Torsional Newton-Cartan geometry in Field Theory, Gravity and Holography

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based on work with:
Jelle Hartong and Elias Kiritsis:
1409.1519 (PLB), 1409.1522 (PRD), 1502.00228 (JHEP) & to appear
Jelle Hartong 1504.0746 (JHEP)

and
Morten Holm Christensen, Jelle Hartong, Blaise Rollier
1311.4794 (PRD) & 1311.6471 (JHEP)
Outline

• **Why Newton-Cartan (NC)** ? (→ non-relativistic space-time)
  - holography,
  - field theory
  - gravity

• **What is NC (& its torsionful generalization TNC) geometry** ?
  - NC from gauging the Bargmann algebra

• How do non-relativistic field theories couple to NC ?

• What theory of gravity does one get when making TNC dynamical ?
  - connection to Horava-Lifshitz gravity

• Outlook
Motivation (Holography)

AdS/CFT has been a very successful tool in studying strongly coupled (conformal) relativistic systems

- holography beyond original AdS-setup?

- How general is holographic paradigm? (nature of quantum gravity, black hole physics, cosmology)

- Examples of potentially holographic descriptions based on non-AdS space-times: Lifshitz, Schrödinger, warped AdS3 (Kerr/CFT), flat space-time.

  - simplest example appears to be Lifshitz spacetimes

\[ ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2} (dr^2 + d\vec{x}^2) \]

characterized by anisotropic (non-relativistic) scaling between time and space

\[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}. \]

* introduced originally to study strongly coupled systems with critical exponent z

[Kachru,Liu,Mulligan]
Motivation (Holography) cont’d

• for standard AdS setup: boundary geometry is Riemannian just like the bulk geometry

• not generic: in beyond-AdS holography bdry. geometry typically non-Riemannian

  Christensen,Hartong,Rollier,NO (1311)
  Hartong,Kiritsis,NO (1409)

  -> need new approach: prime (simplest) example to gain traction = Lifshitz
  (lessons can subsequently be applied to other cases)

Result: torsional Newton-Cartan geometry is the boundary geometry found in
large class of examples in EPD model

  -> Lifshitz holography dual to field theories on TNC space-time

by making the resulting non-Riemannian geometry dynamical one gains access to
other bulk theories of gravity (than those based on Riemannian gravity)
- apply holography (e.g. HL gravity)
- interesting in their own right
Different Holographic setups

bulk: Riemannian gravity (GR)  non-Riemannian gravity (e.g. HL gravity)

AdS  non-AdS  non-AdS

boundary: CFT  NR-FT with scaling  NR-FT with scaling
coupling to: Riemannian geometry  non-Riemannian
Motivation (Field Theory)

• in relativistic FT: very useful to couple to background (Riemannian) geometry
  -> compute EM tensors, study anomalies, Ward identities, etc.

- background field methods for systems with non-relativistic (NR) symmetries require NC geometry (with torsion)
  -> there is full space-time diffeomorphism invariance when coupling to the right background fields

- Recent examples
  * Son’s approach to the effective field theory for the FQHE
    [Son, 2013], [Geracie, Son, Wu, Wu, 2014]
  * non-relativistic (NR) hydrodynamics  [Jensen, 2014]
Motivation (Gravity)

- interesting to make NC geometry dynamical
-> “new” theories of gravity
will see: dynamical Newton-Cartan (NC) = Horava-Lifshitz (HL) gravity

natural geometric framework with full diffeomorphism invariance & possibly non-trivial consequences for HL gravity

such theories of gravity interesting as
- other bulk theories of gravity in holographic setups
- effective theories (cosmology)
Newton-Cartan makes Galilean local

- NC geometry originally introduced by Cartan to geometrize Newtonian gravity
  - both Einstein’s and Newton’s theories of gravity admit geometrical formulations which are diffeomorphism invariant
  - NC originally formulated in “metric” formulation
  - more recently: vielbein formulation (shows underlying sym. principle better) Andringa,Bergshoeff,Panda,de Roo

Riemannian geometry: tangent space is Poincare invariant

Newton-Cartan geometry: tangent space is Bargmann (central ext. Gal.) invariant

- gives geometrical framework and extension to include torsion
  i.e. as geometry to which non-relativistic field theories can couple
  (boundary geometry in holographic setup is non-dynamical)

* will next consider dynamical (torsional) Newton-Cartan
Riemannian geometry from gauging Poincare

Poincare = Lorentz + translations (space & time)

\[
[M_{ab} , P_c] = \eta_{ac} P_b - \eta_{bc} P_a ,
\]
\[
[M_{ab} , M_{cd}] = \eta_{ac} M_{bd} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad} + \eta_{bd} M_{ac} .
\]

[conformal = Poincare + dilatation + special conformal]

gauge Poincare: \[ A_\mu = P_a e_\mu^a + \frac{1}{2} M_{ab} \omega_\mu^{ab} \]

vielbein \hspace{1cm} spin connection

- adjoint action \[ \delta A_\mu = \partial_\mu \Lambda + [A_\mu , \Lambda] \]

\[ \Lambda = P_a \zeta^a + \frac{1}{2} M_{ab} \sigma^{ab} \]

- field strength- \[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu , A_\nu] \]

\[ = P_a R_{\mu\nu}^a (P) + \frac{1}{2} M_{ab} R_{\mu\nu}^{ab} (M) \]
Riemannian geometry from gauging Poincare (cont’d)

• find set of trasfos that replace local translations by diffeomorphisms

\[ \delta A_\mu = \delta A_\mu - \xi^\nu F_{\mu\nu} = \mathcal{L}_\xi A_\mu + \partial_\mu \Sigma + [A_\mu, \Sigma] \]

with \[ \Lambda = \xi^\mu A_\mu + \Sigma \]
\[ \Sigma = \frac{1}{2} M_{ab} \lambda^{ab} \] Lorentz

-> vielbein and spin connection transform accordingly

Lorentz invariant: \[ g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \]

covariant derivative defined via vielbein postulates -> \[ \nabla_\rho g_{\mu\nu} = 0. \]

\[ R_{\mu\nu}^a (P) \sim \Gamma^\rho_{[\mu\nu]} \] encodes the torsion

\[ R_{\mu\nu}^{\alpha\beta} (M) \]

Riemann curvature two-form

- setting torsion = zero gives Riemannian geometry with Levi-Civita connection
(else Riemann-Cartan geometry)

• GR is a diff invariant theory whose tangent space
invariance group is the Poincaré group

* Einstein equivalence principle -> local Lorentz invariance
Relevant non-relativistic algebras

Galilean

\[ [H, G_a] = P_a \quad [P_a, G_b] = 0 \]

(Bargmann algebra is the c to infinity limit of Poincare)

\[ [P_a, G_b] = N \delta_{ab} \]

Bargmann

Lifshitz

\[ [D, H] = zH \quad [D, P_a] = P_a \]

\[ [D, N] = (2 - z)N \]

Schrödinger

\[ \text{Schrödinger} = \text{Bargmann} + \text{dilatations} (+ \text{special conformal for } z=2) \]
Gauging the Bargmann algebra

Galilean

\[ H, P_a, J_{ab}, G_a \quad N \]

Bargmann

\[ [H, G_a] = P_a \quad [P_a, G_b] = 0 \]

\[ [P_a, G_b] = N\delta_{ab} \]

gauge Bargmann and impose curvature constraints

\[ R_{\mu\nu}(H) = R_{\mu\nu}^a(P) = R_{\mu\nu}(N) = 0. \]

independent fields:

\[ \tau_\mu, e_\mu^a, m_\mu \]

\[ h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab} \]

= gauge fields of Hamiltonian, spatial translations and central charge
Newtonian gravity is a diff invariant theory whose tangent space is Bargmann (make **Bargmann local**)

<table>
<thead>
<tr>
<th>symmetry</th>
<th>generators</th>
<th>gauge field</th>
<th>parameters</th>
<th>curvatures</th>
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</thead>
<tbody>
<tr>
<td>time translations</td>
<td>$H$</td>
<td>$\tau_\mu$</td>
<td>$\zeta(x^\nu)$</td>
<td>$R_{\mu\nu}(H)$</td>
</tr>
<tr>
<td>space translations</td>
<td>$P_a$</td>
<td>$e_\mu^a$</td>
<td>$\zeta^a(x^\nu)$</td>
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<tr>
<td>boosts</td>
<td>$G_a$</td>
<td>$\omega_\mu^a$</td>
<td>$\lambda^a(x^\nu)$</td>
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</tr>
<tr>
<td>spatial rotations</td>
<td>$J_{ab}$</td>
<td>$\omega_\mu^{ab}$</td>
<td>$\lambda^{ab}(x^\nu)$</td>
<td>$R_{\mu\nu}^{ab}(J)$</td>
</tr>
<tr>
<td>central charge transf.</td>
<td>$N$</td>
<td>$m_\mu$</td>
<td>$\sigma(x^\nu)$</td>
<td>$R_{\mu\nu}(N)$</td>
</tr>
</tbody>
</table>

Curvature constraints:

$$R_{\mu\nu}(H) = R_{\mu\nu}^a(P) = R_{\mu\nu}(N) = 0.$$  

Leaves as **independent fields**:

$$\tau_\mu, e_\mu^a, m_\mu$$

$$h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$$

Transforming as:

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu$$

$$\delta e_\mu^a = \mathcal{L}_\xi e_\mu^a + \lambda^a \tau_\mu + \lambda^a b e_\mu^b$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda^a e_\mu^a$$
Newton-Cartan geometry

gauge field of central extension of Galilean algebra (Bargmann)

NC geometry = no torsion

TTNC geometry = twistless torsion

TNC geometry

- in TTNC: torsion measured by geometry on spatial slices is Riemannian

spatial metric:
\[ h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab} \]

notion of absolute time

preferred foliation in equal time slices
Adding torsion to NC

- inverse vielbeins

\[ (v^\mu, e^\mu_a) \]

\[ v^\mu \tau_\mu = -1, \quad v^\mu e^a_\mu = 0, \quad e^\mu_a \tau_\mu = 0, \quad e^\mu_a e^b_\mu = \delta^b_a \]

can build Galilean boost-invariants

\[ \hat{v}^\mu = v^\mu - h^{\mu\nu} M_\nu, \]
\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu, \]
\[ \tilde{\Phi} = -v^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu, \]

-introduce Stueckelberg scalar chi
(to ensure N-invariance):

\[ M_\mu = m_\mu - \partial_\mu \chi. \]

affine connection of TNC (inert under G,J,N)

\[ \Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}) \]

with torsion

\[ \Gamma^\rho_{[\mu\nu]} = -\frac{1}{2} \hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) \]

\[ \nabla_\mu \tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0, \quad \text{analogue of metric compatibility} \]
torsion in NC (recent activity)

- NC introduced in problem of FQH  Son (1306)

- TNC first observed as bdry geometry in z=2 Lifshitz holography & generalized to large class with general z  Christensen,Hartong, Rollier, NO (1311)

   Hartong, Kiritsis, NO (1409)

- TTNC introduced in FQH  Geracie, Son, Wu, Wu (1407)

- TNC from gauging Schrödinger algebra  Bergshoeff, Hartong, Rosseel (1409)

- TNC from gauging Bargmann (with torsion)  Hartong, NO (1504)

- coupling of non-relativistic field theories to TNC (independent of holography)  Jensen (1408)

   Hartong, Kiritsis, NO (1409)

- TNC related to warped geometry that couples to 2D WCFT  Hofmann, Rollier (1411)

- other approaches  Banerjee, Mitra, Mukherjee (1407), Brauner, Endlich, Monin, Penco (1407)

   Bekaert, Morand (1412)

- recent activity using NC/TNC in CM (strongly-correlated electron system, FQH)  Gromov, Abanov, [Moroz, Hoyos],[Geracie, Son][Wu, Wu],[Geracie, Golkar, Roberts],

   Jensen, Karch (1412), Bergshoeff, Rosseel, Zojer (1505)

- (T)NC from non-rel limits  Jensen, Karch (1412)
Coupling FTs to TNC

- action functional

\[ S = S[\hat{\theta}^\mu, h^{\mu\nu}, \tilde{\Phi}] . \]

- EM tensor:
  \[ T^\mu_\nu \]
  mass current
  \[ T^\mu \]
  energy current (density + flux)
  momentum current
  spatial stress
  mass density

\[
\delta S \sim \int d^{d+1}x e [\mathcal{E}^\mu \delta \tau_\mu + \mathcal{P}_\mu h^{\mu\nu} \delta v^\nu + T_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} \delta h^{\rho\sigma} + T^\mu \delta m_\mu]
\]

* important to have torsion in order to describe the most general energy current!

- from the various local symmetries:
  particle number conservation (if extra local U(1))
  mass current = momentum current (local boosts)
  symmetric spatial stress (local rotations)
Diffeomorphism and scale Ward identities

- diffeos $\rightarrow$ on-shell WI

\[ \nabla_\nu T^\nu_\mu + \text{torsion terms} + \rho \nabla_\mu \tilde{\Phi} = 0 \]

* conserved currents \[ \partial_\nu (eK^\mu T^\nu_\mu) = 0. \]
  for K a TNC Killing vector:

- if theory has scale invariance:
  can use TNC analogue of dilatation connection

\[ z\mathcal{E} + \text{Tr} T_{\text{spatial}} + 2(z - 1)\rho \Phi = 0 \]

z-deformed trace WI
intermezzo: geodesics on NC space-time

- worldline action of non-rel particle of mass m on NC background

\[ S = \int d\lambda L = \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} \]

\[ \text{[Kuchar], [Bergshoeff et al]} \]

• gives the geodesic equation with NC connection

\[ \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0, \]

* reduces to Newton’s law

\[ \frac{d^2 x^i}{dt^2} + \delta^{ij} \partial_j \Phi = 0, \]

provided we take

\[ M_t = \partial_t M + \Phi, \]
\[ M_i = \partial_i M, \]

for flat NC space-time: zero Newtonian potential

symmetries of flat NC = conformal Killing vectors (spanning Lifshitz) + extra
Global symmetries for non-rel FTs on flat NC

• novel phenomenon: notion of **global symmetries** depends on type of matter fields (and their couplings)

  **two scenarios** when coupling non-rel FT to TNC background
  i) theory has internal local U(1) related to particle #
  ii) not

  one finds for non-rel. FTs on flat NC:
  -> i) mechanism that enhances Lif with:
     particle # + Galilean boosts (+ special conformal)
     example: Schrödinger model (+ deformations)
  -> ii) no sym. enhancement (only Lif symmetry)
     example: Lifshitz model

* interplay between conserved currents and space-time isometries is different compared to relativistic case: same mechanism seen in Lifshitz holography!
Dynamical Newton-Cartan geometry

so far: (T)NC geometry was non-dynamial:
- what happens when we allow it to fluctuate?

• what is the theory of gravity that incorporates local Galilean symmetry?
  (Einstein equivalence principle, but applied to Galilean instead of Lorentz)

recently shown that:

• dynamical NC geometry = projectable HL gravity

• dynamical TTNC geometry = non-projectable HL gravity

* Horava-Lifshitz gravity was originally introduced as non-Lorentz invariant and renormalizable UV completion of gravity
  - phenomenologically viable?
  - interesting theoretically as alternate bulk gravity theories
    relevant to i) holography for strongly coupled non-relativistic systems
    ii) alternate theories in cosmology

[Hartong, NO]
NC/TTNC gravity

TNC geometry is a natural geometrical framework underlying HL gravity

- NC quantities combine into: \( g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu} \)
- ADM parametrization of metric used in HL gravity:
  \[
  ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt)
  \]
  relation:
  \( \tau_\mu \sim \text{lapse} \), \( \hat{h}_{\mu\nu} \sim \text{spatial metric} \), \( m_\mu \sim \text{shift + Newtonian potential} \),

some features:

- khronon field of BPS appears naturally \( \tau_\mu = \psi \partial_\mu \tau \) \( \text{Blas,Pujolas,Sibiryakov(2010)} \)

  NC (no torsion): \( N = N(t) \) \( \text{projectable HL gravity} \)
  TTNC:
  \( N = N(t, x) \) \( \text{non-projectable HL gravity} \)

- U(1) extension of HMT emerges naturally as Bargmann U(1)
- new perspective (via chi field) on nature of U(1) symmetry

Horava,Melby-Thompson(2010)
Effective actions reproduce HL

- covariant building blocks:
  - extrinsic curvature: \( \hat{h}_{\nu\rho} \nabla_\mu \hat{v}^\rho = -K_{\mu\nu} \)
  - spatial curvature \( R_{\mu\nu\sigma} \)
  - covariant derivative, torsion vector \( a_\mu \), inverse spatial metric \( h^{\mu\nu} \)
  - tangent space invariant integration measure \( e = \text{det}(\tau_\mu, e^a_\nu) \)

-> construct all terms that are relevant or marginal (up to dilatation weight \( d+z \))
  - in 2+1 dimensions for \( 1 < z \leq 2 \)

\[
S = \int d^3x e \left[ C (h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda (h^{\mu\nu} K_{\mu\nu})^2) - \mathcal{V} \right]
\]

potential:

\[
-\mathcal{V} = 2\Lambda + c_1 h^{\mu\nu} a_\mu a_\nu + c_2 \mathcal{R} + \delta_{z,2} \left[ c_{10} (h^{\mu\nu} a_\mu a_\nu)^2 + c_{11} h^{\mu\rho} a_\mu a_\rho \nabla_\nu (h^{\nu\sigma} a_\sigma) + c_{12} \nabla_\nu (h^{\mu\rho} a_\rho) \nabla_\mu (h^{\nu\sigma} a_\sigma) + c_{13} \mathcal{R}^2 + c_{14} \mathcal{R} \nabla_\mu (h^{\mu\nu} a_\nu) + c_{15} \mathcal{R} h^{\mu\nu} a_\mu a_\nu \right]
\]

kinetic terms (2\text{nd} order)
Perspectives for HL gravity

new perspectives on HL:
- different vacuum (flat NC space-time): reexamine issues with HL gravity
- IR effective theory for non-relativistic field theories
- insights into non-relativistic quantum gravity corner of $(\hbar, G_N, 1/c)$ cube?

• relevance for cosmology?

alternate theories of gravity in cosmological scenarios, effective theories for inflation

• examine TNC gravity (general torsion)
  * relation with vector khronon of [Janiszewski,Karch]
TNC in NR hydro & fluid/gravity correspondence

- TNC of growing interest in cond-mat (str-el, mes-hall) literature

  developments in Lifshitz holography can drive development of tools to study dynamics and hydrodynamics of non-rel. systems

    Lifshitz hdyro: [Hoyos,Kim,Oz]
    Galilean: [Jensen]

  (in parallel to progress in the last many years in relativistic fluids and superfluids inspired from the fluid/gravity correspondence in AdS)

TNC right ingredients to start constructing effective TNC theories and their coupling to matter (e.g. QH-effect)

- organizing principle for derivative expansion of stress tensor/mass current (transport coefficients)
- consider boosted Lifshitz black branes & perturb

[Kiritsis,Matsuo], in progress: [Hartong,NO,Sanchioni]
Outlook

- employ similar techniques to Schrödinger, warped AdS, flat space holography
  [Andrade,Keeler,Peach,Ross,][Hofman,Rollier][Armas,Blau,Hartong(in progress)]
- adding charge (Maxwell in the bulk)
  adding other exponents (hyperscaling, matter scaling)
  [Kiritsis,Goutereaux][Gath,Hartong,Monteiro,NO]
  [Khveshchenko][Karch][Hartnoll,Karch]
- applications to non-rel. hydrodynamics:
  fluid-gravity: black branes with zero/non-zero particle no. ? Galilean perfect fluids
  Lifshiz: [Hoyos,Kim,Oz]  
  Galilean: [Jensen]  
  in progress: [Kiritsis,Matsuo][Hartong,NO,Sanchioni]
- flat space holography: gauging of Caroll group and ultra-relativistic gravity
  [Hartong]
- NC supergravity, NC in string theory  
  [Bergshoeff et al]
- revisit HL gravity using TNC language/connections with NR String Theory
- effective TNC theories and their coupling to matter (e.g. QH-effect)
  [Son] et al
The end