Classical AdS String Dynamics

ANTAL JEVICKI

PHYSICS DEPARTMENT, BROWN UNIVERSITY
PROVIDENCE, RI 02912, USA

- In collaboration with **Ines Aniceto, Kewang Jin**

Outline

- The polygon problem
- Classical string solutions: spiky strings
- Spikes as sinh-Gordon solitons
- AdS string as a σ-model
- Inverse scattering method
- Open string solutions
- Closed string solutions
- N-soliton (spike) construction
- Motion of Singularities: moduli space
- Conclusion and Outlook

1. Motivation

- Semiclassical analysis of strings in $AdS \times S$ space-time is relevant for large $\lambda = g_{YM}^2 N$ (strong coupling) investigation of AdS/CFT;
- Computing gluon scattering amplitudes can be reduced to finding the minimal area of a classical string solution (Alday-Maldacena program);
- Giant magnon solutions on $R \times S^2$ and $R \times S^3$ can be mapped to soliton solutions in sine-Gordon and complex sine-Gordon, respectively.

Euclidean world Sheet : The Polygon Problem

- Alday & Maldacena (2007) outlined a version of Yang-Mills $\longleftrightarrow$ String duality;
- $\mathcal{N}$=4 Super Yang Mills scattering amplitudes can be alternatively evaluated by AdS strings;
- Strong coupling $(\lambda = g_{YM}^2 N)$: Minimal area surface in AdS:

$$ds^2 = R^2 \left[\frac{dx_{3+1}^2 + dz^2}{z^2} \right]$$

Boundary of AdS : $z = z_{IR} \to 0$

Polygon : $(k_1^\mu, k_2^\mu, \cdots, k_n^\mu)$

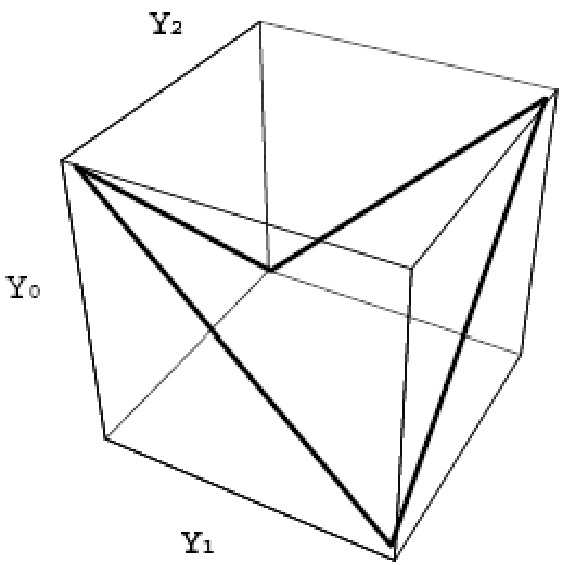
Amplitude : $\mathcal{A}(k_1, k_2, \cdots, k_n) \sim e^{-\text{minimal area}}$

Four-point Solution:

Gauge : $x_1 = \tau$, $x_2 = \sigma$ } Euclidean worldsheet

AdS string action ($z = 1/r$) :

$$S = \frac{R^2}{2\pi}\int dx_1 dx_2 \frac{\sqrt{1+(\partial_i r)^2-(\partial_i y_0)^2-(\partial_1 r\partial_2 y_0-\partial_2 r\partial_1 y_0)^2}}{r^2}$$



$s = t$ case :

$$y_0(x_1,x_2) = x_1 x_2,$$

$$r(x_1,x_2) = \sqrt{(1-x_1^2)(1-x_2^2)}$$

where : $s = -(k_1+k_2)^2 \quad t = -(k_1+k_4)^2$

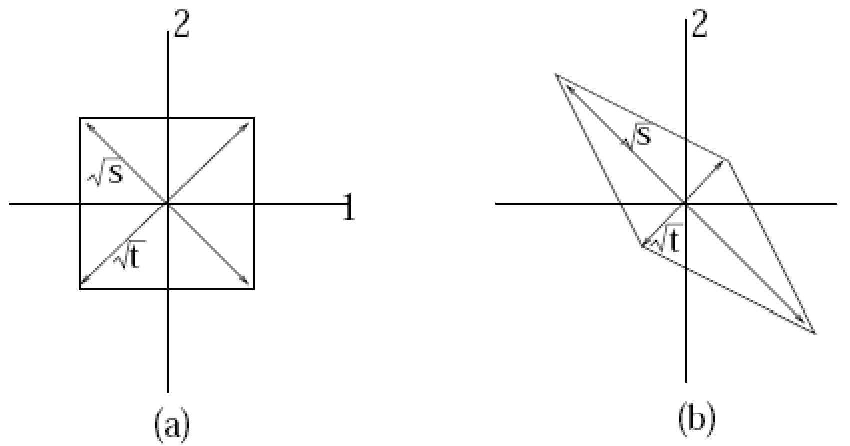
[1] Alday & Maldacena : 0705.0303

The boosted solution:

- Perform a boost in the 04 plane, the solution for s≠t reads:

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad y_0 = \frac{a\sqrt{1+b^2}\sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}$$
$$y_1 = \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad y_2 = \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \tag{3.18}$$

- Projection:



- Area:

$$\mathcal{A} = \mathcal{A}_{tree}(\mathcal{A}_{div,s})^2(\mathcal{A}_{div,t})^2 \exp\left\{\frac{f(\lambda)}{8}[(\log\frac{s}{t})^2 + 4\pi^2/3] + C(\lambda)\right\}$$

- n=8 solution was accomplished in [2] recently.

[2] Alday & Maldacena '09

GKP folded string solution:

Gubser, Klebanov and Polyakov [3] gave a first study of large (spin) angular momentum solutions in conformal gauge.

AdS_3 coordinates : $X^i = (t, \rho, \theta)$

metric : $ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2$

action : $A = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma G_{ij} \partial_\alpha X^i \partial^\alpha X^j$

Virasoro constraints : $T_{++} = \partial_+ X^i \partial_+ X^j G_{ij} = 0$

$T_{--} = \partial_- X^i \partial_- X^j G_{ij} = 0$

Ansatz : $t = c\,\tau$

$\theta = c\,\omega\,\tau$

where c is a constant to rescale the period of σ.

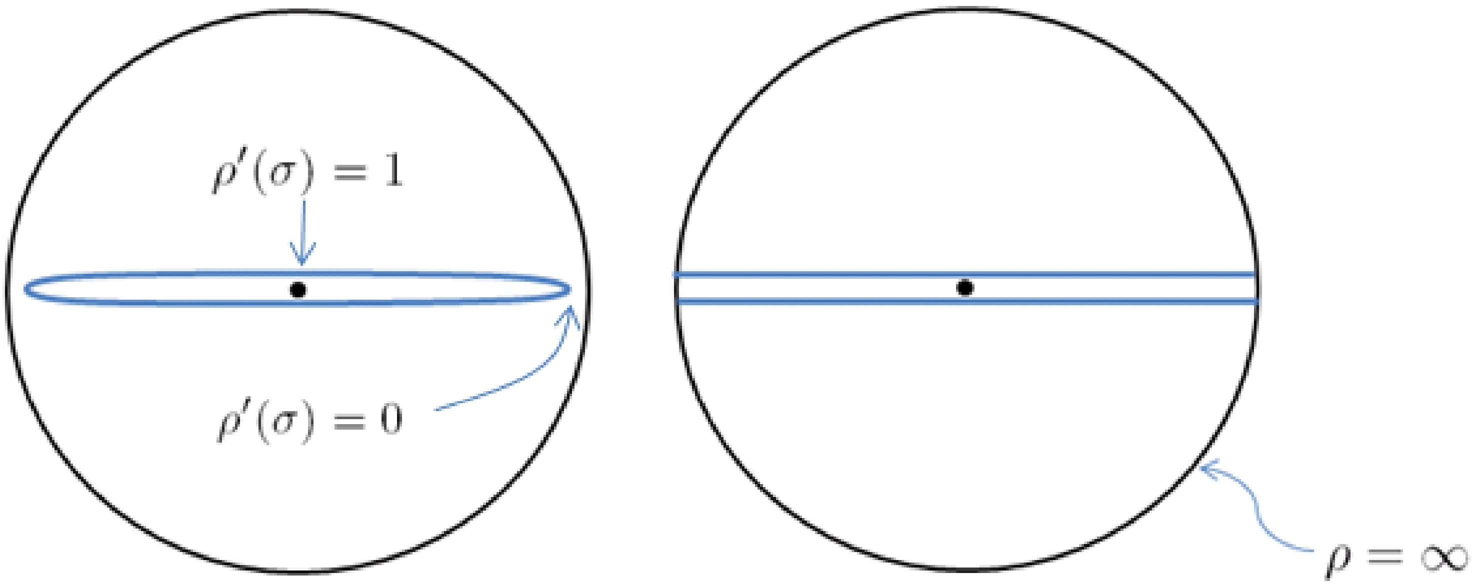
Assumption : $\rho = \rho(\sigma)$ ← rigid rotation

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov '02

Rigid rotating string:

Solution : $$\rho'^2(\sigma) = c^2(\cosh^2\rho - \omega^2\sinh^2\rho)$$

$$\Rightarrow \quad \rho(\sigma) = \operatorname{arccosh}(\operatorname{nd}(\omega\sigma, \frac{1}{\omega}))$$



(a) w>1

(b) w=1

Energy-momentum relation:

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2L} d\sigma \cosh^2\rho = \frac{2\sqrt{\lambda}}{\pi}\Big[\frac{\omega}{\omega^2-1}E(\frac{1}{\omega})\Big],$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2L} d\sigma\ \omega \sinh^2\rho = \frac{2\sqrt{\lambda}}{\pi}\Big[\frac{\omega^2}{\omega^2-1}E(\frac{1}{\omega}) - K(\frac{1}{\omega})\Big].$$

where $\mathrm{E}(\frac{1}{\omega})$ and $\mathrm{K}(\frac{1}{\omega})$ are elliptic functions. Therefore,

$$E - \omega S = \frac{2\omega\sqrt{\lambda}}{\pi}[K(\frac{1}{\omega}) - E(\frac{1}{\omega})]$$

In the large S (spin angular momentum) limit, we have $\omega=1+2\eta$, where $\eta<<1$.

$$\mathrm{E}(\frac{1}{\omega}) \sim 1 + \eta\ln\frac{1}{\eta}, \qquad \mathrm{K}(\frac{1}{\omega}) \sim \frac{1}{2}\ln\frac{1}{\eta}$$

$$E = \frac{\sqrt{\lambda}}{2\pi}(\frac{1}{\eta} + \ln\frac{1}{\eta} + \cdots)$$

$$S = \frac{\sqrt{\lambda}}{2\pi}(\frac{1}{\eta} - \ln\frac{1}{\eta} + \cdots)$$

$\Longrightarrow$

$$\boxed{E - S = \frac{\sqrt{\lambda}}{\pi}\ln(\frac{S}{\sqrt{\lambda}}) + \cdots}$$

Spiky string solution:

Kruczenski [4] gave the spiky string solutions in physical gauge:

Ansatz : $t = \tau$

$\theta = \omega\,\tau + \sigma$

rigid rotation : $\rho = \rho\,(\sigma)$

Nambu-Goto action : $$A = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2}$$

$$= \sqrt{\rho'^2(\cosh^2\rho - \omega^2\sinh^2\rho) + \sinh^2\rho\cosh^2\rho}$$

Spiky string solution:

$$\rho'(\sigma) = \frac{1}{2}\frac{\sinh 2\rho}{\sinh 2\rho_0}\frac{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

where ρ_0 is the minimum value of ρ; the maximum value is ρ_1=arccoth ω.

$$\sigma = \frac{\sinh 2\rho_0}{\sqrt{2}\sqrt{u_0+u_1}\sinh\rho_1}\left\{\Pi(\alpha,\frac{u_1-u_0}{u_1-1},p) - \Pi(\alpha,\frac{u_1-u_0}{u_1+1},p)\right\}$$

[4] M. Kruczenski '04

Energy-momentum relation:

$$E = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho\, \frac{\cosh^2\rho\sinh^2 2\rho - \omega^2\sinh^2\rho\sinh^2 2\rho_0}{\sinh 2\rho\sqrt{(\cosh^2\rho - \omega^2\sinh^2\rho)(\sinh^2 2\rho - \sinh^2 2\rho_0)}}$$

$$S = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho\, \frac{\omega}{2}\frac{\sinh\rho}{\cosh\rho}\frac{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

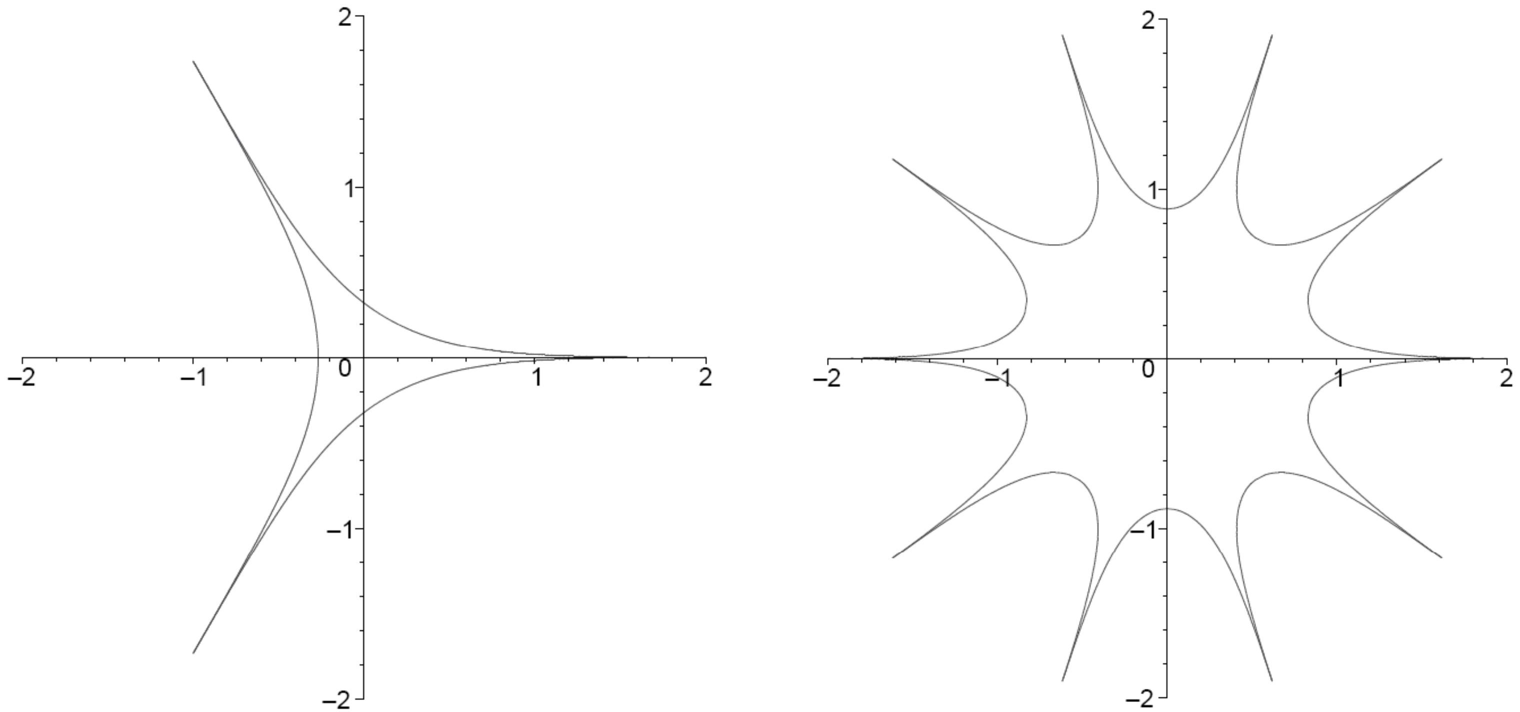
$$E - \omega S = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho\, \sinh 2\rho\frac{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}$$

In the limit $\rho_1 \gg 1$ and $\rho_1 \gg \rho_0$, we have $\omega = \coth\rho_1 \to 1$

Large S energy of n-spike solution: $E - S = n\frac{\sqrt{\lambda}}{2\pi}\ln\frac{S}{\sqrt{\lambda}} + \cdots$

Note : n=2 agrees with the GKP solution.

Spiky strings in AdS



- The main interest is to study the dynamics of spikes
- For this purpose, it is convenient to introduce the soliton picture
- We will show next the soliton picture of the GKP solution
- The same argument works for the Kruczenski n-spike solution

Kruczenski's solution in conformal gauge

ansatz :

$$\begin{aligned} t &= \tau + f(\sigma), \\ \theta &= \omega\tau + g(\sigma), \\ \rho &= \rho(\sigma). \end{aligned}$$

The equations of motion and the Virasoro constraints can be solved by :

$$f'(\sigma) = \frac{\omega \sinh 2\rho_0}{2\cosh^2\rho}, \qquad g'(\sigma) = \frac{\sinh 2\rho_0}{2\sinh^2\rho},$$

$$\rho'^2(\sigma) = \frac{(\cosh^2\rho - \omega^2\sinh^2\rho)(\sinh^2 2\rho - \sinh^2 2\rho_0)}{\sinh^2 2\rho}.$$

Near the spike, we have $\rho \sim \rho_1 \equiv \mathrm{arccoth}\omega$, further assume $\rho_1 \gg \rho_0$,

$$\rho'^2(\sigma) = \cosh^2\rho - \omega^2\sinh^2\rho \qquad \text{(GKP solution)}$$

Therefore, the n-spike configuration is a n-soliton solution in sinh-Gordon picture.

Exact transformation

$$\rho = \frac{1}{2}\operatorname{arccosh}\left(\cosh 2\rho_1 \mathrm{cn}^2(\mathrm{u},\mathrm{k}) + \cosh 2\rho_0 \mathrm{sn}^2(\mathrm{u},\mathrm{k})\right)$$

where : $u \equiv \sqrt{\frac{\cosh 2\rho_1 + \cosh 2\rho_0}{\cosh 2\rho_1 - 1}}\sigma, \quad k \equiv \sqrt{\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + \cosh 2\rho_0}},$

The gauge transformation functions are :

$$f = \frac{\sqrt{2}\omega \sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 + 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}}\Pi\left(\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + 1}, x, k\right)$$

$$g = \frac{\sqrt{2} \sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 - 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}}\Pi\left(\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 - 1}, x, k\right)$$

where : $x = \mathrm{am}(u, k)$

2. Spikes as sinh-Gordon solitons

Asymptotics near the turning point: GKP solution

$$\rho'^2 \quad = \quad \cosh^2\rho - \omega^2 \sinh^2\rho \quad \sim \quad \frac{1}{4}e^{2\rho}(1-\omega^2+(1+\omega^2)2e^{-2\rho})$$

Let $\omega = 1 + 2\eta$ where $\eta \ll 1$, then one gets

$$\rho'^2 \sim e^{2\rho}(e^{-2\rho} - \eta)$$

Denote $u = e^{-\rho}$, we have

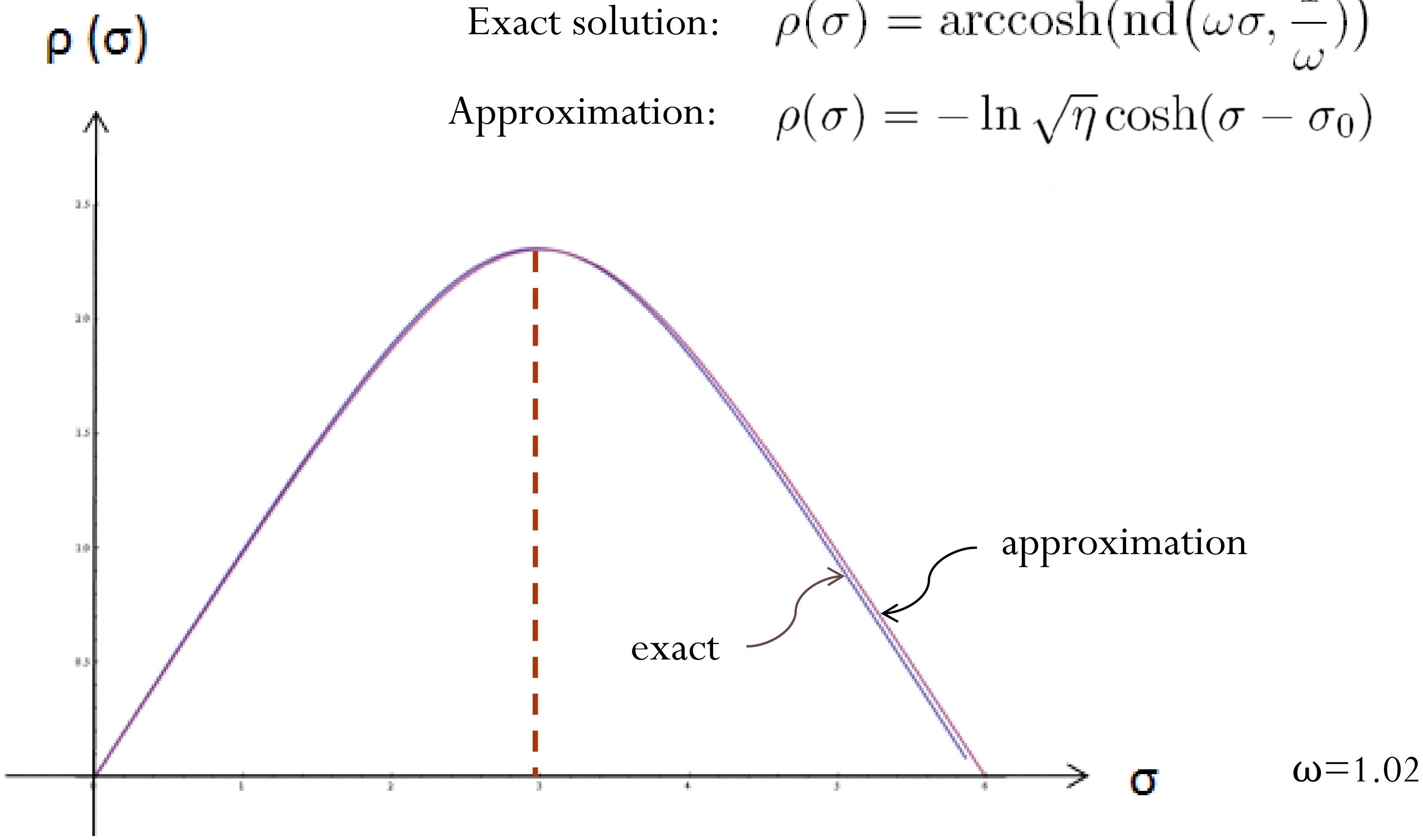
$$u'^2 \sim u^2 - \eta$$

$$\Longrightarrow \quad \boxed{\rho(\sigma) = -\ln\sqrt{\eta}\cosh(\sigma - \sigma_0)}$$

Near-spike approximation

Exact solution: $\rho(\sigma) = \text{arccosh}(\text{nd}(\omega\sigma, \frac{1}{\omega}))$

Approximation: $\rho(\sigma) = -\ln\sqrt{\eta}\cosh(\sigma - \sigma_0)$



Relation to Sinh-Gordon soliton:

One observes the correspondence with the sinh-Gordon soliton.

Define : $\alpha \equiv \ln(q_\xi \cdot q_\eta)$

where q being a AdS_3 string solution with signature: {-1, -1, +1, +1}.

One can check, that for the near turning point GKP solution,

$$\alpha = \ln(2\rho'^2) = \ln(2\tanh^2\sigma) = \ln 2 + \hat{\alpha}$$

satisfies the sinh-Gordon equation:

$$\hat{\alpha}_{\xi\eta} - 4\sinh\hat{\alpha} = 0$$

Therefore, the finite GKP solution is then a two-soliton configuration of sinh-Gordon system !

$$\xi = (\sigma + \tau)/2$$

$$\eta = (\sigma - \tau)/2$$

3. AdS string as a σ-model

We parameterize AdS_d with $d+1$ embedding coordinates q subject to the constraint

$$q^2 = -q_{-1}^2 - q_0^2 + q_1^2 + q_2^2 + \cdots + q_{d-1}^2 = -1$$

Conformal gauge action :

$$A = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left(\partial q \cdot \partial q + \lambda(\sigma, \tau)(q \cdot q + 1)\right)$$

where τ and σ are Minkowski worldsheet coordinates.

Equations of motion : $q_{\xi\eta} - (q_\xi \cdot q_\eta) q = 0$

Virasoro constraints : $q_\xi^2 = q_\eta^2 = 0$

$$\xi = (\sigma + \tau)/2 \qquad \partial_\xi = \partial_\sigma + \partial_\tau$$

$$\eta = (\sigma - \tau)/2 \qquad \partial_\eta = \partial_\sigma - \partial_\tau$$

Equivalence to sinh-Gordon model

Choose a basis : $e_i = (q, q_\xi, q_\eta, b_4, \cdots, b_{d+1})$

where i=1,2, … , d+1 and the vectors b_k with k=4,5, … , d+1 are orthonormal

$$b_k \cdot b_l = \delta_{kl}, \quad b_k \cdot q = b_k \cdot q_\xi = b_k \cdot q_\eta = 0$$

Define: $\alpha \equiv \ln(q_\xi \cdot q_\eta) \quad u_k \equiv b_k \cdot q_{\xi\xi} \quad v_k \equiv b_k \cdot q_{\eta\eta}$

The equations of motion are :

$$\alpha_{\xi\eta} - e^\alpha - e^{-\alpha} \sum_{i=4}^{d+1} u_i v_i = 0 \quad (u_i)_\eta = \sum_{j=4, j\neq i}^{d+1} u_j (b_j) \cdot (b_i)_\eta \quad (v_i)_\xi = \sum_{j=4, j\neq i}^{d+1} v_j (b_j) \cdot (b_i)_\xi$$

Generalized sinh-Gordon model [5].

d=2: Liouville equation   d=3: sinh-Gordon equation   d=4: B_2 Toda model

[5] H. J. de Vega and N. Sanchez, PRD, **47**, 3394 (1993).

AdS$_3$ case in more detail

$$u_\eta = 0 \Rightarrow u = u(\xi)$$

$$v_\xi = 0 \Rightarrow v = v(\eta)$$

$$\alpha_{\xi\eta} - e^\alpha - uve^{-\alpha} = 0 \qquad \hat{\alpha}_{\xi'\eta'} - 2\sinh\hat{\alpha} = 0$$

$$\frac{d\xi'}{d\xi} = \sqrt{u(\xi)} \qquad \frac{d\eta'}{d\eta} = \sqrt{-v(\eta)} \qquad \alpha(\xi,\eta) = \hat{\alpha}(\xi',\eta') + \frac{1}{2}\ln[-u(\xi)v(\eta)]$$

Now we express the derivatives of the basis vectors in terms of the basis itself :

$$\frac{\partial e_i}{\partial \xi} = A_{ij}(\xi,\eta)e_j, \qquad \frac{\partial e_i}{\partial \eta} = B_{ij}(\xi,\eta)e_j$$

we get : $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha_\xi & 0 & u \\ e^\alpha & 0 & 0 & 0 \\ 0 & 0 & -ue^{-\alpha} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ e^\alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha_\eta & v \\ 0 & -ve^{-\alpha} & 0 & 0 \end{pmatrix}$

SO(2,2) symmetry

In order to see the explicit SO(2,2) symmetry, we choose an orthonormal basis

$$e_1 = b, \quad e_2 = \frac{q_\xi + q_\eta}{\sqrt{2}e^{\alpha/2}}, \quad e_3 = \frac{q_\xi - q_\eta}{\sqrt{2}ie^{\alpha/2}}, \quad e_4 = iq.$$

Then A, B matrices become

$$A = \begin{pmatrix} 0 & -\frac{u}{\sqrt{2}}e^{-\alpha/2} & \frac{iu}{\sqrt{2}}e^{-\alpha/2} & 0 \\ \frac{u}{\sqrt{2}}e^{-\alpha/2} & 0 & \frac{i}{2}\alpha_\xi & -\frac{i}{\sqrt{2}}e^{\alpha/2} \\ -\frac{iu}{\sqrt{2}}e^{-\alpha/2} & -\frac{i}{2}\alpha_\xi & 0 & \frac{1}{\sqrt{2}}e^{\alpha/2} \\ 0 & \frac{i}{\sqrt{2}}e^{\alpha/2} & -\frac{1}{\sqrt{2}}e^{\alpha/2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\frac{v}{\sqrt{2}}e^{-\alpha/2} & -\frac{iv}{\sqrt{2}}e^{-\alpha/2} & 0 \\ \frac{v}{\sqrt{2}}e^{-\alpha/2} & 0 & -\frac{i}{2}\alpha_\eta & -\frac{i}{\sqrt{2}}e^{\alpha/2} \\ \frac{iv}{\sqrt{2}}e^{-\alpha/2} & \frac{i}{2}\alpha_\eta & 0 & -\frac{1}{\sqrt{2}}e^{\alpha/2} \\ 0 & \frac{i}{\sqrt{2}}e^{\alpha/2} & \frac{1}{\sqrt{2}}e^{\alpha/2} & 0 \end{pmatrix}.$$

4. Inverse Scattering Method

Remember the isometry :

$$SO(2,2) = SO(2,1) \times SO(2,1)$$

Introduce two commuting sets of SO(2,1) generators :

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_3, \quad [K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3, \quad [J_i, K_j] = 0,$$

Expand A, B matrices as

$$A = w^i_{1,(+)} J_i + w^i_{1,(-)} K_i, \qquad B = w^i_{2,(+)} J_i + w^i_{2,(-)} K_i,$$

with coefficients

$$\vec{w}_{1,(\pm)} = \Big(\frac{i}{2}\alpha_\xi, \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \mp e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \pm e^{\alpha/2}\Big),$$

$$\vec{w}_{2,(\pm)} = \Big(\frac{-i}{2}\alpha_\eta, \frac{i}{\sqrt{2}}(ve^{-\alpha/2} \pm e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ve^{-\alpha/2} \mp e^{\alpha/2}\Big).$$

Spinor representation

Remember SO(2,1)=SU(1,1), we can define two spinors as

$$\phi_\xi = w^i_{1,(+)}\sigma_i\phi = A_1\phi, \qquad \phi_\eta = w^i_{2,(+)}\sigma_i\phi = A_2\phi,$$
$$\psi_\xi = w^i_{1,(-)}\sigma_i\psi = B_1\psi, \qquad \psi_\eta = w^i_{2,(-)}\sigma_i\psi = B_2\psi.$$

where the matrices are given by

$$A_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) \\ -\frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) & -\frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) \\ \frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) \\ -\frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) & -\frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) \\ \frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) \end{pmatrix}.$$

Reconstructing the string solution:

Then the string solution is given by:

$$
\begin{aligned}
q_{-1} &= \frac{1}{2}(\phi_1\psi_1^* - \phi_2\psi_2^*) + c.c. \qquad & q_0 &= \frac{i}{2}(\phi_1\psi_1^* - \phi_2\psi_2^*) + c.c. \\
q_1 &= \frac{1}{2}(\phi_2\psi_1 - \phi_1\psi_2) + c.c. \qquad & q_2 &= \frac{i}{2}(\phi_2\psi_1 - \phi_1\psi_2) + c.c.
\end{aligned}
$$

5. Open string solutions

Vacuum solution: $u = 2, v = -2, \alpha_0 = \ln 2,$

Matrices :

$$A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad A_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Spinors :

$$\phi_1 = e^{-i\tau} \quad \phi_2 = 0 \quad \psi_1 = \cosh\sigma \quad \psi_2 = -\sinh\sigma.$$

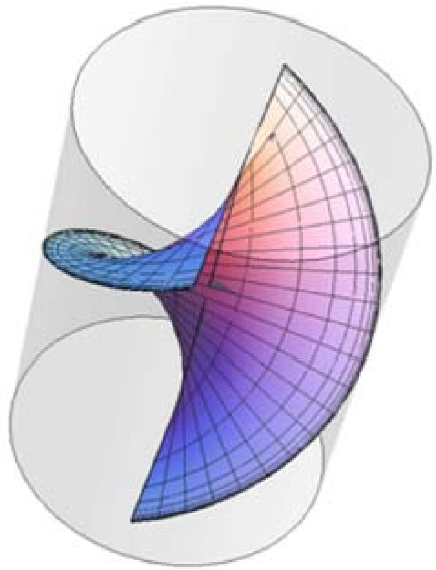
String solution :

$$q = \begin{pmatrix} \cosh\sigma\cos\tau \\ \cosh\sigma\sin\tau \\ \sinh\sigma\cos\tau \\ \sinh\sigma\sin\tau \end{pmatrix}$$

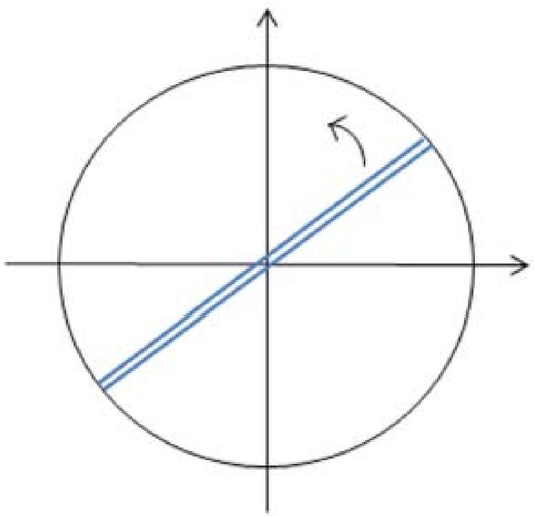
[6] A. Jevicki, K. Jin, C. Kalousios and A. Volovich: 0712.1193.

Vacuum :

$$t = \tau$$
$$\theta = \tau$$
$$\rho = \sigma$$
$$\omega = 1$$



(a)



(b)

$$E = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \cosh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda},$$
$$S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \sinh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda},$$

$$E - S \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{4\pi}{\sqrt{\lambda}}S$$

One-soliton solution:

Sinh-Gordon : $\alpha_s = \ln(2\tanh^2\sigma)$

Spinors :

$$\phi_1 = e^{-i\tau}\cosh(\frac{1}{2}\ln\tanh\sigma),$$
$$\phi_2 = -e^{-i\tau}\sinh(\frac{1}{2}\ln\tanh\sigma),$$
$$\psi_1 = (\tau+i)\cosh(\frac{1}{2}\ln\sinh 2\sigma) - \tau\sinh(\frac{1}{2}\ln\sinh 2\sigma),$$
$$\psi_2 = -(\tau+i)\sinh(\frac{1}{2}\ln\sinh 2\sigma) + \tau\cosh(\frac{1}{2}\ln\sinh 2\sigma).$$

Linear !

String solution :

$$q_s = \frac{1}{2\sqrt{2}\cosh\sigma}\begin{pmatrix} 2\tau\cos\tau - \sin\tau(\cosh 2\sigma + 2) \\ 2\tau\sin\tau + \cos\tau(\cosh 2\sigma + 2) \\ -2\tau\cos\tau + \sin\tau\cosh 2\sigma \\ -2\tau\sin\tau - \cos\tau\cosh 2\sigma \end{pmatrix}$$

Energy of one-soliton solution

$$\mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{16\pi}(1+8\tau^2+4\cosh 2\sigma+\cosh 4\sigma)\,\mathrm{sech}^2\,\sigma \qquad \mathcal{P}_t^\sigma = \frac{\sqrt{\lambda}}{\pi}\tau\tanh\sigma$$

$$\mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{16\pi}(1+8\tau^2-4\cosh 2\sigma+\cosh 4\sigma)\,\mathrm{sech}^2\,\sigma \qquad \mathcal{P}_\theta^\sigma = \frac{\sqrt{\lambda}}{\pi}\tau\tanh\sigma$$

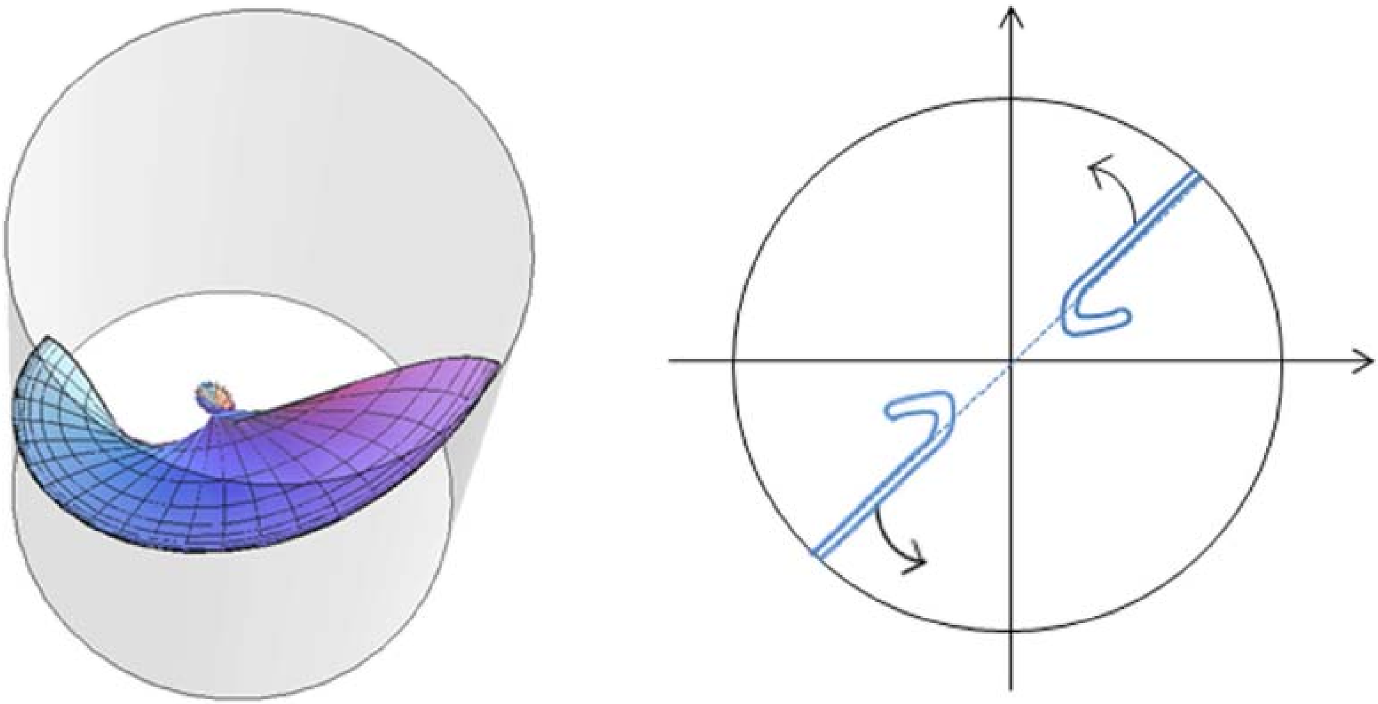
$$E = \int_{-\Lambda}^{\Lambda} d\sigma \mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{\pi}\left(\frac{1}{4}\sigma+\frac{1}{8}\sinh 2\sigma-\frac{1}{8}\tanh\sigma+\frac{1}{2}\tau^2\tanh\sigma\right)\Big|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{\pi}\left(\frac{1}{8}e^{2\Lambda}+\tau^2\right)$$

$$S = \int_{-\Lambda}^{\Lambda} d\sigma \mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{\pi}\left(-\frac{3}{4}\sigma+\frac{1}{8}\sinh 2\sigma+\frac{3}{8}\tanh\sigma+\frac{1}{2}\tau^2\tanh\sigma\right)\Big|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{\pi}\left(\frac{1}{8}e^{2\Lambda}+\tau^2\right)$$

If we neglect the τ dependence since the exponential term increases much faster,

$$\Rightarrow \quad \boxed{E - S = \int_{-\Lambda}^{\Lambda} \frac{\sqrt{\lambda}}{2\pi}\cosh 2\sigma\,\mathrm{sech}^2\,\sigma\, d\sigma \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{8\pi}{\sqrt{\lambda}}S}$$

One-soliton solution



(a) (b)

Two-soliton solution

sinh-Gordon : $\alpha_{ss} = \ln 2\left(\frac{v\cosh X - \cosh T}{v\cosh X + \cosh T}\right)^2$

where $X = \frac{2\sigma}{\sqrt{1-v^2}}$, $T = \frac{2v\tau}{\sqrt{1-v^2}}$, and v is the relative velocity of two solitons.

$$q = \frac{1}{\cosh T + v\cosh X}\begin{pmatrix} (v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{sh}\sigma)\cos\tau + \gamma^{-1}\mathrm{sh}T\mathrm{ch}\sigma\sin\tau \\ -(v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{sh}\sigma)\sin\tau + \gamma^{-1}\mathrm{sh}T\mathrm{ch}\sigma\cos\tau \\ (v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{ch}\sigma)\cos\tau + \gamma^{-1}\mathrm{sh}T\mathrm{sh}\sigma\sin\tau \\ -(v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{ch}\sigma)\sin\tau + \gamma^{-1}\mathrm{sh}T\mathrm{sh}\sigma\cos\tau \end{pmatrix}$$

$$E = \int d\sigma \mathcal{P}_t^\tau = \frac{1}{8v(\cosh T + v\cosh X)}\Big\{-4v^2\gamma^{-1}\sinh X - (2 - v^2 + 2\gamma^{-1})\sinh(2-2\gamma)\sigma$$
$$+ (2 - v^2 - 2\gamma^{-1})\sinh(2+2\gamma)\sigma + 2v\cosh T(2\sigma + \sinh 2\sigma) + 4v^2\sigma\cosh X\Big\}$$

$$S = \int d\sigma \mathcal{P}_\theta^\tau = \frac{1}{8v(\cosh T + v\cosh X)}\Big\{4v^2\gamma^{-1}\sinh X + (2 - v^2 + 2\gamma^{-1})\sinh(2-2\gamma)\sigma$$
$$+ (2 - v^2 - 2\gamma^{-1})\sinh(2+2\gamma)\sigma + 2v\cosh T(-2\sigma + \sinh 2\sigma) - 4v^2\sigma\cosh X\Big\}$$

$$\boxed{E - S = \int d\sigma(\mathcal{P}_t^\tau - \mathcal{P}_\theta^\tau) = \sigma - \frac{v\gamma^{-1}\sinh X}{\cosh T + v\cosh X}}$$

where $\gamma = (1 - v^2)^{-1/2}$.

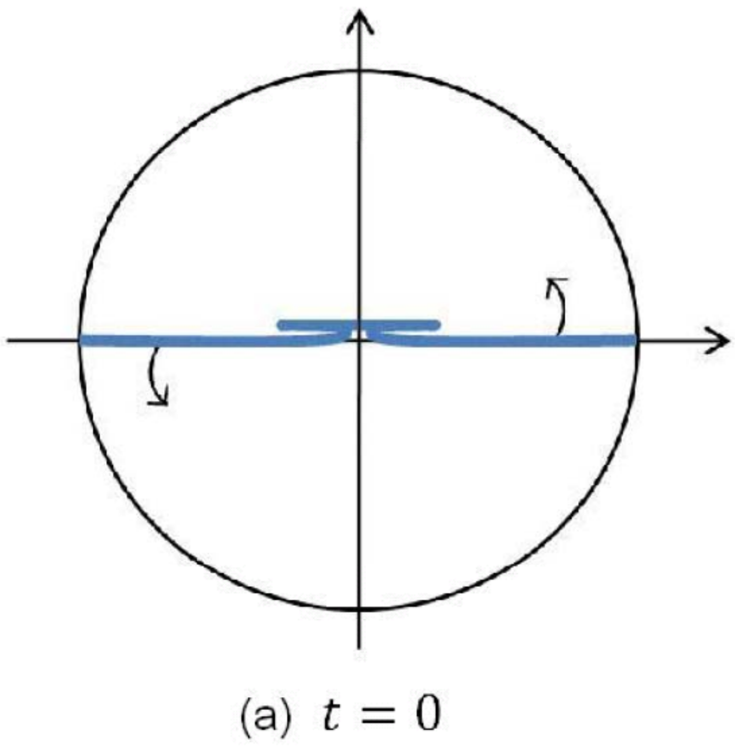


Figure 3: The Minkowskian two-soliton solution with $v = \frac{1}{\sqrt{5}}$ at different global time (a) $t = 0$, (b) $t = \pi/4$. The thick line denotes double-string.

Note: Solitons are localized near the center of AdS space.

Properties of the soliton solutions

- Solitons (spikes) are located in the bulk of AdS
- Near the boundary the solution reduces to vacuum
- These solutions defined on an open line (ω=1) are simple but not fully satisfactory :
  1) Energy is not conserved because there is momentum flow at the asymptotic ends of the string
  2) String is not closed
- To make the physical quantities conserved, and also to clarify the ω=1 limit, we need to build string solutions on a closed circle.

6. Closed string solutions

$$\hat{\alpha}_{\xi\eta} - 2\sqrt{-uv}\sinh\hat{\alpha} = 0$$

$$u = 2, v = -2$$

$$\hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)] \Rightarrow q_1 = \begin{pmatrix} \frac{1}{\sqrt{1-k^2}}\mathrm{dn}(\frac{\sigma}{\sqrt{k}}, k)\cos\sqrt{k}\tau \\ \frac{1}{\sqrt{1-k^2}}\mathrm{dn}(\frac{\sigma}{\sqrt{k}}, k)\sin\sqrt{k}\tau \\ \frac{k}{\sqrt{1-k^2}}\mathrm{cn}(\frac{\sigma}{\sqrt{k}}, k)\cos\frac{1}{\sqrt{k}}\tau \\ \frac{k}{\sqrt{1-k^2}}\mathrm{cn}(\frac{\sigma}{\sqrt{k}}, k)\sin\frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

Period : $L = 2\sqrt{k}K(k) \quad (\ 0 < k < 1\ )$

k=1 limit : $\hat{\alpha}_{1,k=1} = \ln[\tanh^2\sigma]$

$$\Rightarrow q_{1,k=1} = \frac{1}{2\sqrt{2}\cosh\sigma}\begin{pmatrix} 2\tau\cos\tau - \sin\tau(\cosh 2\sigma + 2) \\ 2\tau\sin\tau + \cos\tau(\cosh 2\sigma + 2) \\ -2\tau\cos\tau + \sin\tau\cosh 2\sigma \\ -2\tau\sin\tau - \cos\tau\cosh 2\sigma \end{pmatrix}$$

[7] A. Jevicki and K. Jin: 0804.0412 .

Another solution ?

$$\hat{\alpha}_2 = \ln[k\ \mathrm{cn}^2(\frac{\sigma}{\sqrt{k}}, k)\ \mathrm{nd}^2(\frac{\sigma}{\sqrt{k}}, k)]$$

$$\Rightarrow \quad q_2 = \begin{pmatrix} \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \cos\sqrt{k}\tau \\ \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \sin\sqrt{k}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \cos\frac{1}{\sqrt{k}}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \sin\frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

k=1 limit :

$$\hat{\alpha}_{2,k=1} = 0 \quad \Rightarrow \quad q_{2,k=1} = \begin{pmatrix} \cosh\sigma\cos\tau \\ \cosh\sigma\sin\tau \\ \sinh\sigma\cos\tau \\ \sinh\sigma\sin\tau \end{pmatrix}$$

Relation of two solutions: σ translation

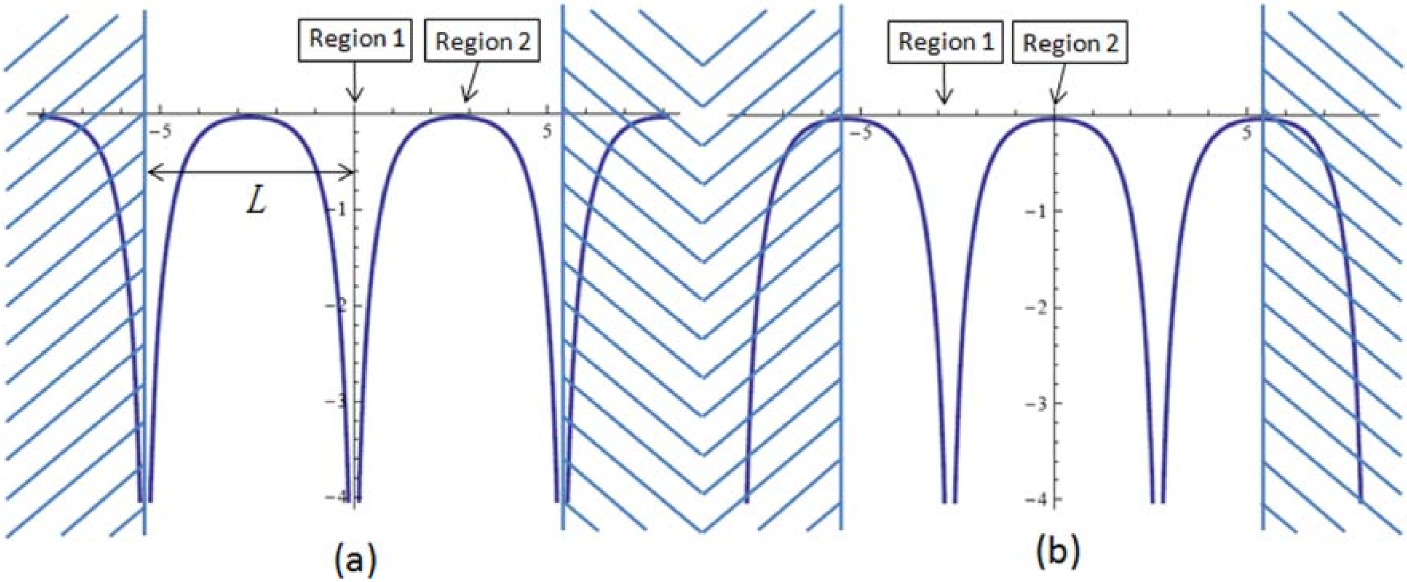


Fig. 1. (a) First periodic sinh-Gordon solution $\hat{\alpha}_1$ when $k = 0.964$; (b) Second periodic sinh-Gordon solution $\hat{\alpha}_2$ when $k = 0.964$. They are related by a translation of $\sigma \to \sigma + \sqrt{k}K(k)$.

$$\hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)] \qquad \hat{\alpha}_2 = \ln[k\ \mathrm{cn}^2(\frac{\sigma}{\sqrt{k}}, k)\ \mathrm{nd}^2(\frac{\sigma}{\sqrt{k}}, k)]$$

$$\sigma \to \sigma + \sqrt{k}K(k)$$

Reduction to the GKP solution

$$\begin{pmatrix} \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \cos \sqrt{k}\tau \\ \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \sin \sqrt{k}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \cos \frac{1}{\sqrt{k}}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \sin \frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

Do the rescaling $\sqrt{k}\tau \to \tau, \sqrt{k}\sigma \to \sigma$ and write $k = 1/\omega$,

$$\Rightarrow \quad \boxed{\rho'^2(\sigma) = \cosh^2 \rho - \omega^2 \sinh^2 \rho}$$

This is exactly the GKP solution.
Folded rotating string along a straight line !

Therefore, the energy reads : $E - S = \frac{\sqrt{\lambda}}{\pi} \ln(\frac{S}{\sqrt{\lambda}}) + \cdots$

7. N-soliton (spike) construction

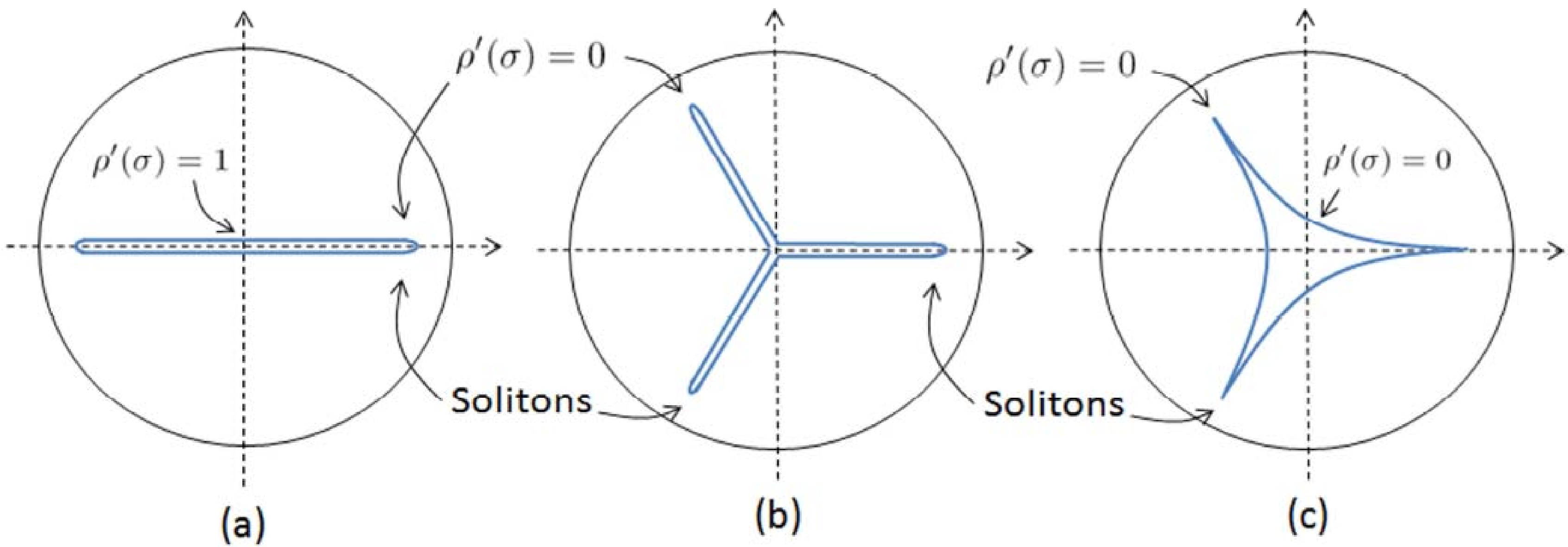


Fig. 2. (a) GKP two-soliton configuration plotted in the plane $x = \rho\cos\theta, y = \rho\sin\theta$ where ρ, θ are the global coordinates; (b) A attempt to construct the GKP type three-soliton solution; (c) Kruczenski's three-spike string solution.

ρ as a function of σ

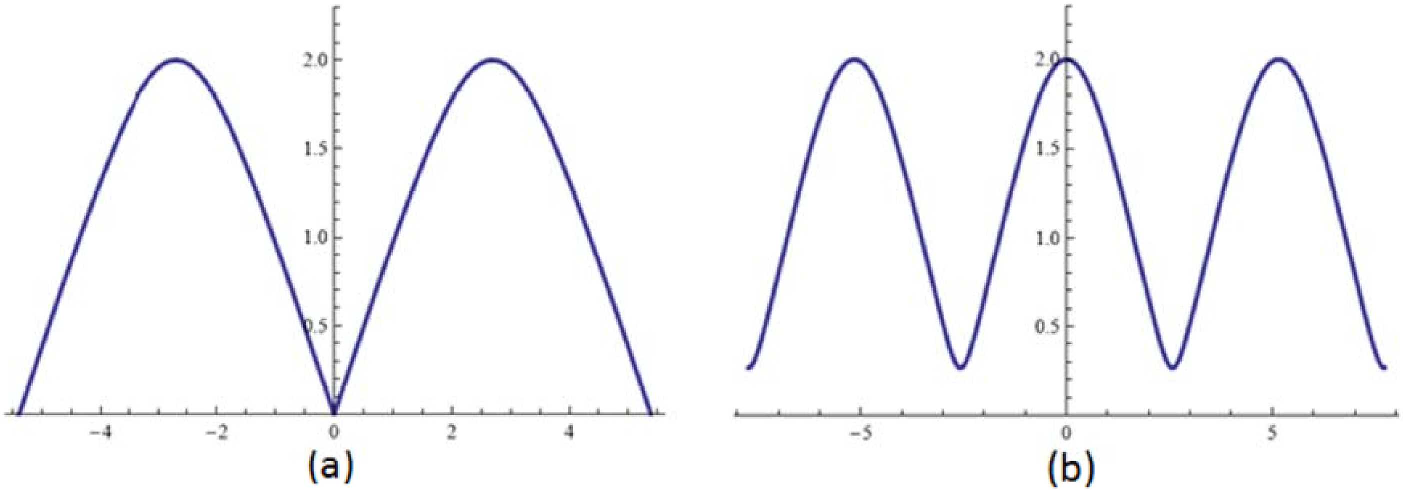
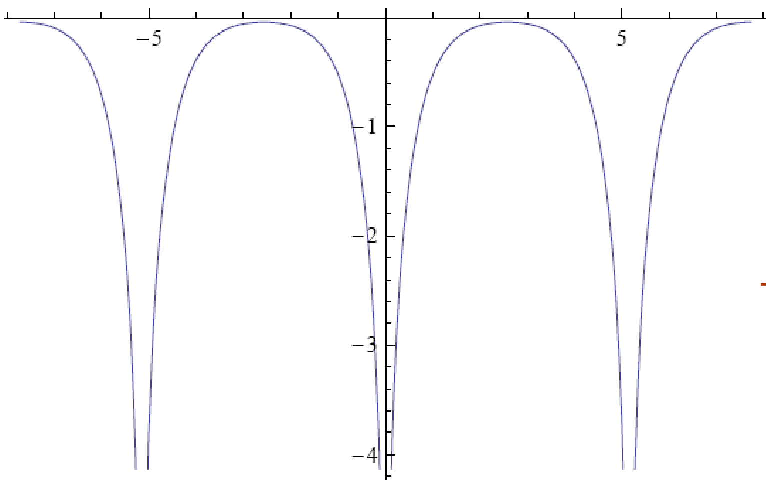


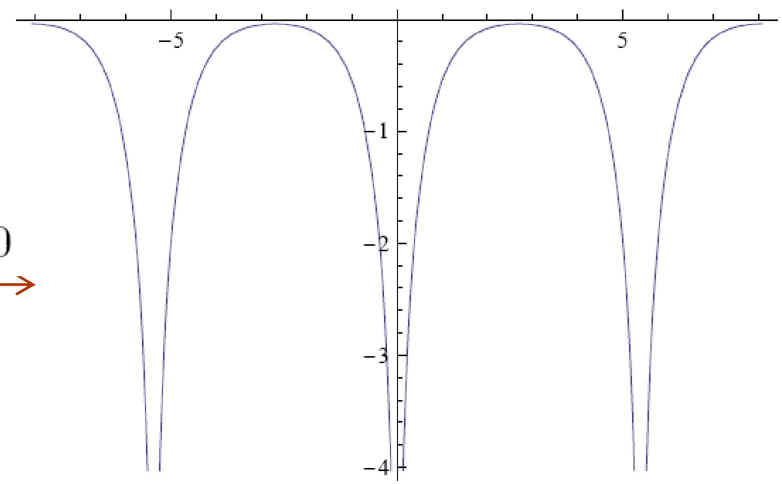
Fig. 3. (a) GKP ρ as a function of σ when $k = 0.964$; (b) Kruczenski ρ as a function of σ when $\rho_1 = 2, \rho_0 = 0.2688735$.

Sinh-Gordon picture

$$\hat{\alpha} = \ln\Big[\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\sqrt{\sinh^2 2\rho_1 - \sinh^2 2\rho_0}} \mathrm{sn}^2(u,k)\Big] \quad \xrightarrow{\rho_0 = 0} \quad \hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)]$$



$\rho_0 = 0$ →



There is a tiny shift along y axis and a tiny expansion of the period : nonzero b.c.

Similarly, there is a σ shifted solution reducing to $\hat{\alpha}_2$ in the limit of $\rho_0 = 0$.

> In this sense, we say Kruczenski's solution is a generalization of the GKP solution by lifting the minimum value of ρ.

N-spike solution

Sinh-Gordon :

$$\varphi_1(\zeta,z,\bar z) = -\Big(\sum_{j,l=1}^{N}\frac{\lambda_j}{\zeta+\zeta_j}(1-A)^{-1}_{jl}\lambda_l\Big)e^{i\zeta\bar z - iz/4\zeta},$$

$$\varphi_2(\zeta,z,\bar z) = \Big(1+\sum_{j,l,k=1}^{N}\frac{\lambda_j}{\zeta+\zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j+\zeta_l}(1-A)^{-1}_{lk}\lambda_k\Big)e^{i\zeta\bar z - iz/4\zeta},$$

where $A_{ij} = \sum_l a_{il}a_{lj}, \qquad a_{il} = \frac{\lambda_i\lambda_l}{\zeta_i+\zeta_l}, \qquad \lambda_k = \sqrt{c_k(0)}e^{i\zeta_k\bar z - iz/4\zeta_k}.$

Spiky Strings :

$$Z_1 = \frac{1+i}{4}e^{-\frac12(i\lambda\xi - i\eta/\lambda)}\Big\{-i(\tilde\varphi_2-\tilde\varphi_1)\Big[e^{-\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2+\tilde\varphi_1)_+ + ie^{\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2+\tilde\varphi_1)_-\Big] + (\tilde\varphi_2+\tilde\varphi_1)\Big[e^{-\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2-\tilde\varphi_1)_+ - ie^{\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2-\tilde\varphi_1)_-\Big]\Big\}, \quad (3.67)$$

$$Z_2 = \frac{1-i}{4}e^{-\frac12(i\lambda\xi - i\eta/\lambda)}\Big\{-i(\tilde\varphi_2-\tilde\varphi_1)\Big[e^{-\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2+\tilde\varphi_1)_+ - ie^{\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2+\tilde\varphi_1)_-\Big] + (\tilde\varphi_2+\tilde\varphi_1)\Big[e^{-\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2-\tilde\varphi_1)_+ + ie^{\frac12(\lambda\xi+\eta/\lambda)}(\tilde\varphi_2-\tilde\varphi_1)_-\Big]\Big\}. \quad (3.68)$$

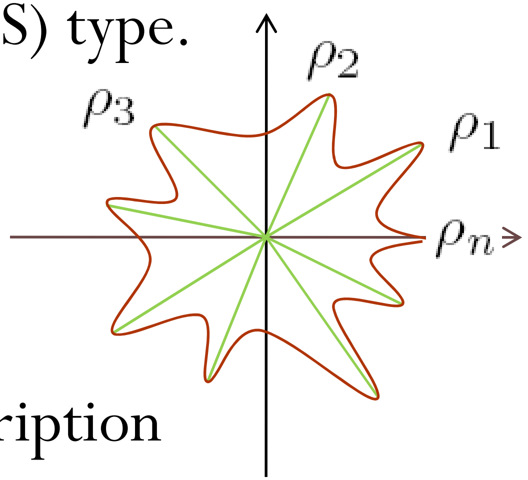
8. Moduli Dynamics

- Spike locations can be described by (collective) coordinates:

  n-spike solution: $\rho_i(t)$ with $i = 1, 2, \cdots, n$

  An interacting Lagrangian $L(\rho, \dot{\rho})$

  ⟹ Dynamical system of Calogero (RS) type.



$$Z_1^i(\tau) = Z_1(\tau, \sigma_i(\tau)),$$

$$Z_2^i(\tau) = Z_2(\tau, \sigma_i(\tau)),$$

- In the flat limit where $R \to \infty$: simple description

$$(\partial_\tau^2 - \partial_\sigma^2)\alpha - 2\sinh\alpha = 0 \quad \Longrightarrow \quad (\partial_\tau^2 - \partial_\sigma^2)\alpha - e^{-\alpha} = 0$$

(sinh-Gordon) (Liouville)

Dynamics of singularities: Liouville case

- Complete description of (singular) solutions of Liouville is known: Singular solutions $\longleftrightarrow$ Motion of poles

$$\alpha = \ln\left[\frac{2}{u(\sigma^+)v(\sigma^-)}\frac{f'(\sigma^+)g'(\sigma^-)}{[f(\sigma^+)+g(\sigma^-)]^2}\right]$$

where $\quad f(\sigma^+) = \sum_{j=1}^{N_A} \frac{c_j}{y_j - \sigma^+}, \qquad g(\sigma^-) = \sum_{j=1}^{N_B} \frac{d_j}{z_j - \sigma^-},$

Singularities: $\quad \sum_{j=1}^{N_A} \frac{c_j}{y_j - \sigma_i^+} + \sum_{j=1}^{N_B} \frac{d_j}{z_j - \sigma_i^-} = 0, \qquad i = 1, 2, \cdots, N.$

Equations of motion: $\quad \ddot{x}_i = 2(1+\dot{x}_i)\frac{1+\dot{x}_j}{x_i - x_j} \qquad i, j = 1, 2$

$$L = -\frac{1}{|x_2 - x_1|}\sqrt{(1+\dot{x}_1)(1+\dot{x}_2)} - m\sqrt{1-(\dot{x}_1+\dot{x}_2+1)^2}$$

- This can be generalized to n-body case;

Dynamics of spikes

One-spike solution :

$$X^0 = \frac{u}{\sqrt{2}d_1\tilde{v}_1}\left(\frac{1}{3}(\tilde{\sigma}^+)^3 + \frac{1}{2}d_1^2\tilde{v}_1^2\tilde{\sigma}^+\right) + \frac{v}{\sqrt{2}d_1}\left(\frac{1}{3}(\tilde{\sigma}^-)^3 + \frac{1}{2}d_1^2\tilde{\sigma}^-\right),$$

$$X^1 = \frac{u}{\sqrt{2}d_1\tilde{v}_1}\left(\frac{1}{3}(\tilde{\sigma}^+)^3 - \frac{1}{2}d_1^2\tilde{v}_1^2\tilde{\sigma}^+\right) + \frac{v}{\sqrt{2}d_1}\left(\frac{1}{3}(\tilde{\sigma}^-)^3 - \frac{1}{2}d_1^2\tilde{\sigma}^-\right),$$

$$X^2 = \frac{u}{2}(\tilde{\sigma}^+)^2 + \frac{v}{2}(\tilde{\sigma}^-)^2,$$

where $\tilde{\sigma}^+ \equiv \sigma^+ - \dfrac{2\sigma_1^0}{1-v_1} - z_1\dfrac{1+v_1}{1-v_1}, \qquad \tilde{\sigma}^- \equiv \sigma^- - z_1, \qquad \tilde{v}_1 \equiv \dfrac{1+v_1}{1-v_1}.$

N-spike dynamics :

$$Z_1^i(\tau) = Z_1(\tau, \sigma_i(\tau)), \qquad Z_2^i(\tau) = Z_2(\tau, \sigma_i(\tau)),$$

Dynamics of singularities: sinh-Gordon case

For soliton-soliton scattering : $\phi_{ss} = \ln\left[\frac{v\cosh(\gamma x) - \cosh(\gamma v t)}{v\cosh(\gamma x) + \cosh(\gamma v t)}\right]^2,$

Following the poles of the Hamiltonian density [8],

Trajectory of the poles : $x(t) = \pm\frac{1}{\gamma}\cosh^{-1}\left[\frac{1}{v}\cosh(\gamma v t)\right]$

N-body Hamiltonian [9]: $H = \sum_{j=1}^{N}\cosh\theta_j \prod_{k\neq j} f(q_j - q_k),$

Soliton-soliton scattering potential : $W_r(q) = \left|\coth\left(\frac{q}{2}\right)\right|$

[8] G. Bowtell and A. E. G. Stuart, "Interacting sine-Gordon solitons and classical particles: A dynamic equivalence," Phys. Rev. D **15**, 3580 (1977).

[9] S. N. M. Ruijsenaars and H. Schneider, "A new class of integrable systems and its relation to solitons," Ann. of Phys. **170**, 370 (1986).

Some comments

- This gives a 0-brane description of AdS_3 string
- Different from the spin-chain picture
- Exact ?
- Holographic

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2$$

$$\{\rho_i(t)\} \xrightarrow[\text{collective boson}]{} \rho(\theta, t)$$

AdS string in the physical gauge

$$\tau = t \quad \sigma = \theta$$

Giant magnons on R × S^2

- For strings on R×S^2 the 0-brane description is given in [10]

$$H_{RS} = \mathrm{tr}(\mathcal{L}) \qquad \omega_0 = dq_i \wedge d\theta_i$$

where L is the Lax matrix.

- Poisson structure :

$$\dot{q}_i = \{q_i, H\} \qquad \dot{p}_i = \{p_i, H\}$$

- String magnon energy is given in [11] : $E_m = \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}$

- Hamiltonian : $H_{\text{string}} = \mathrm{tr}(\mathcal{L}^{-1})$

[10] I. Aniceto and A. Jevicki, "N-body Dynamics of Giant Magnons in $R \times S_2$," arXiv:0810.4548 [hep-th].

[11] D. M. Hofman and J. M. Maldacena, "Giant magnons," J. Phys. A **39**, 13095 (2006) [arXiv:hep-th/0604135].

Magnon dynamics

- ω_0 does not produce the correct dynamics;
- The correct scattering phase shift : $$\delta \sim \int p dq$$
- Integrable models possess multi-Poisson structures;
- In [10], the second Poisson structure was identified which gives the magnon dynamics;
- An analogous situation exists for sine-Gordon/sinh-Gordon theory itself;
- Standard Poisson structure (Light-cone variables)

$$\int dx^- \, \partial_- \varphi \partial_+ \varphi$$

9. Conclusion and Outlook

- ✓ Inverse scattering method is useful for finding the classical string solutions in AdS;
- ✓ Spikes in AdS are related to solitons in sinh-Gordon theory;
- ✓ The GKP solution is a two-soliton configuration with solitons localized at the boundary of AdS;
- ✓ Statoc N-spike solution (Kruczenski's solution) can be constructed from the GKP solution by lifting the minimum value of ρ;
- ✓ We constructed new string solutions with N spikes in the bulk of AdS corresponding to N solitons of sinh-Gordon;
- ✓ Dynamics of the spikes: Moduli space;
- ✓ 0-brane description of AdS string.