_2Q_3 + \frac{N^2}{2}J_1J_2 = \left(\frac{N^2}{2} + Q_1 + Q_2 + Q_3\right)\left(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)\right)$$

Have zero Hawking temperature, so do not radiate.

- However, for equilibrium thermodynamics, Q's and J's play the role of BPS energy, so their chemical potentials in many senses behave like inverse-temperature
- E.g. recall from yesterday's talk that Q's of extremal BH's played the role of mass.

BPS states from index

We consider a "Witten index" version of the thermal partition function.

- Most generally, consider

$$Z = Tr[(-1)^F e^{-\Delta_1 Q_1 - \Delta_2 Q_2 - \Delta_3 Q_3 - \omega_1 J_1 - \omega_2 J_2}]$$

- This function becomes g_{YM}-independent when $\sum_I \Delta_I - \sum_i \omega_i = 0$ (mod $4\pi i$).
- Or, absorb $(-1)^F$ into half-period $2\pi i$ shift (e.g. $(-1)^F = e^{2\pi i J_1}$): $\sum_I \Delta_I - \sum_i \omega_i = 2\pi i$ (mod $4\pi i$)

- Can compute from the free limit: ($\Delta_I = -2\pi i\, \delta_I$, $\omega_1 = -2\pi i \sigma$, $\omega_2 = -2\pi i \tau$, $\alpha_a = 2\pi u_a$)

$$Z(\delta_I, \sigma, \tau) = \frac{1}{N!}\int_0^1 d^N u \frac{\prod_{I=1}^3 \prod_{a,b=1}^N \Gamma(\delta_I + u_a - u_b, \sigma, \tau)}{\prod_{a\neq b}\Gamma(u_a - u_b, \sigma, \tau)} = \exp[\textstyle\sum_n \frac{1}{n} f(n\Delta_I, n\omega_i) \sum_{a,b} e^{in\alpha_{ab}}]$$

$$\Gamma(z, \sigma, \tau) \equiv \prod_{m,n=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{2\pi i((m+1)\sigma + (n+1)\tau)}}{1 - e^{2\pi i z} e^{2\pi i(m\sigma + n\tau)}}$$

$$f(\Delta_I, \omega_i) = -\frac{(1-e^{-\Delta_1})(1-e^{-\Delta_2})(1-e^{-\Delta_3})}{(1-e^{-\omega_1})(1-e^{-\omega_2})}$$

The task: Find the large N saddle point & approximate the integral.

- About this, there is a bit technical aspects, all found in the past 5 years or so.
- I'll skip most of them and explain those which can hopefully be understood more intuitively.

Very large black holes

The intuitions: [Choi, J. Kim, SK, Nahmgoong] (2018)

- "High T limit" $\omega_{1,2} \to 0$ w/ $\sum_I \Delta_I - \sum_i \omega_i = \pm 2\pi i$: Integrand diverges & all α_a's want to be close.

$$\frac{1}{N!}\oint \prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\prod_{a<b}\left(2\sin\frac{\alpha_{ab}}{2}\right)^2 \exp\left[\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\left(1+\sum_{s_1,s_2,s_3=\pm1}\frac{s_1s_2s_3(-1)^{n-1}e^{\frac{ns_I\Delta_I}{2}}}{2\sinh\frac{n\omega_1}{2}\cdot 2\sinh\frac{n\omega_2}{2}}\right)e^{in\alpha_{ab}}\right]$$

$$Z \sim \frac{1}{N!}\oint\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\exp\left[-\frac{1}{\omega_1\omega_2}\sum_{a\neq b}\sum_{s_1,s_2,s_3=\pm1}s_1s_2s_3\mathrm{Li}_3\left(-e^{\frac{s_I\Delta_I}{2}}e^{i\alpha_{ab}}\right)\right] \qquad \mathrm{Li}_3(x)\equiv\sum_{n=1}^{\infty}\frac{x^n}{n^3}$$

- Deconfining free energy in the "Cardy" limit: $\log Z \sim N^2$

$$\mathrm{Li}_3(-e^x)-\mathrm{Li}_3(-e^{-x}) = -\frac{x^3}{6}-\frac{\pi^2 x}{6}$$

$$\log Z \sim -\frac{N^2}{\omega_1\omega_2}\sum_{s_1s_2s_3=+1}\left[\mathrm{Li}_3\left(-e^{\frac{s_I\Delta_I}{2}}\right)-\mathrm{Li}_3\left(-e^{-\frac{s_I\Delta_I}{2}}\right)\right] \longrightarrow \log Z \sim \frac{N^2\Delta_1\Delta_2\Delta_3}{2\omega_1\omega_2}$$

More exact/technical results

Various alternative approaches to compute the large N free energy:

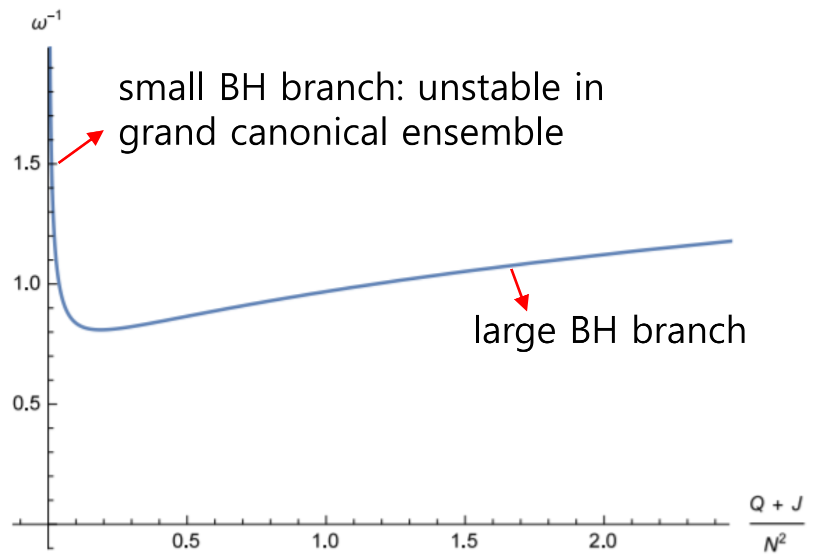
- Large N saddles points: [Choi, SK, Jeong E. Lee] …
- The "Bethe ansatz" approach: [Benini, Milan] …
- Other approaches: [Cabo Bizet, Murthy] [Goldstein, Jejjala, Lei, van Leuven, Li] …

Large N free energy formally continues the Cardy formula: (at $\sum_I \Delta_I - \sum_i \omega_i = \pm 2\pi i$)

$$\log Z \sim \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2\omega_1 \omega_2}$$

Thermodynamic plot

(taking all Q's equal and two J's equal, for simplicity)



Question:

- Very large BH's: gluon plasma
- Small black holes…?

What are the small black holes made of?

To better understand small BH's, helpful to know "giant gravitons" in $AdS_5 \times S^5$:

- D3-branes expanded in $S^5 \subset AdS_5 \times S^5$, carrying charges Q_1, Q_2, Q_3.
- ∃moduli [Mikhailov] of its shape: special limit → intersecting "maximal" giants

  Viewing $S^5 \subset C^3$ as $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, special solutions are $(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3} = 0$.

- Roughly speaking, each maximal giant is a "baryon-like" object in the dual CFT

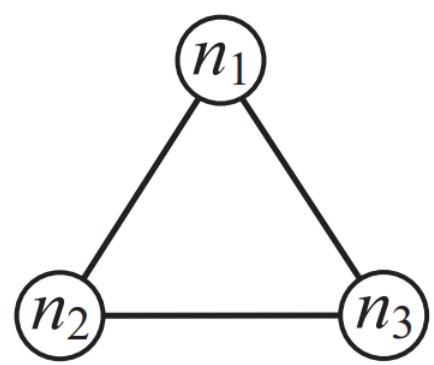
  For instance, $(z_3)^{n_3} = 0 \quad \leftrightarrow \quad [\det(Z)]^{n_3}$ operator

The N=4 index can be expanded in the giant graviton numbers n_1, n_2, n_3.

[Imamura] [Gaiotto, Lee] (2021), [Murthy] [Lee] (2022)

$$Z(\Delta_I, \omega_i) = Z_{\rm KK}(\Delta_I, \omega_i) \sum_{n_1,n_2,n_3=0}^{\infty} e^{-N\sum_{I=1}^{3} n_I \Delta_I} Z_{n_1,n_2,n_3}(\Delta_I, \omega_i)$$

matrix integral of a $U(n_1) \times U(n_2) \times U(n_3)$ quiver, consisting of the 4d/2d fields on giant gravitons & their intersections



Large n_1, n_2, n_3 $(\sim N)$ saddle points of Z_{n_1,n_2,n_3}…?

Giant graviton saddles

Strategy: Compute entropy $S(Q_I + \frac{J_1+J_2}{2}, J_1 - J_2; n_I)$ of each Z_{n_1,n_2,n_3} at fixed giant graviton numbers. Then maximize it in n_I's. (Assumed the dominance of single term Z_{n_1,n_2,n_3}.)

$$Z(\Delta_I, \omega_i) = Z_{\text{KK}}(\Delta_I, \omega_i) \sum_{n_1,n_2,n_3=0}^{\infty} e^{-N\sum_{I=1}^{3} n_I \Delta_I} Z_{n_1,n_2,n_3}(\Delta_I, \omega_i)$$

Expectations:

- Small BH's ($Q \ll N^2$): Expect giant gravitons are good variables. (Yet confining)
- larger BH's ($Q \gg N^2$): Distinction between micro-/grand-canonical ensembles ignorable due to small thermal fluctuations. → Expect the gluon plasma to be the right variable.

Indeed,

$$S(q) = \frac{\pi(2q)^{\frac{3}{2}}}{N} - \frac{9\pi q^{\frac{5}{2}}}{\sqrt{2}N^3} + \frac{351\pi q^{\frac{7}{2}}}{8\sqrt{2}N^5} - \frac{8937\pi q^{\frac{9}{2}}}{32\sqrt{2}N^7} + \frac{1048059\pi q^{\frac{11}{2}}}{512\sqrt{2}N^9} + \cdots \quad \text{for } q \ll N^2$$

$$S_{\text{BH}}(q) = \frac{\pi(2q)^{\frac{3}{2}}}{N} - \frac{21\pi q^{\frac{5}{2}}}{\sqrt{2}N^3} + \frac{1287\pi q^{\frac{7}{2}}}{8\sqrt{2}N^5} - \frac{46189\pi q^{\frac{9}{2}}}{32\sqrt{2}N^7} + \frac{7243275\pi q^{\frac{11}{2}}}{512\sqrt{2}N^9} + \cdots \quad \text{for } q \ll N^2$$

- Yields the correct entropy in the very small BH limit, but otherwise over-estimates.
- We can view (very) small AdS black holes as bound states of D-branes & open strings. (Similar to the asymptotically flat black holes. [Strominger, Vafa])

Exotic states of matter

Recall the connection: hot black holes ↔ strongly interacting quark-gluon plasma

More generally, black holes in holography probe various interesting states of matter.

- For instance, the "no-hair theorem" of black holes which was a cornerstone of my talk today can be interestingly violated to represent the rich structures of matter.

Thermodynamic interpretations:

- "no hair": Most information is not accessible, except a few macroscopic ones.
- "hair": With phase transitions, there may be extra parameters labelling the new phases

A picture (oversimplified): "black hole hairs ↔ order parameters of novel phases"

- E.g. this picture is beautifully realized with black branes as holographic superconductor / superfluid [Gubser] (2008) …
- charged hairs outside the horizon ~ order parameter of U(1) symmetry breaking

Hairy black holes in AdS

The subject of “hairy black holes” in AdS have some history.

- Tightly connected with the novel states of matter in the holographically dual QFT.

- Charged hairy black holes ($Q \neq 0$): superconductor/superfluid & their variants

  [Basu, Bhattacharya, Bhattacharyya, Loganayagam, Minwalla] (2010)

  [Bhattacharyya, Minwalla, Papadodimas] (2010) ……

- Rotating hairy black holes ($J \neq 0$):

  [SK, Kundu, E. Lee, J. Lee, Minwalla, Patel] (2023)

- Has generalizations to hairy BPS black holes

  [Markeviciute, Santos] (2018) [Jain, SK, Krishna, E. Lee, Minwalla, Patel] in progress

Better holographic understandings of these general black objects in AdS/CFT…?

Lessons from black holes

Asymptotically flat black holes (or small BH's in AdS)

- Black hole entropy grows fast & suggests a large # of states of at high E.
- Quantum gravity is more delicate than naive particle physics of quantizing general relativity.
- In my mind, this makes string theory a more natural description of quantum gravity.

Large AdS BH's & novel phase of quantum gravity

- Quantum gravity may undergo a phase transition at high T.
- Objects in traditional description of gravity (gravitons, strings, D-branes, …) lose meanings.
- May shed lights on how our universe should behave in extreme conditions.

Hairy BH's & novel states of matter

- Black holes can probe interesting phases of matter.
- Sometimes by violating the celebrated no-hair theorem.
- Recent explorations on hairy charged and/or rotating black holes.