Marginal deformations and quasi-Hopf algebras

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Based on arXiv:1602.08061 (with H. Dlamini) and work to appear
Outline

• Discuss symmetries and integrability of the exactly marginal deformations of $\mathcal{N} = 4$ SYM

• Explain why it is useful to generalise from Lie algebras to (quasi-) Hopf algebras

• Define these algebras using a Drinfeld twist

• Use the twist to define a star product on the representation space of the algebra

• Relate deformed and undeformed gauge theory quantities using the star product
\[ \mathcal{N} = 4 \text{ Super–Yang–Mills} \]

- Fundamental role in the study of gauge and string theory
- Unique four–dimensional gauge theory with maximal global supersymmetry (16 supercharges)
- Many interesting properties:
  - Perturbative Finiteness
  - Nonperturbative Finiteness (4d SCFT)
  - AdS/CFT Correspondence
  - Planar Integrability
  - Planar Amplitudes

- Is it the unique theory with these features?
- Are there theories which share only some of these features?
- How does the knowledge accumulated for \[ \mathcal{N} = 4 \text{ SYM} \] help to understand more realistic theories?
Marginal Deformations of $\mathcal{N} = 4$ SYM

- Look for theories as close as possible to $\mathcal{N} = 4$ SYM
- Preserve conformal invariance $\Rightarrow$ Marginal Deformations
- Focus on superpotential deformations $\Rightarrow \mathcal{N} = 1$ SUSY
- In $\mathcal{N} = 1$ superspace:

$$
\mathcal{L} = \int d^4 \theta \text{Tr} e^{-gV} \Phi^i i \Phi^i + \left( \int d^2 \theta \mathcal{W} + \int d^2 \bar{\theta} \bar{\mathcal{W}} \right) + \cdots
$$

- Chiral Superfields $\Phi^i = \phi^i + \theta^\alpha \psi^i_\alpha + \theta^2 F^i, \; i = 1, 2, 3$
- $\mathcal{N} = 4$ superpotential:

$$
\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k
$$

- Most general classically marginal deformation:

$$
\delta \mathcal{W} = h_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k,
$$

where $h_{ijk}$ is a symmetric tensor
Exactly Marginal Deformations

- Which of these deformations are exactly marginal?
- Perturbative approaches [Parkes, West ’84],[Jones,Mezincescu ’84]
- Non–perturbative proof by Leigh and Strassler ('95) using the NSVZ $\beta$ function
- The Leigh–Strassler superpotential:
  \[
  \mathcal{W}_{LS} = \kappa \text{Tr} \left( \Phi^1[\Phi^2, \Phi^3]_q + \frac{h}{3} \left( (\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)
  \]
- $q$–commutator $[X, Y]_q = XY - q YX$
- Recover $\mathcal{N} = 4$ SYM for $q = 1, h = 0, \kappa = g$
- Finite if $f(g, \kappa, q, h) = 0$, where $f$ unknown in general
- 1–loop finiteness condition
  \[
  2g^2 = \kappa \bar{\kappa} \left[ \frac{2}{N^2} (1 + q)(1 + \bar{q}) + \left( 1 - \frac{4}{N^2} \right) \left( 1 + q\bar{q} + h\bar{h} \right) \right]
  \]
- An interesting case: $q = e^{i \beta}, h = 0$ ("Real $\beta$ deformation")
$\mathcal{N} = 4$ SYM vs. Leigh–Strassler?

- How do the LS theories compare with $\mathcal{N} = 4$ SYM?

<table>
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<th>$\mathcal{N} = 4$ SYM</th>
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<td>Planar Integrability</td>
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- What makes the real $\beta$ deformation so special?
- Take a closer look at the symmetries
- Work at the level of the classical lagrangian
Symmetries: $\mathcal{N} = 4$ SYM

- 4d Superconformal group: $\text{PSU}(2, 2|4)$
- Focus on the $R$–symmetry subgroup $\text{SU}(4) \sim \text{SO}(6)$
- In $\mathcal{N} = 1$ superspace notation, the $\mathcal{N} = 4$ theory has manifest $\text{SU}(3) \times \text{U}(1)_R$ symmetry

$$\mathcal{W} = g \text{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \text{Tr} \Phi^i \Phi^j \Phi^k$$

- $\epsilon_{ijk}$ is the invariant tensor of $\text{SU}(3)$

$$\epsilon_{ijk} U^i_m U^j_n = (\det U) \epsilon_{lmn} = \epsilon_{lmn}$$

- Transforming $\Phi^i \rightarrow U^i_j \Phi^j$ leaves the superpotential invariant
Symmetries: Leigh–Strassler

- The generic LS deformation breaks $SU(3)$ to a discrete subgroup

$$\mathcal{W}_{LS} = \kappa \text{Tr} \left( \Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} ((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3) \right)$$

- This superpotential has the following $\mathbb{Z}_3$ symmetries:

$$\mathbb{Z}_3^A : \Phi^1 \rightarrow \Phi^2 , \quad \Phi^2 \rightarrow \Phi^3 , \quad \Phi^3 \rightarrow \Phi^1$$

$$\mathbb{Z}_3^B : \Phi^1 \rightarrow \omega \Phi^1 , \quad \Phi^2 \rightarrow \omega^2 \Phi^2 , \quad \Phi^3 \rightarrow \Phi^3 \quad (\omega^3 = 1)$$

- Together with a third $\mathbb{Z}_3$ within $U(1)_R (\Phi^i \rightarrow \omega \Phi^i)$, they form a trihedral group known as $\Delta_{27}$ [Aharony et al. '02]

- For real $\beta$ the symmetry group is enhanced to $U(1)^3$

- Is this all?
**q-deforming SU(3)**

- In 0811.3755 (with T. Månsson) it was claimed that the global SU(3) is not actually broken
- Rather, it is *deformed* to a Hopf algebra.

\[
\Phi^i \rightarrow T^i_j \Phi^j \quad \text{where} \quad T = \begin{pmatrix}
t^1_1 & t^1_2 & t^1_3 \\
t^2_1 & t^2_2 & t^2_3 \\
t^3_1 & t^3_2 & t^3_3
\end{pmatrix}
\]

is a symmetry if the components of \( T \) satisfy:

1. \( t^a_c t^{a+1}_c - q t^{a+1}_c t^a_c + h t^{a-1}_c t^{a-1}_c = h \left( t^a_{c+1} t^{a+1}_{c-1} - \bar{q} t^a_{c-1} t^{a+1}_{c+1} + \bar{h} t^a_c t^{a+1}_c \right) \)
2. \( q[t^{a+1}_{c+1}, t^a_c] = -q^2 t^{a+1}_c t^a_{c+1} + h q t^{a-1}_c t^{a-1}_{c+1} + h t^{a-1}_c t^{a-1}_{c+1} + t^a_{c+1} t^{a+1}_c \)
3. \( -q t^{a+1}_c t^a_{c+1} + \bar{q} t^a_{c+1} t^{a+1}_c = \bar{h} t^a_{c-1} t^{a+1}_{c-1} - h t^{a-1}_c t^{a-1}_{c+1} \)
4. \( h(t^{a+1}_{c+1} t^a_{c-1} - \bar{q} t^a_{c-1} t^{a+1}_c) = \bar{h}(t^{a+1}_c t^{a-1}_c - q t^{a-1}_c t^{a+1}_c) \)
Quantum Algebras

- Recall an **algebra** $C$ is a vector space together with a product $\cdot : C \otimes C \to C$ and a unit map $\eta : k \to C$
- A **coalgebra** $C$ is instead equipped with a coproduct $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to k$

\[
\begin{array}{ccc}
C \otimes C & \overset{\Delta \otimes \text{Id}}{\longrightarrow} & C \otimes C \\
\downarrow \text{Id} \otimes \Delta & & \downarrow \Delta \\
C \otimes C & \overset{\Delta}{\longrightarrow} & C
\end{array}
\quad
\begin{array}{ccc}
C \otimes C & \overset{\epsilon \otimes \text{Id}}{\longrightarrow} & C \\
\downarrow \Delta & & \downarrow \text{Id} \otimes \epsilon \\
k \otimes C = C = C \otimes k
\end{array}
\]

- A **bialgebra** is both an algebra and a coalgebra in a compatible way
- A **Hopf Algebra** is a bialgebra equipped with an antipode $S : C \to C$

\[
\eta \circ \epsilon = (S \otimes \text{id}) \circ \Delta = (\text{id} \otimes S) \circ \Delta.
\]
The coproduct

• Lie algebras have a trivial coproduct

\[ \Delta(X) = X \otimes 1 + 1 \otimes X \]

• (Recall from QM:

\[ \Delta(\vec{S})(|\psi_1\rangle \otimes |\psi_2\rangle) = (\vec{S} |\psi_1\rangle) \otimes |\psi_2\rangle + |\psi_1\rangle \otimes (\vec{S} |\psi_2\rangle) \]

• Commutative: \( \tau \circ \Delta(X) = \Delta(X) \)

• Non-commutative co-product (and product): Hopf Algebra
The $R$-matrix

- One can construct Hopf algebras using the RTT relations
  [Faddeev, Reshetikhin, Takhtajan]

\[ R_{a b}^{i k} t_{j}^{a} t_{l}^{b} = t_{b}^{k} t_{a}^{i} R_{j l}^{a b}, \]

where $R : C \otimes C \to C \otimes C$ is called an $R$-matrix.

- Quantum Yang–Baxter Equation (QYBE): Quasitriangular Hopf Algebra

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad \left( R_{s r}^{i j} R_{l p}^{s k} R_{m n}^{r p} = R_{s p}^{i j} R_{r n}^{k l} R_{l m}^{r s} \right) \]

- YBE guarantees the resulting algebra is not too trivial

- $R$ controls $\Delta$ non-commutativity: $\tau \circ \Delta(h) = R(\Delta(h)) R^{-1}$
Drinfeld Twists

- Given a Hopf algebra, can produce a new one through a process of twisting
- A Drinfeld twist $F : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ acts on two copies of the algebra
- Twisted coproduct
  \[ \tilde{\Delta}(a) = F \Delta(a) F^{-1} \]
- Twisted $R$-matrix
  \[ \tilde{R} = F_{21} RF_{12}^{-1} \]
- Not a similarity transformation!
- To ensure that the twisted algebra is a Hopf algebra, $F$ needs to satisfy the cocycle condition
  \[ (F \otimes 1)(\Delta \otimes \text{id})(F) = (\text{id} \otimes F)(\text{id} \otimes \Delta)(F) \]
Quasi-Hopf algebras

- Can consider more general twists, not satisfying the cocycle condition
- $R$-matrix does not satisfy YBE $\Rightarrow$ Non-associative algebra
- Structure is that of a quasi-Hopf algebra [Drinfel'd '89]
- Non-associativity controlled by coassociator:

$$ (\text{id} \otimes \Delta)(\Delta(a)) = \Phi^{-1}(\Delta \otimes \text{id})(\Delta(a))\Phi $$

- quasi-Hopf QYBE:

$$ R_{12} \Phi_{312} R_{13} \Phi^{-1}_{132} R_{23} \Phi_{123} = \Phi_{321} R_{23} \Phi^{-1}_{231} R_{13} \Phi_{213} R_{12} $$

- Coassociator defined through the twisting procedure

$$ \Phi^F = F_{12}(\Delta \otimes \text{id})(F)\Phi(\text{id} \otimes \Delta)(F)^{-1}F_{23}^{-1} $$
\[ R \text{-matrix for the general LS deformation} \]

- Read off 1–loop spin chain Hamiltonian [Roiban '03]

\[
H_{l,l+1} = \frac{1}{2d^2} \begin{pmatrix}
    h \bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}q & 0 \\
    0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\
    0 & 0 & q \bar{q} & 0 & -h \bar{q} & 0 & -\bar{q} & 0 & 0 \\
    0 & -\bar{q} & 0 & q \bar{q} & 0 & 0 & 0 & 0 & -h \bar{q} \\
    0 & 0 & -\bar{h}q & 0 & h \bar{h} & 0 & \bar{h} & 0 & 0 \\
    h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\
    0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\
    -h \bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q \bar{q} & 0 \\
    0 & \bar{h} & 0 & -\bar{h}q & 0 & 0 & 0 & 0 & h \bar{h}
\end{pmatrix}
\]

- Define \( \hat{R}^{ij}_{kl} = \delta^i_k \delta^j_l - H^{ij}_{kl} \implies R^{i j}_{k l} = \hat{R}^{j i}_{k l} \)

\[
R = \frac{1}{2d^2} \begin{pmatrix}
    1 + q \bar{q} - h \bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\
    0 & 2\bar{q} & 0 & 1 - q \bar{q} + h \bar{h} & 0 & 0 & 0 & 0 & 2h \bar{q} \\
    0 & 0 & 2q & 0 & -2h & 0 & q \bar{q} + h \bar{h} - 1 & 0 & 0 \\
    0 & q \bar{q} + h \bar{h} - 1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\
    0 & 0 & 2\bar{h}q & 0 & 1 + q \bar{q} - h \bar{h} & 0 & -2h & 0 & 0 \\
    2h \bar{q} & 0 & 0 & 0 & 0 & 2\bar{h}q & 0 & 2q & 0 \\
    0 & 0 & 1 - q \bar{q} + h \bar{h} & 0 & 2h \bar{q} & 0 & q \bar{q} + h \bar{h} - 1 & 0 & 0 \\
    -2h & 0 & 0 & 0 & 0 & q \bar{q} + h \bar{h} - 1 & 0 & 2q & 0 \\
    0 & -2\bar{h} & 0 & 0 & 2\bar{h}q & 0 & 0 & 0 & 1 + q \bar{q} - h \bar{h}
\end{pmatrix}
\]

\((2d^2 = 1 + q \bar{q} + h \bar{h})\)
General LS deformation

- $R$ does not satisfy YBE in general
- YBE $\iff$ known integrable deformations
- Can we derive $R$ from a Drinfeld twist?
- What does it tell us about the Leigh-Strassler theories?
The general twist

- Found a twist which produces the \((q, h)\) \(R\)-matrix starting from the trivial \(\mathcal{N} = 4\) \(R\)-matrix \(I \otimes I\).

\[
F_{12} = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 & e & 0 & f & 0 \\
  0 & b & 0 & c & 0 & 0 & 0 & 0 & g \\
  0 & 0 & i & 0 & j & 0 & d & 0 & 0 \\
  0 & d & 0 & i & 0 & 0 & 0 & 0 & j \\
  0 & 0 & f & 0 & a & 0 & e & 0 & 0 \\
  g & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\
  0 & 0 & c & 0 & g & 0 & b & 0 & 0 \\
  j & 0 & 0 & 0 & 0 & d & 0 & i & 0 \\
  0 & e & 0 & f & 0 & 0 & 0 & 0 & a
\end{pmatrix}
\]
The general twist

\[ a = \frac{\sqrt{2} \sqrt{q+1} \sqrt{\bar{q}+1}}{\sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} + \frac{(q - 1) (\bar{q} - 1)}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} \]

\[ b = \frac{h \bar{h}}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} + \frac{q^2 \bar{q}^2 - q \bar{q}^2 + q^2 \bar{q} + h \bar{h} q \bar{q} - h h \bar{q} - \bar{q} - 2 q^2 + 3 h \bar{h} q + q + h \bar{h} + 1}{\sqrt{2} \sqrt{q+1} \sqrt{\bar{q}+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ c = \frac{h \bar{h}}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} + \frac{q^2 \bar{q}^2 - q \bar{q}^2 - q^2 \bar{q} + h \bar{h} q \bar{q} + \bar{q} - h \bar{h} q + q + 3 \bar{h} - 1}{\sqrt{2} \sqrt{q+1} \sqrt{\bar{q}+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ d = \frac{h \bar{h}}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} - \frac{q^2 \bar{q}^2 - q \bar{q}^2 - q^2 \bar{q} + 3 h \bar{h} q \bar{q} + h \bar{h} q + \bar{q} + h \bar{h} q + q - 2 \bar{h} - 1}{\sqrt{2} \sqrt{q+1} \sqrt{\bar{q}+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ e = \frac{\bar{h} (q - 1)}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} - \frac{\sqrt{2} \bar{h} \sqrt{q+1} (q - \bar{h} - 1)}{\sqrt{q+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ f = \frac{\bar{h} (q - 1)}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} - \frac{\sqrt{2} \bar{h} \sqrt{\bar{q}+1} (q \bar{q} - \bar{q} + h \bar{h})}{\sqrt{q+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ g = \frac{h (\bar{q} - 1)}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} - \frac{\sqrt{2} h \sqrt{\bar{q}+1} (q \bar{q} - \bar{q} + h \bar{h})}{\sqrt{\bar{q}+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]

\[ i = \frac{h \bar{h}}{q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1} + \frac{q^2 \bar{q}^2 + q \bar{q}^2 - 2 \bar{q}^2 - q^2 \bar{q} + h \bar{h} q \bar{q} + 3 h \bar{h} q + \bar{q} - h \bar{h} q - q + h \bar{h} + 1}{\sqrt{2} \sqrt{q+1} \sqrt{\bar{q}+1} \sqrt{q \bar{q} + h \bar{h} + 1} (q \bar{q} - \bar{q} - q + 2 h \bar{h} + 1)} \]
The general twist

• Factorising twist: \( R = F_{21}F_{12}^{-1} \)
• Triangular: \( F_{21}F_{12} = I \otimes I \)
• Becomes a Hopf twist for known cases \((q = e^{i\beta}, q = 1 + w, h = w)\)
• Does not satisfy cocycle condition in general \(\Rightarrow\) quasi-Hopf algebra?
• To check, construct the coassociator \( \Phi \)

\[
\tilde{\Phi} = F_{23}(1 \otimes \Delta)(F)\Phi(\Delta \otimes 1)(F^{-1})F_{12}^{-1}
\]

• To do this, need to make sense of the coproduct acting on \( F \)
Exponential form

- Only know how to act on algebra elements
- Write the twist in exponential form

\[ F_{q,h} = \exp \left( \alpha_{\beta_r} \tilde{f}_{\beta_r} + \alpha_{\beta_i} \tilde{f}_{\beta_i} + \alpha_{h_r} \tilde{f}_{h_r} + \alpha_{h_i} \tilde{f}_{h_i} \right) \]

- \( f \)'s are classical twists
- \( \alpha \)'s are known functions of \( q, h \)
- Can now compute

\[ \Phi_F = F_{23} \ (\text{id} \otimes \Delta)(F) \ (\Delta \otimes \text{id})(F^{-1}) \ F_{12}^{-1} \]

- Very complicated in general, still trying to simplify
- Quasi-Hopf YBE is satisfied \( \Rightarrow \) quasi-Hopf algebra

\[ R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} = \Phi_{321} R_{23} \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \]
Example: Imaginary $\beta$

- Illustrate for a simple case
- $\bar{q} = q, \bar{h} = h = 0$

$$F = e^\alpha[\lambda^1 \wedge \lambda^2 - \lambda^4 \wedge \lambda^5 + \lambda^6 \wedge \lambda^7]$$

- Here $x \wedge y = x \otimes y - y \otimes x$.

$$(\text{id} \otimes \Delta) F = e^\alpha[\lambda^1 \otimes \Delta(\lambda^2) - \lambda^2 \otimes \Delta(\lambda^1) - \lambda^4 \otimes \Delta(\lambda^5) + \lambda^5 \otimes \Delta(\lambda^4) + \lambda^6 \otimes \Delta(\lambda^7) - \lambda^7 \otimes \Delta(\lambda^6)]$$

$$= \exp [\alpha[\lambda^1 \otimes (\lambda^2 \otimes I + I \otimes \lambda^2) - \lambda^2 \otimes (\lambda^1 \otimes I + I \otimes \lambda^1) - \lambda^4 \otimes (\lambda^5 \otimes I + I \otimes \lambda^5) + \lambda^5 \otimes (\lambda^4 \otimes I + I \otimes \lambda^4) + \lambda^6 \otimes (\lambda^7 \otimes I + I \otimes \lambda^7) - \lambda^7 \otimes (\lambda^6 \otimes I + I \otimes \lambda^6)]]$$

- Similarly for $(\Delta \otimes \text{id}) F$
- Explicitly construct the coassociator
Imaginary $\beta$ coassociator

\[ \Phi_{1111}^{112} = \Phi_{122}^{1112} = \frac{4 \cos \left( \sqrt{2} \alpha \right) - 2 \frac{3}{2} \sin \left( \sqrt{2} \alpha \right) + \left( \sqrt{2} \sin \left( \frac{3}{2} \alpha \right) + 4 \right) q}{2^{\frac{5}{2}} \sqrt{q^2 + 1}} \]

\[ \Phi_{1111}^{113} = \Phi_{133}^{1113} = \frac{4 - \sqrt{2} \sin \left( \frac{3}{2} \alpha \right) + 4 \cos \left( \sqrt{2} \alpha \right) q + 2 \frac{3}{2} \sin \left( \sqrt{2} \alpha \right) q}{2^{\frac{5}{2}} \sqrt{q^2 + 1}} \]

\[ \Phi_{2113}^{132} = \Phi_{312}^{132} = \frac{4 - \sqrt{3} \cos \left( 2 \left( \sqrt{3} - 1 \right) \alpha \right) + 2 \cos \left( 2 \sqrt{3} \alpha \right) + \sqrt{3} \cos \left( 2 \left( 1 + \sqrt{3} \right) \alpha \right) - \frac{12 q}{q^2 + 1}}{12} \]

\[ \Phi_{121}^{112} = \Phi_{311}^{131} = \frac{-2 - 2^{\frac{5}{2}} \sin \left( \sqrt{2} \alpha \right) q + q^2 + \cos \left( \frac{3}{2} \alpha \right) q^2}{4 \left( q^2 + 1 \right)} \]

\[ \Phi_{221}^{122} = \Phi_{331}^{133} = -\frac{\sin^2 \left( \sqrt{2} \alpha \right)}{2} \]

\[ \Phi_{212}^{122} = \Phi_{112}^{121} = \frac{2^{\frac{3}{2}} + \sin \left( \frac{3}{2} \alpha \right) - 2^{\frac{3}{2}} \cos \left( \sqrt{2} \alpha \right) q + 2 \sin \left( \sqrt{2} \alpha \right) q}{4 \sqrt{q^2 + 1}} \]

etc...

with \( \alpha(q) = \arccos \left( \frac{1 + q}{\sqrt{2 \left( q^2 + 1 \right)}} \right) = \frac{q - 1}{2} - \frac{(q - 1)^2}{4} + \ldots \).
What is this good for?

- The twist can be used to define a star product

\[ g \triangleright (x \cdot y) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y). \]

- Hopf algebra acts as derivation

\[ g \triangleright (x \cdot y) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y). \]

- Compatibility of algebra and module product

\[
\begin{align*}
g \triangleright (x \cdot y) &= m(\Delta(g) \triangleright [x \otimes y]) = m([g \otimes 1 + 1 \otimes g] \triangleright [x \otimes y]) \\
&= m((g \triangleright x) \otimes y + x \otimes (g \triangleright y)) = (g \triangleright x) \cdot y + x \cdot (g \triangleright y)
\end{align*}
\]

- Twisting \(\Delta\) also twists the module product

\[
\begin{align*}
m_F(x \otimes y) &= m(F^{-1} \triangleright x \otimes y) = (F_{(1)}^{-1} \triangleright x) \cdot (F_{(2)}^{-1} \triangleright y)
\end{align*}
\]

- Star product

\[
x \star y = m_F(x \otimes y).
\]
What is this good for?

- In index form:

\[ z^i \star z^j = (F^{-1})^i_{\ j\ k} z^k z^j = F^i_{\ j\ k\ l} z^k z^l. \]

- Takes LS expressions to \( \mathcal{N} = 4 \) SYM ones

\[ z^1 \star z^2 - qz^2 \star z^1 + h(z^3)^2 = \frac{\sqrt{1+q} \sqrt{1 + q\bar{q} + h\bar{h}}}{\sqrt{2} \sqrt{1 + \bar{q}}} \left( z^1 z^2 - z^2 z^1 \right) \]

- In gauge theory language:

\[ \Phi^1 \star \Phi^2 - q\Phi^2 \star \Phi^1 + h \Phi^3 \star \Phi^3 = \frac{\sqrt{1+q} \sqrt{1 + q\bar{q} + h\bar{h}}}{\sqrt{2} \sqrt{1 + \bar{q}}} \left[ \Phi^1, \Phi^2 \right] \]
Cubic star product

- Two cubic expressions

\[(z^i \star z^j) \star z^k = [F_{3,L}]^{i,j,k}_{i',j',k'} z^i z^j z^k \quad \text{and} \quad z^i (z^j \star z^k) = [F_{3,R}]^{i,j,k}_{i',j',k'} z^i z^j z^k\]

where

\[[F_{3,L}] = (\Delta \otimes \text{id})(e^{-f})(F^{-1} \otimes \text{id}) \quad \text{and} \quad [F_{3,R}] = (\text{id} \otimes \Delta)(e^{-f})(\text{id} \otimes F^{-1})\]

- Show for imaginary beta

\[[F_{3,L}]^{1,2,3}_{1,1,2} = [F_{3,L}]^{1,1,3}_{1,1,2} = [F_{3,R}]^{3,1,1}_{3,1,1} = [F_{3,R}]^{2,1,1}_{2,1,1} = \cos \left( \sqrt{2} \alpha \right)\]

\[[F_{3,L}]^{1,2,3}_{1,2,3} = [F_{3,L}]^{1,3,2}_{1,1,2} = [F_{3,R}]^{3,2,1}_{3,2,1} = [F_{3,R}]^{3,1,2}_{3,1,2} = \frac{(1 + q) \left( 1 + 2 \cos \left( \sqrt{3} \alpha \right) \right)}{3 \sqrt{2} \sqrt{q^2 + 1}}\]

\[[F_{3,L}]^{1,1,2}_{1,1,2} = -[F_{3,L}]^{1,1,3}_{3,1,1} = [F_{3,R}]^{3,1,1}_{3,1,1} = -[F_{3,R}]^{2,1,1}_{1,1,2} = -\frac{q \sin \left( \sqrt{2} \alpha \right)}{\sqrt{q^2 + 1}}\]

\[[F_{3,L}]^{1,2,1}_{1,1,2} = -[F_{3,L}]^{1,2,2}_{2,1,1} = -[F_{3,R}]^{1,1,2}_{2,1,1} = [F_{3,R}]^{2,1,2}_{1,2,2} = \frac{\sin \left( \sqrt{2} \alpha \right)}{\sqrt{2}}\]

etc.
Cubic relations

• The star product is nonassociative in general

\[ z^1 \star (z^2 \star z^3) \neq (z^1 \star z^2) \star z^3 \]

• The coassociator relates the two placements of parentheses

\[
m((\Delta \otimes \text{id})(F^{-1})F_{12}^{-1} \triangleright [x \otimes y \otimes z]) \\
= m((\text{id} \otimes \Delta)(F^{-1})F_{23}^{-1} \Phi_{123} \triangleright [x \otimes y \otimes z])
\]
Inverse star product

- Can also define
  \[ x \star y = m_{F^{-1}}(x \otimes y) = m(F \triangleright x \otimes y) = (F(1) \triangleright x) \cdot (F(2) \triangleright y) \]

- In index form:
  \[ z^i \star z^j = F^j_{\ lk} z^k z^l \]

- Relates \( \mathcal{N} = 4 \) expressions to LS-deformed ones
  \[ z^1 \star z^2 - z^2 \star z^1 = \frac{\sqrt{2} \sqrt{1 + \bar{q}}}{\sqrt{1 + q \sqrt{1 + q\bar{q} + hh}}} \left( z^1 z^2 - qz^2 z^1 + h(z^3)^2 \right) \]

- In gauge theory language:
  \[ [\Phi^1, \Phi^2]_{q,h} = \Phi^1 \Phi^2 - q \Phi^2 \Phi^1 + h(\Phi^3)^2 \]

- Still non-associative. However, \( \Phi \) drops out of cyclic relations!
  \[ z^1 \star (z^2 \star z^3) + z^2 \star (z^3 \star z^1) + z^3 \star (z^1 \star z^2) = (z^1 \star z^2) \star z^3 + (z^2 \star z^3) \star z^1 + (z^3 \star z^1) \star z^2 \]
Cubic inverse product

- Generate the LS superpotential from the $\mathcal{N} = 4$ SYM one
- Obtain:
  \[
  \Phi_1^* \Phi_2^* \Phi_3 + \Phi_2^* \Phi_3^* \Phi_1 + \Phi_3^* \Phi_1^* \Phi_2 - \Phi_1^* \Phi_3^* \Phi_2 - \Phi_3^* \Phi_2^* \Phi_1 - \Phi_2^* \Phi_1^* \Phi_3
  = \frac{A}{3} \left[ \Phi_1^2 \Phi_3^2 + \Phi_2^2 \Phi_1^2 + \Phi_3^2 \Phi_1^2 - q \left[ \Phi_1^3 \Phi_2^2 - \Phi_3^2 \Phi_1^2 - \Phi_2^2 \Phi_1^3 \right] 
  + h \left( (\Phi_1^3)^3 + (\Phi_2^3)^3 + (\Phi_3^3)^3 \right) \right]
  \]
- Take the gauge theory trace
  \[
  \text{Tr} \left( \Phi_1^* \Phi_2^* \Phi_3 - \Phi_1^* \Phi_3^* \Phi_2 \right)
  = A \text{ Tr} \left( \Phi_1^2 \Phi_3^2 - q \Phi_1^3 \Phi_2^2 + \frac{h}{3} \left( (\Phi_1^3)^3 + (\Phi_2^3)^3 + (\Phi_3^3)^3 \right) \right)
  \]
- Can also show kinetic terms are not deformed
  \[
  \overline{\Phi}_i \Phi^i = \overline{\Phi}_i \Phi^i
  \]
- LS theories are just $\ast$ deformations of $\mathcal{N} = 4$ SYM
Deformed dual background?

- Work in generalised geometry description
- Use gauge-theory star product, expanded to first order, to twist flat-space pure spinors

\[ dz^I \wedge_* dz^J = \left( 1 - \frac{i}{2} \Theta^{KL}_{\ell_K \wedge \ell_L} \right) dz^I \wedge dz^J = dz^I \wedge dz^J - i \Theta^{IJ} \]

- For integrable cases, constructed the dual background [DZ '16]

\[ \Phi_* = dz^1 \wedge_* dz^2 \wedge_* dz^3 \quad \text{and} \]

\[ \Phi^*_+ = 1 + \frac{1}{2} \sum_{i=1}^{3} dz^i \wedge_* d\bar{z}^i + \frac{1}{4} \sum_{i=1}^{3} dz^i \wedge_* d\bar{z}^i \wedge_* dz^{i+1} \wedge_* d\bar{z}^{i+1} \]

\[ + \frac{1}{8} dz^1 \wedge_* d\bar{z}^1 \wedge_* dz^2 \wedge_* d\bar{z}^2 \wedge_* dz^3 \wedge_* d\bar{z}^3 \]

- Geometric manifestation of the CFT quantum symmetry
- Can we repeat this for general \((q, h)\)?
Summary

- Found a twist that relates the $R$-matrix of $\mathcal{N} = 4$ SYM to that of the Leigh-Strassler theories
- Non-trivial coassociator $\Rightarrow$ Quasi-Hopf algebra
- The global $SU(3)_R$ is not broken, just deformed
- Used the twist to define a star product
- Relate any expression in the LS theories to $\mathcal{N} = 4$ SYM
- In particular, recover the LS superpotential by inserting $\star$ in the $\mathcal{N} = 4$ SYM superpotential
Outlook

• Study conservation laws from the quantum algebra perspective

• Revisit non-integrability of LS theories (quasitriangular quasi-Hopf)

• Does the quasi-Hopf symmetry manifest itself on the dual gravity side?

• Extend the results of [DZ'16] to the generic LS case

• Relation to solution-generating techniques (CYBE, $\lambda$-deformations etc.)

• Deeper reason for appearance of quantum algebra in 4d SCFT? (Perturbative finiteness?)