Finite N black hole cohomologies

Seok Kim
(Seoul National University)

Maldelstam School & Workshop
Jan 12, 2024

Talk based on various collaborations with

- Jaehyeok Choi (SNU) [1)]
- Sunjin Choi (KIAS) [2)]
- Eunwoo Lee (SNU) [3)]
- Jehyun Lee (SNU) [4)]
- Siyul Lee (Univ. of Michigan) [5)]
- Jaemo Park (Postech) [6)]

"The shape of non-graviton operators for SU(2)" arXiv:2209.12696. [2), 3), 6)]
"Towards quantum black hole microstates" arXiv.2304.10155. [2), 3), 5), 6)]
"Finite N black hole cohomologies" arXiv:2312.16443. [1), 2), 4), 5)]

See also

- Chi-Ming Chang, Ying-Hsuan Lin,
  "Words to describe a black hole" arXiv:2209.06728.
- Budzik, Gaiotto, Kulp, Williams, Wu, Yu,
  "Semi-chiral operators in 4d N=1 gauge theories" arXiv:2306.01039.
- Chang, Feng, Lin, Tao,
  "Decoding stringy near-supersymmetric black holes" arXiv:2306.04673.
- Budzik, Murali, Vieira,
  "Following black hole states" arXiv:2306.04693.

Introduction

General question: Better understand black hole microstates.

- $S_{BH} = A/4G = \log(\#\text{states w/ given energy, charges})$

  Counted it from the quantum systems of black holes. [Strominger, Vafa] (1996), …

- Construct & better characterize the individual microstates?

Concrete question: Black hole microstates in AdS/CFT

- Provides a definition of quantum gravity (in principle)
- But requires strong coupling QFT calculations: hard in general
- BPS black holes: SUSY helps, but still hard to construct exact eigenstates.

I will explain a modest version of "constructing" BPS black hole microstates.

- 4d SU(N) maximal super-Yang-Mills: From classical (weak-coupling) cohomologies.
- Want to eventually study $SU(N \gg 1)$. ↔ Today I will report SU(2), SU(3), SU(4).
- Qualitative features of these states & roughly "compare" w/ black hole physics.

Maximal SYM & BPS operators

SU(N) MSYM on R^4: fields in adjoint rep. (written in N=1 language)

3 chiral multiplets: ϕ_m , $\bar{\phi}^m$ and $\psi_{m\alpha}$, $\bar{\psi}^m_{\dot{\alpha}}$ $(m = 1,2,3)$

vector multiplet: $A_\mu \sim A_{\alpha\dot{\beta}}$ and λ_α , $\bar{\lambda}_{\dot{\alpha}}$ $(\mu = 1, \cdots, 4)$ $(\alpha = \pm, \dot{\alpha} = \dot{\pm})$

- 32 supercharges: Q^i_α , $\bar{Q}_{i\dot{\alpha}}$ & $S^\alpha_i = \left(Q^i_\alpha\right)^\dagger$, $\bar{S}^{i\dot{\alpha}} = (\bar{Q}_{i\dot{\alpha}})^\dagger$ $(i = 1, \cdots, 4)$
- Operator-state map for CFT: {local operators on R^4} ↔ {states on S^3 x R}

Gauge-invariant local BPS operators: (at $x^\mu = 0$ on R^4)

- Pick $Q \equiv Q^4_-$, $S \equiv S^-_4 = Q^\dagger$: invariant operators $[Q, O(0)\} = [Q^\dagger, O(0)\} = 0$.

- Free ($g_{YM} \to 0$): Trivially constructed with invariant fields under $Q, Q^\dagger$:

$\bar{\phi}^m$, ψ_{m+}, $\bar{\lambda}_{\dot{\alpha}}$, $f_{++} \equiv F_{1+i2\,,\,3+i4}$ & derivatives $\partial_{1+i2} \equiv \partial_1 - i\partial_2$, $\partial_{3+i4} \equiv \partial_3 - i\partial_4$ acting on them

→ Too many states: more than BH entropy & more than low E gravitons.

- Many of them acquire anomalous dimensions when $g_{YM} \neq 0$: At $g_{YM} \ll 1$,

$Q\,\bar{\phi}^m = 0$, $Q\psi_{m+} \sim g_{YM}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]$, $Qf_{++} \sim g_{YM}\sum_m\,[\psi_{m+}, \bar{\phi}^m]$, $Q\,\bar{\lambda}_{\dot{\alpha}} = 0$, $[Q, D_{+\dot{\alpha}}] \sim g_{YM}[\,\bar{\lambda}_{\dot{\alpha}}\, ,\ \}$

Q & $Q^\dagger \sim O(g^1_{YM})$ → $QQ^\dagger + Q^\dagger Q \sim E - E_{BPS} \geq 0$ at 1-loop, $O(g^2_{YM})$.

The cohomology problem

Cohomology problem

- The supercharges are nilpotent: $Q^2 = 0$, $(Q^\dagger)^2 = 0$
- The equation

$$[Q, O(0)\} = [Q^\dagger, O(0)\} = 0 \leftrightarrow [QQ^\dagger + Q^\dagger Q, O(0)] = 0$$

for the BPS states is formally like harmonic form equation (where $Q \sim d$ and $Q^\dagger \sim d^\dagger$)

- 1-to-1 map of the spectrum: harmonic forms $\leftrightarrow$ Q-cohomology class.

(Local operator $O(0)$ satisfying $QO(0) = 0$, with equivalence $O \sim O + Q\Lambda$.)

1-loop BPS spectrum ↔ classical Q-cohomology class

Weak-coupling (1-loop) vs. strong-coupling BPS spectrum?

- Originally, assumed that the BPS spectrum does not jump. [Minwalla] (2006)
- Perturbative non-renormalization argued (w/ certain assumptions). [Chang, Lin] (2022)
- Index counts these cohomologies & captures black holes. [Cabo Bizet, Cassani, Martelli, Murthy] [Choi, J. Kim, SK, Nahmgoong] [Benini, Milan] (2018) → At least some of them protected.

Gravitons vs. black holes

Two different types of cohomologies:

- "gravitons" vs. "all the rest" (could possibly be "black holes")
- Gravitons in practice (well-defined even at finite N):
  1) Construct single-trace (~single-particle) cohomologies:
     → Chiral primaries $\mathrm{tr}[\bar{\phi}^{(m_1} \cdots \bar{\phi}^{m_n)}]$ & their superconformal descendants in PSU(1,2|3)
  2) Construct multi-trace (~multi-particle) cohomologies by multiplying them.

"Gravitons at finite N" (including N=2,3,4, …)

- #(states) reduces, due to trace relations: E.g. for SU(2), $2tr(X^4) = [tr(X^2)]^2$

  ↔ "stringy exclusion principle" [Maldacena, Strominger] (1998) due to giant gravitons
- This definition reflects all spectral aspects (that I know) of finite-N-corrected supergravitons.

With "all the rest":

- Hopefully wish to study "quantum" black holes in "quantum" gravity (even at N=2,3,4).
- Newton constant, controlling the quantumness, $G_N \sim (\text{radius of AdS})^3/N^2$

The problem & strategies

The problem at finite N:

- Grade operators with a charge, like energy. Or in our studies,

$$j \equiv 6(R+J) = 2(R_1+R_2+R_3)+3(J_1+J_2) \geq 0.$$

- At fixed j, construct all "Q-closed", remove "Q-exact" & remove gravitons: ∃remainders?
- Increase j & repeat: Very painful procedure. Done till $j \leq 25$ for SU(2). [Chang, Lin] (2022)

Streamlined studies [Choi, Eunwoo Lee, Siyul Lee, SK, Park] (2023) :

- Compute the index of finite N gravitons.
- Subtract it from the full index to get the index of the rest, to detect where they exist.
- E.g. the full index $Z(t) = Tr[(-1)^F t^j]$ and $Z_{\rm grav}(t)$ for SU(2) theory:

$$\begin{aligned} Z(t) &= 1+6t^4-6t^5-7t^6+18t^7+6t^8-36t^9+6t^{10}+84t^{11}-80t^{12}-132t^{13}+309t^{14}-18t^{15}-567t^{16} \\ &\quad +516t^{17}+613t^{18}-1392t^{19}-180t^{20}+2884t^{21}-1926t^{22}-4242t^{23}+7890t^{24}+792t^{25}-15876t^{26} \\ &\quad +13804t^{27}+15177t^{28}-37536t^{29}+7049t^{30}+57522t^{31}-58704t^{32}+\cdots \\ Z_{\rm grav}(t) &= 1+6t^4-6t^5-7t^6+18t^7+6t^8-36t^9+6t^{10}+84t^{11}-80t^{12}-132t^{13}+309t^{14}-18t^{15}-567t^{16} \\ &\quad +516t^{17}+613t^{18}-1392t^{19}-180t^{20}+2884t^{21}-1926t^{22}-4242t^{23}+7891t^{24}+786t^{25}-15864t^{26} \\ &\quad +13804t^{27}+15138t^{28}-37476t^{29}+7048t^{30}+57414t^{31}-58566t^{32}+\cdots \end{aligned}$$

$$\boxed{Z - Z_{\rm grav} = -t^{24}+6t^{25}-12t^{26}+0t^{27}+39t^{28}-60t^{29}+t^{30}+108t^{31}-138t^{32}+\cdots}$$

The BMN sector

Calculation within the streamlined strategy is still cumbersome.

- In classical cohomology, ∃consistent truncation of 4d QFT to 1d:

  BMN matrix model [Berenstein, Maldacena, Nastase] (2002) [Nakwoo Kim, Klose, Plefka] (2003)

  (Originally found as a D0-brane theory on a pp-wave backfound)

- The truncation allows us to exclude derivatives & gauginos: $D_{+\dot{\alpha}}$, $\bar{\lambda}_{\dot{\alpha}}$
- Q-algebra closes w/ remaining fields: $Q\bar{\phi}^m = 0$, $Q\psi_{m+} \sim \epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]$, $Qf_{++} \sim [\psi_{m+}, \bar{\phi}^m]$

Truncation still exhibits black hole like entropy growth (if the charge is not too large).

- "Small black hole" regime $j/N^2 \ll 1$: $S_{BMN} = \pi/9\sqrt{j^3/2N^2} < S_{BH} = \pi\sqrt{j^3/27N^2}$
- "Large black hole" regime $j/N^2 \gg 1$: much smaller entropy (derivatives discarded).

SU(2) studied generally beyond BMN, while SU(3) & SU(4) only in BMN sector.

- In this talk, for concise presentation, I will mostly focus on the BMN sector.

SU(2)

The graviton-depleted BMN index is surprisingly simple for SU(2):

$$[Z - Z_{\rm grav}]_{\rm BMN} = -\frac{t^{24}}{1-t^{12}} \cdot \frac{1}{(1-t^8)^3} \cdot (1-t^2)^3$$

"core black hole" primary operators O_n ($n = 0,1,2,\cdots$)

Limited dressings by gravitons $tr(2\bar{\phi}^m f + \epsilon^{mnp}\psi_n\psi_p)$ (only 3 out of 17 gravitons in BMN sector)

SU(1|3) descendants within BMN

Representatives of the cohomologies accounting for $-t^{24} - t^{36} - t^{48} - t^{60} - \cdots$:

$$\begin{aligned} O_n \ = \ & (f\cdot f)^n \epsilon^{c_1c_2c_3}(\phi^a\cdot\psi_{c_1})(\phi^b\cdot\psi_{c_2})(\psi_a\cdot\psi_b\times\psi_{c_3}) \\ & +n(f\cdot f)^{n-1}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f\cdot\psi_{b_1})(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3}) \\ & -\left(\frac{n}{72}+\frac{n(n-1)}{108}\right)(f\cdot f)^{n-1}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3}) \end{aligned}$$

[Used 3d vector notation for SU(2) adjoints: $A\cdot B \sim \mathrm{tr}(AB)$ and $A\times B \sim [A,B]$.]

- The threshold non-graviton at t^{24} order: $O_0 = \epsilon^{p_1p_2p_3}(\phi^m\cdot\psi_{p_1})(\phi^n\cdot\psi_{p_2})(\psi_m\cdot\psi_n\times\psi_{p_3})$

How did we find them?

- For $n = 0, 1$: "computer + guesses" to find "nice" representatives. Then analytic proofs.
- For $n \geq 2$: Carefully watching how $n = 1$ became Q-closed, could generalize to infinite tower.

Finite N "black holes" ?

Are they "black hole states" in the "most quantum" AdS/CFT?

- Beyond semiclassical regime, unclear to what extent they behave like black holes, if any.
- For instance, $S = \log 1 = 0$ at the threshold. Not like semi-classical black holes.

- Here, our attitude is rather phenomenological/observational.
- Just observe some novel spectral properties reminiscent of black holes.

A basic property of black holes: no-hair theorem.

- Stationary black holes don't want hairs outside horizon, but rather absorb them.

The theorem is often violated, especially in AdS (w/ certain matters, charge, spin…).

- Especially in interesting phases of holographic matters, like superconductor (for the modes dual to the order parameters). [Gubser] (2008) …
- However, still true that black holes don't want to be dressed by many (often most) of the surrounding matter modes.

A no-hair theorem?

To appreciate the last point at SU(2), recall the BMN index:

- Q satisfies Leibniz rule → The product (BH) x (graviton) is another cohomology.
- 17 different species of graviton particles (single trace ops.) in SU(2) BMN sector.
- But "black hole operators" O_n abhor dressings by all but 3 gravitons: $tr\left(2\bar{\phi}^m f + \epsilon^{mnp}\psi_n\psi_p\right)$.

$$[Z - Z_{\text{grav}}]_{\text{BMN}} = \boxed{-\frac{t^{24}}{1-t^{12}}} \cdot \boxed{\frac{1}{(1-t^8)^3}} \cdot (1-t^2)^3$$

- So there is in a sense "partial" no-hair phenomenon in this index.

This feature continues in the general SU(2) index, beyond BMN sector

- Checked till t^{40} order. Indeed the index indicates a partial no-hair behavior

  Conformal primaries of gravitons: 29 of 32 don't dress O_0 (at least invisible in the index).

  Conformal descendants: Index admits the possibilities of these graviton hairs

Possibility: All Q-exact (absent) when index does not see them.

← Checked explicitly for several product operators (next slide).

Checks: Q-exactness

t^{28}:

$$
\begin{aligned}
O_0(\bar\phi^{(m}\cdot\bar\phi^{n)}) = -\frac{1}{14}Q[&20\epsilon^{rs(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^{p}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})\\
&-20\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})\\
&+30\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{s+})(f_{++}\cdot\psi_{q+})\\
&-7\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^{q}\cdot\psi_{q+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+18\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{q+})(\bar\phi^{q}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]
\end{aligned}
$$

t^{29}:

$$
\begin{aligned}
O_0(\bar\phi^{m}\cdot\bar\lambda_{\dot\alpha}) = \frac{1}{8}Q[&40\epsilon^{mnp}(f_{++}\cdot\psi_{q+})(\bar\lambda_{\dot\alpha}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{n+})(\bar\phi^{r}\cdot\psi_{p+})\\
&-4\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^{p}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+6\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{p+})(\bar\phi^{p}\cdot\psi_{n+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+\epsilon^{na_1a_2}\epsilon^{pb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^{m}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]
\end{aligned}
$$

t^{30}:

$$
\begin{aligned}
&O_0\left(\bar\phi^{m}\cdot\psi_{n+}-\tfrac{1}{3}\delta^m_n\bar\phi^{p}\cdot\psi_{p+}\right)\\
&=\frac{1}{4}Q\left[\epsilon_{npq}\epsilon^{ra_1a_2}\epsilon^{qb_1b_2}\epsilon^{mc_1c_2}(\bar\phi^{p}\cdot\psi_{r+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})(\psi_{c_1+}\cdot\psi_{c_2+})\right]
\end{aligned}
$$

They lift to non-BPS operators: Non-BPS/near-BPS hairs may exist.

[Bhattacharyya, Minwalla, Papadodimas] [Markeviciute, Santos]

SU(∞) & black hole hairs

$N = \infty$ from BPS black hole solutions in $AdS_5 \times S^5$.

- For simplicity, set $R_1 = R_2 = R_3 \equiv R$, $J_1 = J_2 \equiv J$. [Gutowski, Reall] (2004)
- 1-parameter solution exists: the "size" parameter q & charges $R(q)$, $J(q)$.

Scalar hair: Φ dual to operator $tr(X^2 + Y^2 + Z^2)$ (& its conformal descendants)

- SU(2) QFT implied no-hair for this operator in QFT (in s-wave)
- $N = \infty$: Can we turn on small hair, $\Phi(x) \sim \varepsilon \ll 1$?

- Linearized BPS solutions in the black hole background: (x: radius & θ, ϕ, ψ: 3-sphere)

"size" parameter of the black hole

$$\Phi(x,\theta,\phi,\psi) = \varepsilon x^{\frac{m-2q/\ell^2}{1+3q/\ell^2}} \left(1 + \frac{3q}{\ell^2} + \frac{x}{\ell^2}\right)^{-\frac{1+m+q/\ell^2}{1+3q/\ell^2}} (\cos\tfrac{\theta}{2} e^{i\phi_1})^{m_1} (\sin\tfrac{\theta}{2} e^{i\phi_2})^{m_2}$$

$$m_1 + m_2 = 2m \quad m_1, m_2 = 0, 1, 2, \cdots$$

- Singular at horizon $x = 0$ for $m < 2q/\ell^2$. (Includes "s-wave" ↔ conformal primary at $m = 0$.)
- Regular hair for highly-spinning descendants: similar between $N = \infty$ & $N = 2$.

SU(3) index

$Z - Z_{\text{grav}} \equiv f(t) \cdot 1/(1-t^8)^3 \cdot (1-t^2)^3$ has rich structures, even in BMN.

descendants

limited graviton hair $tr(2\bar{\phi}^m f + \epsilon^{mnp}\psi_n\psi_p)$

Roughly, "core black hole" primaries

- Numerical results till t^{54}.
- Coefficients of $f(t)$ better arranged in a table, grouped in various towers.

$[m, n]$: irreps of $SU(3) \subset SO(6)_R$

j	F_0	F_1	F_2	F_3	F_4	F_{exc}	B_1	B_2	B_3	B_{exc}
24	[0, 0]									
26										
28										
30	[0, 0]	[3, 0]								
32		[4, 0]								
34		[5, 0]					[3, 1]			
36	[0, 0]	[6, 0]					[4, 1]			[3, 0]
38		[7, 0]				[1, 0]	[5, 1]			
40		[8, 0]	[5, 0]		[3, 1]		[6, 1]			
42	[0, 0]	[9, 0]	[6, 0]		[4, 1]		[7, 1]			[1, 1]
44		[10, 0]	[7, 0]		[5, 1]		[8, 1]	[5, 1]		
46		[11, 0]	[8, 0]		[6, 1]	[2, 0]	[9, 1]	[6, 1]		[5, 0]
48	[0, 0]	[12, 0]	[9, 0]		[7, 1]	[3, 0]	[10, 1]	[7, 1]		[4, 1]
50		[13, 0]	[10, 0]	[7, 0]	[8, 1]		[11, 1]	[8, 1]		[4, 0]
52		[14,0]	[11,0]	[8,0]	[9,1]	[2,0]	[12, 1]	[9,1]		[3,1]
54		[15,0]	[12,0]	[9,0]	[10,1]	[4,1]	[13, 1]	[10,1]	[7,1]	

Features:

- Many graviton hairs absent.
- Unlike SU(2), many nontrivial towers of primaries. Mostly R-symmetry towers.

  $\leftrightarrow$ Relation to the giant graviton expansion? [Imamura] [Gaiotto, Lee] [Murthy] [Lee]
- Would these towers last forever or disappear beyond certain charges?

SU(3) threshold cohomology

The threshold cohomology at $j = 24$:

$$
\begin{aligned}
&288 v^{(j}{}_a v^{k)a}{}_i \times \epsilon_{c_1 c_2 (j} \mathrm{tr}\left(\bar\phi^{c_1}\bar\phi^{c_2}\bar\phi^{i}\psi_{k+)}\right) - 72 v^a{}_b v^{bk}{}_a \times \epsilon_{c_1 c_2 (k} \mathrm{tr}\left(\bar\phi^{c_1}\bar\phi^{c_2}\bar\phi^{d}\psi_{d+)}\right) \\
&+ 36\epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} \times \Big[2\mathrm{tr}\left(\phi^{(i}\phi^{c}\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\mathrm{tr}\left(\phi^{(i|}\phi^{c}\phi^{|j)}\psi_{(c}\psi_{k)}\right) \\
&\qquad\qquad +9\mathrm{tr}\left(\phi^{(i}\phi^{j}\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^{c}\psi_{k)}\right)\Big] \\
&- 9\epsilon_{a_1 a_2 j} u^{a_1 b} v^{a_2}{}_b \times \Big[2\mathrm{tr}\left(\phi^{(j}\phi^{c}\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\mathrm{tr}\left(\phi^{(j|}\phi^{c}\phi^{|d)}\psi_{(c}\psi_{d)}\right) \\
&\qquad\qquad +9\mathrm{tr}\left(\phi^{(j}\phi^{d}\psi_{(c}\phi^{c)}\psi_{d)}\right) - 6\mathrm{tr}\left(\phi^{(j}\phi^{d)}\psi_{(c}\phi^{c}\psi_{d)}\right)\Big] \\
&- 20 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 a_3} \left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right] \\
&- 36 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 (i} \left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&- 36 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 (i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&- 36 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 (i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right] \\
&- 36 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 (i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right] \\
&+ 60 u^{a(i} v^{j)}{}_a \times \epsilon_{a_1 a_2 (i} \left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right)\right]
\end{aligned}
$$

There is a long story on how we found this, combining

- Novel "ansatz" to construct operators which become Q-closed only after applying trace relations (so that it does not fall into graviton type)
- A numerics-assisted test of (non-)Q-exactness, applying all possible trace relations
- See [Jaehyeok Choi, Sunjin Choi, SK, Jehyun Lee, Siyul Lee] (2023)

Conclusion

I explained a kind of "construction" program of individual BH microstates.

- Some weak-coupling cohomologies constructed for SU(2), SU(3).
- Higher $SU(N)$? Higher charges? SU(4) non-graviton index in BMN sector

$$Z - Z_{grav} = -[2,0]t^{28} + [1,1]t^{30} + \cdots$$

- Thresholds for $j = 6(R+J) = 24, 24, 28, \cdots$ for $N = 2, 3, 4, \cdots \rightarrow$ Thresholds grows in N.

More physics?

- Information theoretic difference between graviton/black hole states? [Budzik, Murali, Vieira]
- We saw a hint of partial "no-hair theorem" already. Continues to higher N?

When can we reach large N? Need more computational breakthroughs.

- But important signals (like no-hair, towers) may be observed at reasonable N.
- More "approximation" techniques at large N? "BPS-ness" is an exact property so that $1/N$ corrections ignored might affect the conclusion.
- One can still study "approximate" BPS states. They may also shed lights on the "near-BPS" states, recently studied through JT gravity. [Boruch, Heydeman, Iliesu, Turiaci] (2022)