1+3 COVARIANT PERTURBATIONS IN MULTIFLUID f(R)-GRAVITY

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Motivation
1.f(R)

2.The Universe is not composed of a single component of matter alone;it consists of many fluids(radiation, dust, 'CDM', etc) thermodynamically interacting with each other

Outline

- Motivation
- Basics: f(R)+Cosmological perturbations
- 1+3 Covariant formalism
- Perturbation Equations
- Applications to a Radiation/Dust-dominated Universe in R^n -gravity
- Quasi-static approximations
- Conclusion

Motivation

- Why f(R)?
- The Universe is not composed of a single component of matter alone; it consists of many fluids (radiation, "CDM", etc) thermodynamically interacting with each other.
- Analyzing how the perturbations of the different components of the total cosmic fluid evolve in time can tell us how structures grow in the Universe.
- Extending the single-fluid formulation in f(R) to one where the equation of state is no longer constant but evolves with time, i.e., $w = w(t)$.

Why modify Gravity?!

- Cosmic acceleration: missing energy?
- Possible explanations:
  - ✳ Λ ,the "cosmological constant"
  - ✳"dark energy''
  - ✳ modification of GR
    - Braneworlds
    - Gauss-Bonnet gravities
    - ✓ f(R) models...
  - ✳ Other approaches in the pipeline
    - Inhomogeneous cosmologies
    - Cosmological backreaction...

f(R) theories of Gravity

- Einstein –Hilbert action: $\mathcal{A} = \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \mathcal{L}_m\right]$
- Einstein field equations: $G_{ab} = \kappa T_{ab}$
- Generalized higher-order-gravity action:

$$\mathcal{A}_{f(R)} = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{L}_m\right]$$

- Generalized field equations

$$f' G_{ab} = f'(R_{ab} - \frac{1}{2} g_{ab} R) = T^m_{ab} + \frac{1}{2} g_{ab}(R - Rf') + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f',$$

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} = T^{tot}_{ab}$$

Cosmological perturbation theory

- Can help us understand the evolution of structure in the Big Bang model
- Uses gravitational theories to compute the gravitational forces causing small perturbations to grow and eventually seed the formation of galaxies, clusters, superclusters etc...
- Applicable largely to a homogeneous universe
- The theory is a good approximation on the large-intermediate scales
- On smaller scales, more sophisticated techniques such as N-body simulations are widely in use

Two approaches...

Metric Gauge Invariant Perturbation Theory

- Based on foliating a space-time with hyper-surfaces (Lifshitz 1946,Bardeen 1980 Kodama & Sasaki,1984)
- Starts with a background, perturbs away
- Nonlocal
- Coordinate dependent
- Does not treat nonlinearities

Covariant Gauge Invariant Perturbation Theory

- Based on threading a space-time with frames

(Hawking 1966,Ehlers,Ellis 1971,Ellis,Bruni,Dunsby 1989,1991,1992...)

- Starts with theory, reduces down to linearities in a particular background
- Local
- Covariant
- Nonlinearities treated

1+3 covariant decomposition

- A fluid approach to spacetime with the time-like flow:

$$u^a = \frac{dx^a}{d\tau}, \qquad u^a u_a = -1$$

- Projections onto surfaces orthogonal to the flow :

$$h_{ab} = g_{ab} + u_a u_b$$

- Covariant convective derivative on a scalar: $\dot{f} = u^a \nabla_a f$
- Spatial covariant derivative: $\tilde{\nabla}_a f = h^b{}_a \nabla_b f$
- Kinematics of $u^a \Rightarrow$ geometry of fundamental worldlines

acceleration, expansion, shear, vorticity

$$\nabla_a u_b = -u_a \dot{u}_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}$$

Effective total EMTs

- The total energy-momentum tensor is given by

$$T_{ab} = \tilde{T}^{m}_{ab} + T^{R}_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

from which the thermodynamical quantities

$$\mu = \tilde{\mu}^{m} + \mu^{R}, \quad p = \tilde{p}^{m} + p^{R}, \quad q_a = \tilde{q}^{m}_{a} + q^{R}_{a}, \quad \pi_{ab} = \tilde{\pi}^{m}_{ab} + \pi^{R}_{ab}$$

$$\text{where} \quad \tilde{\zeta}^{m} = \frac{\zeta^{m}}{f'} \quad \text{for} \quad \zeta = \{\mu, p, q_a, \pi_{ab}\}$$

are decomposed.

energy density — isotropic pressure — heat flux — anisotropic pressure

The energy-momentum tensor of the curvature "fluid" can be decomposed as follows:

$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(Rf'-f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f''\,\dot{u}_b\tilde{\nabla}R\right],$$

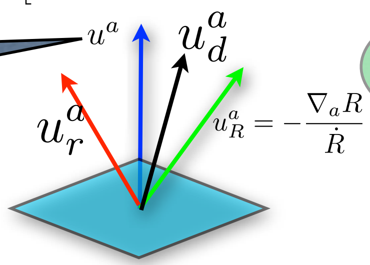
$$p^R = \frac{1}{f'}\left[\frac{1}{2}(f-Rf') + f''\ddot{R} + 3f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R + \right.$$
$$\left. -\frac{2}{3}f'''\tilde{\nabla}^a R\tilde{\nabla}_a R - \frac{1}{3}f''\,\dot{u}_b\tilde{\nabla}R\right],$$

No background contribution.

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\tilde{\nabla}_a R\right],$$

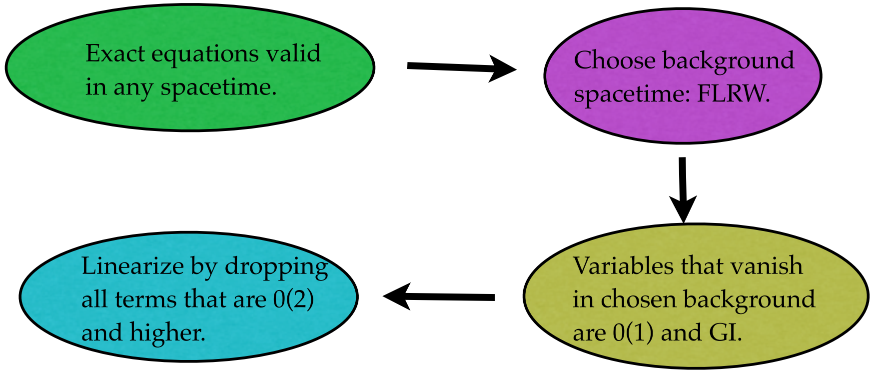
$$\pi^R_{ab} = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R + \sigma_{ab}\dot{R}\right].$$

Taken to be the energy frame of the total matter



So one can think of this as a curvature "fluid" moving relative to u^a

Linearisation



$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \frac{1}{2}(\mu^{tot} + 3p^{tot}) = 0$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 - \tilde{\nabla}^a\dot{u}_a + \frac{1}{2}(\mu^{tot} + 3p^{tot}) = 0 \longrightarrow$$

Almost FLRW model.

The linear gravitational equations

Propagation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a A_a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R),$$

$$\dot{\omega}_a + 2H\omega_a + \tfrac{1}{2}\mathrm{curl}\, A_a = 0\,,$$

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = -q_a^R,$$

$$\dot{E}_{ab} + 3HE_{ab} - \mathrm{curl}\, H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} = -\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab},$$

$$\dot{H}_{ab} + 3HH_{ab} + \mathrm{curl}\, E_{ab} = \tfrac{1}{2}\mathrm{curl}\, \pi^R_{ab}\,,$$

Constraint

$$\tilde{\nabla}^b\sigma_{ab} - \mathrm{curl}\,\omega_a - \tfrac{2}{3}\tilde{\nabla}_a\Theta = -q_a^R,$$

$$\mathrm{curl}\,\sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0\,,$$

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m = -\tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu^R - \tfrac{1}{3}\Theta q_a^R,$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = -\tfrac{1}{2}\mathrm{curl}\, q_a^R + (\mu^R + p^R)\omega_a\,,$$

$$\tilde{\nabla}^a\omega_a = 0\,,$$

The linear conservation equations

The Bianchi identities:
$$\begin{cases} \tilde{T}^{M;b}_{ab} = \dfrac{T^{m;b}_{ab}}{f'} - \dfrac{f''}{f'^2}\, T^{m}_{ab}\; R^{;b} \,, \\ T^{R;b}_{ab} = \dfrac{f''}{f'^2}\, \tilde{T}^{M}_{ab}\; R^{;b} \,, \end{cases}$$

Matter
$$\begin{cases} \dot{\mu}^{m} = -\,\Theta\,(\mu^{m} + p^{m}) \,, \\ \tilde{\nabla}^{a} p^{m} = -(\mu^{m} + p^{m})\,\dot{u}^{a} \,, \end{cases}$$

Curvature
$$\begin{cases} \dot{\mu^{R}} + \tilde{\nabla}^{a} q^{R}_{a} = -\,\Theta\,(\mu^{R} + p^{R}) + \boxed{\mu^{m}\dfrac{f''\,\dot{R}}{f'^2} \,,} \\ \dot{q}^{R}_{\langle a\rangle} + \tilde{\nabla}_{a} p^{R} + \tilde{\nabla}^{b}\pi^{R}_{ab} = -\,\tfrac{4}{3}\,\Theta\, q^{R}_{a} - (\mu^{R} + p^{R})\,\dot{u}_{a} + \boxed{\mu^{m}\dfrac{f''\,\tilde{\nabla}_{a} R}{f'^2}} \,, \end{cases}$$

Inhomogeneity variables

- Define co-moving, gauge-invariant inhomogeneity quantities
- Matter inhomogeneities in the total fluid

$$\mathcal{D}^m_a = \frac{a}{\mu_m}\tilde{\nabla}_a \mu_m, \qquad Z_a = a\tilde{\nabla}_a \Theta$$

$$C_a = a\tilde{\nabla}_a \tilde{R}, \qquad \varepsilon_a = \frac{a}{\mu_m}(\frac{\partial p_m}{\partial s})\tilde{\nabla}_a s$$

- Curvature inhomogeneities

$$\mathcal{R}_a = a\tilde{\nabla}_a R, \qquad \Re_a = a\tilde{\nabla}_a \dot{R}$$

- Matter inhomogeneities in the component fluids

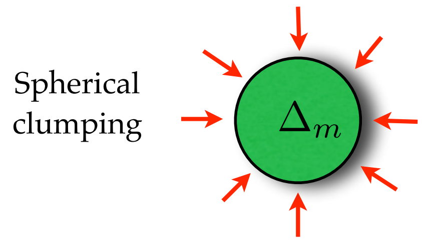
$$\mathcal{D}^i_a = \frac{a}{\mu_i}\tilde{\nabla}_a \mu_i, \quad \varepsilon^i_a = \frac{a}{\mu_i}(\frac{\partial p_i}{\partial s})\tilde{\nabla}_a s, \quad V^i_a = u^i_a - u_a$$

- Relative variables

$$S^{ij}_a = \frac{\mathcal{D}^i_a}{1+w_i} - \frac{\mathcal{D}^j_a}{1+w_j}, \quad \varepsilon^{ij}_a = \frac{w_i}{1+w_i}\varepsilon^i_a - \frac{w_j}{1+w_j}\varepsilon^j_a, \quad V^{ij}_a = V^i_a - V^j_a$$

Extracting the scalar modes

$$a\tilde{\nabla}_a \mathcal{D}^m_b = \Sigma^m_{ab} + W^m_{ab} + \frac{1}{3}\Delta_m h_{ab}$$

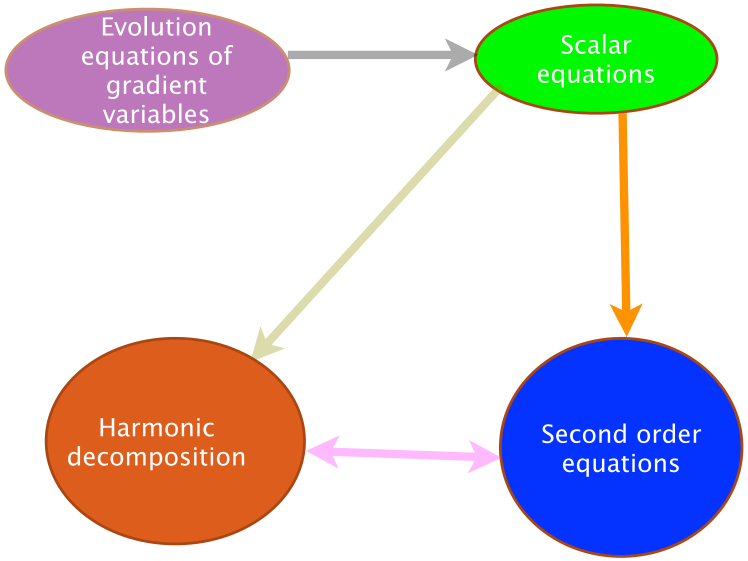


$$\Delta_m = a\tilde{\nabla}^a \mathcal{D}^m_a$$

$C = a\tilde{\nabla}^a C_a, \quad Z = a\tilde{\nabla}^a Z_a, \quad \varepsilon = a\tilde{\nabla}^a \varepsilon_a, \quad \mathcal{R} = a\tilde{\nabla}^a \mathcal{R}_a, \quad \Re = a\tilde{\nabla}^a \Re_a$

$\Delta_i = a\tilde{\nabla}^a \mathcal{D}^i_a, \quad \varepsilon_i = a\tilde{\nabla}^a \varepsilon^i_a, \quad V_i = a\tilde{\nabla}^a V^i_a$

Perturbation equations



Harmonic decomposition

- Given a second order equation:

Damping term · Restoring term · Source forcing term

$$\ddot{X} + A\dot{X} + BX + C = 0$$

- Separation of variables: $X(x,t) = X(x)X(t)$

$$X = \sum_k X^k(t)Q^k(x)$$

where

$$\tilde{\nabla}^2 Q = -\frac{k^2}{a^2}Q \quad \text{such that} \quad k = \frac{2\pi a}{\lambda}, \quad \dot{Q}_x = 0$$

Total fluid equations

$$\ddot{\Delta}^k_m + \left[(c_s^2 + \frac{2}{3} - 2w)\Theta - \dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_m + \left[\left(\frac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\frac{\mu_m}{f'} + \frac{1}{2}(3w - 5c_s^2)\frac{f}{f'}\right.$$

$$\left. +(c_s^2 - w)\left(2R - 4\dot{R}\Theta\frac{f''}{f'} - \frac{12K}{a^2}\right) + c_s^2\frac{k^2}{a^2}\right]\Delta^k_m + \left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2} + \frac{k^2}{a^2}\right]w\varepsilon^k$$

$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta(\frac{f''}{f'})^2 - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k - (1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k$$

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k + \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}^k_m$$

$$+ \left[\frac{(3c_s^2 - 1)\mu_m}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c}_s^2 + c_s^2(c_s^2 - w)\Theta\right)\right]\Delta^k_m$$

$$+ \frac{w}{1+w}\dot{R}\dot{\varepsilon}^k + \left[\frac{w\mu_m}{f''} + \frac{2w - c_s^2}{1+w}\dot{R}\Theta + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k = 0$$

Component & relative equations

The density perturbations of the **i**-th component:

$$\ddot{\Delta}_i^k + (\frac{2}{3} - w_i)\Theta\dot{\Delta}_i^k - \frac{1+w_i}{1+w}\left(\dot{R}\frac{f''}{f'} + (c_s^2 - c_{si}^2)\Theta\right)\dot{\Delta}^k + c_{si}^2\frac{k^2}{a^2}\Delta_i^k$$
$$- \frac{1+w_i}{1+w}\left[(1+w)\frac{\mu_m}{f'} - \left(2\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'}\right)c_s^2 + \dot{c}_s^2\Theta\right.$$
$$\left. +(c_s^2 - c_{si}^2)(c_s^2 - w)\Theta^2 - (c_s^2 + w)\dot{R}\Theta\frac{f''}{f'}\right]\Delta^k - \frac{1+w_i}{1+w}w\Theta\dot{\varepsilon}^k$$
$$- \frac{1+w_i}{1+w}\left[(w - c_s^2 - c_{si}^2 w)\Theta^2 - w\left(2\frac{\mu_m}{f'} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'}\right)\right]\varepsilon^k$$
$$+ (1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k + (1+w_i)\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0$$

And the entropy and velocity perturbations:

$$\ddot{S}_{ij}^k = \frac{k^2}{a}\dot{V}_{ij} - \frac{k^2}{3a}\Theta V_{ij}$$
$$\ddot{V}_{ij}^k = (c_z^2 - \frac{1}{3})\Theta\dot{V}_{ij} + \left[\dot{c}_z^2\Theta - (c_z^2 - \frac{1}{3})\left(\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3w)\frac{\mu_m}{f'} + \frac{1}{2}(\mu_R + 3p_R)\right)\right]V_{ij}$$
$$- \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\dot{}_m + \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}(\frac{1}{3} + w - c_s^2)\Theta_m - \frac{c_z^2}{a}\dot{S}_{ij} + \frac{c_z^2\Theta - 3\dot{c}_z^2}{3a}S_{ij}.$$

The background equations

Friedmann $\left\{ H^2 + \frac{K}{a^2} = \frac{1}{3f'}\left\{\frac{1}{2}\left[f'R - f(R)\right] - 3H\dot{f}' + \mu_m\right\} ,\right.$

Raychaudhuri $\left\{ 2\dot{H} + H^2 + \frac{K}{a^2} = -\frac{1}{f'}\left\{\frac{1}{2}\left[f'R - f(R)\right] + \ddot{f}' - 3H\dot{f}' + p_m\right\} ,\right.$

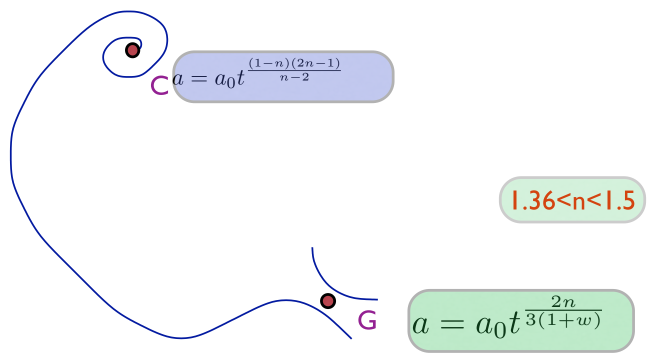
Conservation $\left\{ \dot{\mu}_m + 3H(\mu_m + p_m) = 0 ,\right.$

$$R = -6(2H^2 + \dot{H} + \frac{K}{a^2})$$

.......Any background solutions?

R^n - gravity

... Not yet, but...



- Carloni, Dunsby Capozziello, Troisi (CQG, 2005)
- Carloni, Dunsby, Troisi (PRD 77: 024024)

Background quantities

If we choose point $\mathrm{G}: \ a = a_0 t^{\frac{2n}{3(1+w)}}, \quad K = 0, \quad \mu_m = \mu_0 t^{-2n}$,

then the background quantities will be

$$\Theta = \frac{2n}{(1+w)t}$$

$$R = \frac{4n\left[4n - 3(1+w)\right]}{3(1+w)^2 t^2}$$

$$\mu_R = \frac{2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2 t^2}$$

$$p_R = \frac{2(n-1)\left[n(6w^2+8w-2) - 3w(1+w)\right]}{3(1+w)^2 t^2}$$

$$\mu_m = \left(\frac{3}{4}\right)^{1-n} n\chi \left(\frac{n(4n-3(1+w))}{(1+w)^2 t^2}\right)^{n-1} \frac{4n^2 - 2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2 t^2}$$

Applications to a Radiation/Dust-dominated Universe

- Background model: R^n –gravity
- Non-interacting radiation-dust background mixture
- Flat (K=0) FLRW spacetime
- Conservation equations: $\begin{cases} \dot{\mu}_d + \Theta\mu_d = 0 \\ \dot{\mu}_r + \frac{4}{3}\Theta\mu_r = 0 \end{cases}$
- Equation of state of the mixture: $w = \frac{1}{3}\frac{\mu_r}{\mu_d + \mu_r}$
- Speed of sound in the mixture: $c_s^2 = \frac{4\mu_r}{3(3\mu_d + 4\mu_r)}$

Total equations

$$\ddot{\Delta}_m^k + \left[(c_s^2 + \frac{2}{3} - 2w)\Theta - \dot{R}\frac{f''}{f'}\right]\dot{\Delta}_m^k + \left[\left(\frac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\frac{\mu_m}{f'} + \frac{1}{2}(3w - 5c_s^2)\frac{f}{f'}\right.$$
$$\left. +(c_s^2 - w)\left(2R - 4\dot{R}\Theta\frac{f''}{f'} - \frac{12K}{a^2}\right) + c_s^2\frac{k^2}{a^2}\right]\Delta_m^k - 4wc_z^2\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2} + \frac{k^2}{a^2}\right]S_{dr}^k$$
$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta(\frac{f''}{f'})^2 - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k - (1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k$$

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k + \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}_m^k$$
$$+ \left[\frac{(3c_s^2 - 1)\mu_m}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right)\right]\Delta_m^k$$
$$- \frac{4w}{1+w}c_z^2\dot{R}\dot{S}_{dr}^k - 4wc_z^2\left[\frac{\mu_m}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\frac{f'''}{f''}\right)\right]S_{dr}^k = 0$$

$$\ddot{S}_{dr}^k + (c_s^2 + \frac{1}{3})\Theta\dot{S}_{dr}^k + \frac{k^2}{a^2}c_z^2 S_{dr}^k - \frac{k^2}{a^2}(c_z^2 + \frac{3}{4}c_s^2)\Delta_m^k = 0$$

$$\Delta_m = \frac{\mu_d\Delta_d + \mu_r\Delta_r}{\mu_m}, \qquad S_{dr} = \Delta_d - \frac{3}{4}\Delta_r$$

The short wavelength limit

- The radiation energy density taken to be almost homogeneous:

$$\begin{cases} \Delta_r << \Delta_d \\ c_s^2 \mu_m \Delta_m^k + p_m \varepsilon^k = \frac{1}{3}\mu_r \Delta_r^k \approx 0 \\ S_{dr}^k \approx \Delta_d^k \end{cases}$$

- Radiation affects the growth of density perturbations by speeding up cosmic expansion.
- "Meszaros effect": constraints on n

Radiation-dominated epoch

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0$$

In GR these equations reduce to

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k + \frac{1}{2}\mathcal{R}^k = 0$$

$$\mathcal{R}^k = 0$$

$$\Rightarrow \quad \ddot{\Delta}_d^k + \frac{1}{t}\dot{\Delta}_d^k = 0$$

$$\Downarrow$$

$$\Delta_d^k(t) = C_1 + C_2 \ln t$$

Quasi-static approximation

For $n \neq 1$,with $w = \frac{1}{3}$ in the radiation-dominated epoch,

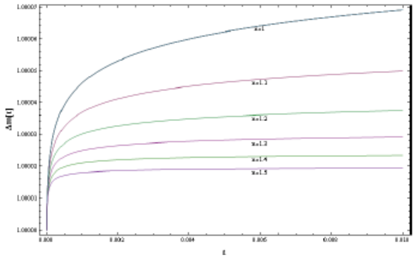
$$\begin{cases} \ddot{\Delta}_d^k + \frac{n}{t}\dot{\Delta}_d^k + \frac{t}{2}\dot{\mathcal{R}}^k + \left[\frac{12-5n}{4} + \frac{n}{12}\left(\frac{\lambda_H}{\lambda}\right)_{eq}^2 t^{2-n}\right]\mathcal{R}^k = 0 \\ \ddot{\mathcal{R}}^k - \left(\frac{5n-16}{2t}\right)\dot{\mathcal{R}}^k + \left[\frac{n^2}{4}\left(\frac{\lambda_H}{\lambda}\right)_{eq}^2 t^{-n} - \frac{6(n-2)}{t^2}\right]\mathcal{R}^k = 0 \end{cases}$$

$$\ddot{\Delta}_d^k + \frac{n}{t}\dot{\Delta}_d^k = 0$$

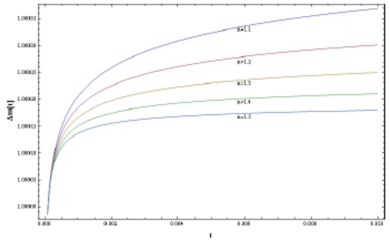
$$\Delta_d^k(t) = C_1 + C_2 t^{1-n}$$

$$\frac{d}{dt}\left[\frac{\Delta_d^k(t)}{a(t)}\right] \propto \frac{d}{dt}\left[\frac{t^{1-n}}{t^{\frac{n}{2}}}\right] < 0 \Rightarrow 1 - \frac{3n}{2} < 0$$

$$n > \frac{2}{3}$$



Analytic



Numerical

Dust-dominated epoch

- In this case

$$\ddot{\Delta}_d^k + \left(2H - \dot{R}\frac{f''}{f'}\right)\dot{\Delta}_d^k - \frac{\mu_d}{f'}\Delta_d^k + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k$$
$$+\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_d}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0$$
$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k - \dot{R}\dot{\Delta}_d^k - \frac{\mu_d}{3f''}\Delta_d^k$$
$$+\left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0$$

The GR equations :

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k - \mu_d\Delta_d^k + \frac{1}{2}\mathcal{R}^k = 0$$
$$-\frac{\mu_d}{3}\Delta_d^k + \frac{1}{3}\mathcal{R}^k = 0$$

$\Rightarrow$

$$\ddot{\Delta}_d^k + \frac{4}{3t}\dot{\Delta}_d^k - \frac{2}{3t^2}\Delta_d^k = 0$$

$\Downarrow$

$$\Delta_d^k(t) = C_1 t^{-1} + C_2 t^{\frac{2}{3}}$$

Quasi-static analysis

For $n \neq 1$ the equations take the form

$$\ddot{\Delta}_d^k + \left(\frac{10n-6}{3t}\right)\dot{\Delta}_d^k + \left(\frac{2(8n^2-13n+3)}{3t^2}\right)\Delta_d^k + \frac{3(n-1)}{2(4n-3)}t\dot{\mathcal{R}}^k + \left[\frac{n(n-1)}{3(4n-3)}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-\frac{4n}{3}} + \frac{27n^2-8n^3-18n}{2n(4n-3)}\right]\mathcal{R}^k = 0$$

$$\ddot{\mathcal{R}}^k + \left[\frac{8n\left([n(8n-13)+3](4n-3)\right)}{27(n-1)t^4}\right]\Delta_d^k + \frac{8n(4n-3)}{3t^3}\dot{\Delta}_d^k + \frac{8-2n}{t}\dot{\mathcal{R}}^k + \left[\frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}} - \frac{2[n(8n+5)-69]+54}{9(n-1)t^2}\right]\mathcal{R}^k = 0$$

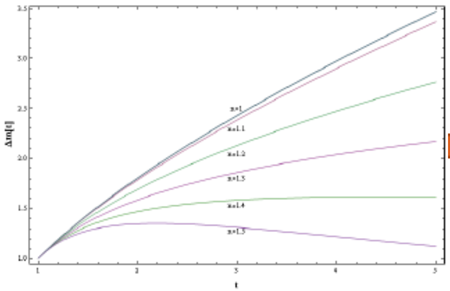
⬇

$$\ddot{\Delta}_d^k + \frac{4n}{3t}\dot{\Delta}_d^k + \left[\frac{4(8n^2-13n+3)}{9t^2}\right]\Delta_d^k = 0$$

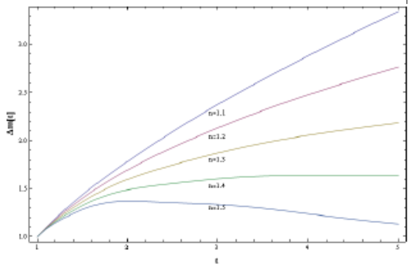
The general solution of the previous equation is

$$\Delta_d^k(t) = C_1 t^{-\frac{2n}{3}+\frac{1}{2}+\frac{\sqrt{-112n^2+184n-39}}{6}} + C_2 t^{-\frac{2n}{3}+\frac{1}{2}-\frac{\sqrt{-112n^2+184n-39}}{6}}$$

Analytic



Numerical



The long wavelength limit

$$\left\{\begin{array}{l}\lambda = \frac{2\pi a}{k} >> \lambda_H \Rightarrow \dfrac{k^2}{a^2} \to 0 \quad \text{Drop all Laplacian terms} \\ \dot{C} = 0. \\ \text{Adiabatic perturbations} : S_{ij} = 0 \; = \dot{S}_{dr} + a\tilde{\nabla}^2 V_{dr} = 0 \end{array}\right.$$

$$\dot{\Delta}_m = \left[\frac{1+w-2n}{1+w} - \frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t} - \frac{3(1+w)^2}{4a_0^2\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^{1-\frac{4n}{3(1+w)}}C_0$$

$$-\frac{9(n-1)(1+w)^3 t^2}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^2\Re + \left[\frac{3(n-1)(1+w)^2\left[n(6w+8)-15(1+w)\right]}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}\right]t\mathcal{R}$$

$$\dot{\mathcal{R}} = \Re + \frac{8nc_s^2\left[4n-3(1+w)\right]}{3(1+w)^3}\frac{\Delta_m}{t^3}$$

$$\dot{\Re} = -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re + \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]}\frac{C_0}{a_0^2}t^{-\frac{4n}{3(1+w)}-2}$$

$$-2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)} + 6\right]\frac{\mathcal{R}}{t^2}$$

$$+\frac{16n\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]\left[\left(9w(1+w)+8\right)n^2-\left(3w(9w+8)+13\right)n+3(1+w)(1+6w)\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}\frac{\Delta_m}{t^4}$$

Radiation-dominated epoch

- In this regime

$$\ddot{\Delta}^k + \frac{n(9n-14)+4}{2(n-2)t}\dot{\Delta}^k + \frac{n\left(n(n(19n-54)+58)-32\right)+8}{2(n-2)^2t^2}\Delta^k$$
$$+\frac{2\left(n(3n-4)+2\right)}{3(n-2)^2}t\dot{\mathcal{R}}^k - \frac{\left(n(15n-22)+14\right)}{3(n-2)}\mathcal{R}^k + \frac{4(n^2-1)}{3(n-2)^2a_0^2}t^{-n}C_0 = 0$$

$$\ddot{\mathcal{R}}^k - \frac{n(11n-32)+32}{2(n-2)t}\dot{\mathcal{R}}^k + \frac{3\left(n(5n-9)+8\right)}{2t^2}\mathcal{R}^k - \frac{3n\left(n(n-3)+2\right)}{2(n-2)t^3}\dot{\Delta}^k$$
$$-\frac{3n(n-1)\left(n(19n-28)+4\right)}{4(n-2)t^4}\Delta^k - \frac{3n(n-1)}{(n-2)a_0^2}t^{-(n+2)}C_0 = 0$$

Decoupled third- order system :

$$\dddot{\Delta}^k - \frac{n-5}{t}\ddot{\Delta}^k + \frac{(n(24-19n)+8)}{4t^2}\dot{\Delta}^k + \frac{(n-2)\left(n(5n-8)+2\right)}{2t^3}\Delta^k - \frac{12-7n}{3a_0^2}t^{-(n+1)}C_0 = 0$$

- The equation admits the solution:

$$\Delta^k(t) = \frac{2\left(24-14n\right)}{9(7n^3-18n^2+16)}t^{2-n}C_0 + C_1t^{\frac{n}{2}-1} + C_2t^{-\frac{1}{2}+\frac{n}{4}+\frac{\sqrt{3(81n^2-44n+12)}}{4}}$$
$$+C_3t^{-\frac{1}{2}+\frac{n}{4}-\frac{\sqrt{3(81n^2-44n+12)}}{4}}$$

Dust-dominated epoch

- The governing equations:

$$\ddot{\Delta}^k + \frac{n(8n-13)+3}{(n-3)t}\dot{\Delta}^k + \frac{(n(8n-13)+3)\,(n(16n-15)+9)}{3(n-3)^2t^2}\Delta^k + \frac{3(n-1)\,(n(16n-15)+9)}{4(n-3)^2(4n-3)}t\dot{\mathcal{R}}^k$$
$$-\frac{n\left[(n(16n(8n-31)+711)-540\right]+189}{4(n-3)^2(4n-3)}\mathcal{R}^k - \frac{n\left(27+54n-56n^2\right)-27)}{4(n-3)^2(4n-3)a_0^2}t^{-\frac{4n}{3}}C_0 = 0$$

$$\ddot{\mathcal{R}}^k - \frac{4(n-1)\,(n(2n-5)+6)}{(n(n-4)+3)\,t}\dot{\mathcal{R}}^k + \frac{4\left[n\,(n(2n(16n-65)+213)-198)+81\right]}{9\,(n(n-4)+3)\,t^2}\mathcal{R}^k$$
$$-\frac{16n(3-4n)^2\,(n(8n-13)+3)}{27\,((n-4)n+3)\,t^4}\Delta^k - \frac{2n\,(n(4n-7)+3)}{(n(n-4)+3)\,a_0^2}t^{-(n+2)}C_0 = 0$$

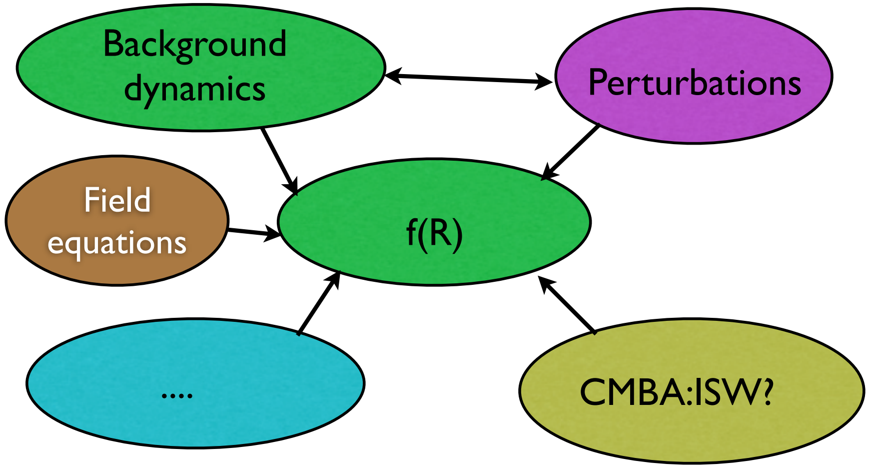
$$\dddot{\Delta}^k + \frac{5}{t}\ddot{\Delta}^k - \frac{2\left[n\,(4n(8n-19)+33)+9\right]}{9(n-1)t^2}\dot{\Delta}^k - \frac{2(4n-3)\,(n(8n-13)+3)}{9(n-1)t^3}\Delta^k$$
$$-\frac{(n(12n-31)+18)}{6(n-1)a_0^2}t^{-(n+1)}C_0 = 0,$$

$$\Delta^k(t) = \frac{9\left[12n^2-31n+18\right]}{8(48n^4-184n^3+159n^2+63n-81)}t^{2-\frac{4n}{3}}C_0 + C_1t^{-1} + C_2t^{-\frac{3n-3+\sqrt{(n-1)(256n^3-608n^2+417n-81)}}{6(n-1)}}$$
$$+ C_3t^{-\frac{3n-3-\sqrt{(n-1)(256n^3-608n^2+417n-81)}}{6(n-1)}}$$

Conclusion

- ✓ 1+3 covariant theory of cosmological perturbations: a good tool kit for f(R)
- ✓ for R^n models in the short wavelength limits, exact solutions found in quasi-static limit for background solutions obtained from dynamical approach to FLRW models
- ✓ Growing modes observed for range of values of n considered
- ✓ Meszaros effect holds; can be used to constrain n
- ✓ The quasi-static approximation: reasonably good
- ✓ Long wavelength analysis of component adiabatic perturbations give the same result as those for single fluid perturbations, deep in their respective era.
- More work ahead!...

Future work...


Background dynamics
Perturbations
Field equations
f(R)
....
CMBA:ISW?

THANK YOU FOR YOUR TIME!