AdS Strings, Minimal Surfaces and Scattering

Antal Jevicki

Brown University

December, 2009

In collaboration with Kewang Jin

Outline

1. Wilson loops and scattering amplitudes

- The vacuum expectation value of Wilson loops at strong coupling can be computed using the area of minimal surfaces in AdS; [Maldacena '98; Rey & Yee '98]
- Alday and Maldacena gave a Wilson loop representation of YM scattering amplitudes : [AM '07]

$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A} \sim \mathcal{A}$$

- This formalism is inspired by the T-duality of string theory;
- For the AdS metric :

$$ds^2 = w^2(z) dx_\mu dx^\mu + \cdots$$

one has the T-dual variables

$$\partial_\alpha y^\mu = i w^2(z) \epsilon_{\alpha\beta} \partial_\beta x^\mu$$

Polygonal boundary conditions

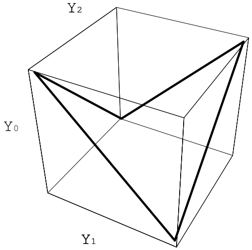
- AdS_5 metric under T-duality transformation : $r = R^2/z$

$$ds^2 = R^2\left[\frac{dx_{3+1}^2 + dz^2}{z^2}\right] \Rightarrow d\tilde{s}^2 = R^2\left[\frac{dy_\mu dy^\mu + dr^2}{r^2}\right]$$

- The boundary conditions for the original coordinates x^μ, which carry momenta k^μ, translate into the conditions that y^μ have "winding"

$$\Delta y^\mu = 2\pi k^\mu$$

- Planar scattering amplitude at strong coupling: minimal area surface with the boundaries specified by lightlike segments.



Simple example: Classical solution with four cusps

- Four-particle scattering: (k_1, k_2, k_3, k_4).
  The Mandelstam variables are defined as

$$\begin{aligned} s &\equiv -(k_1+k_2)^2 = -2k_1 \cdot k_2 \\ t &\equiv -(k_1+k_4)^2 = -2k_1 \cdot k_4 \end{aligned}$$

- Embedding the scattering process in AdS_3 to AdS_4: (r, y_0, y_1, y_2).
- Nambu-Goto action : $y_1 = \tau, y_2 = \sigma$

$$S = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

- The solution :

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}$$

Four cusp solution in conformal gauge

- Conformal gauge action :

$$iS = -\frac{R^2}{2\pi}\int d\tau d\sigma \frac{1}{2}\frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2}$$

- Perfect square solution : $s = t$

$$y_1 = \tanh\tau, \quad y_2 = \tanh\sigma, \quad y_0 = \tanh\tau\tanh\sigma, \quad r = \frac{1}{\cosh\tau\cosh\sigma}$$

- Performing a boost in the 04 plane : $s \neq t$

$$\begin{aligned} r &= \frac{a}{\cosh\tau\cosh\sigma + b\sinh\tau\sinh\sigma} \\ y_0 &= \frac{a\sqrt{1+b^2}\sinh\tau\sinh\sigma}{\cosh\tau\cosh\sigma + b\sinh\tau\sinh\sigma} \\ y_1 &= \frac{a\sinh\tau\cosh\sigma}{\cosh\tau\cosh\sigma + b\sinh\tau\sinh\sigma} \\ y_2 &= \frac{a\cosh\tau\sinh\sigma}{\cosh\tau\cosh\sigma + b\sinh\tau\sinh\sigma} \end{aligned}$$

Projection of four cusp solutions

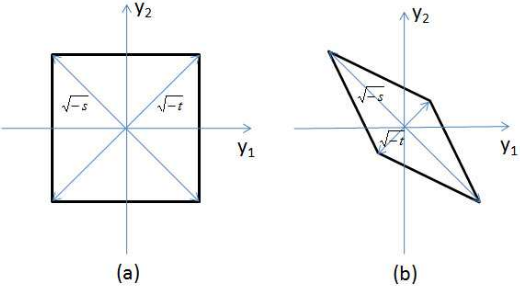


Figure: Projection on the (y_1, y_2) plane for (a) $s = t$ and (b) $s \neq t$.

- Kinematics:

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \quad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$$

Evaluation of the area

- Conformal gauge Lagrangian (four points) : $\mathcal{L} = 1$

$$A = \int d\tau d\sigma \frac{1}{2} \frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2} \Rightarrow A_4 = \int d\tau d\sigma$$

- Introduce the radial cutoff $r_c \to 0$, we have the boundary of the worldsheet determined by the equation

$$\frac{a}{\cosh\tau\cosh\sigma + b\sinh\tau\sinh\sigma} = r_c$$

- Defining the lightcone coordinates

$$\sigma_+ = \tau + \sigma, \quad \sigma_- = \tau - \sigma, \quad b_+ \equiv 1 + b, \quad b_- \equiv 1 - b, \quad \epsilon \equiv r_c/a$$

- The area

$$A_4 = \frac{1}{2}\int d\sigma_+ d\sigma_- = \int_{-L_+}^{L_+} \cosh^{-1}\left[\frac{2/\epsilon - b_+\cosh\sigma_+}{b_-}\right] d\sigma_+$$

The result

- In the limit of $\epsilon \to 0$:

$$\cosh^{-1}[\ ] \sim \ln \frac{4}{\epsilon b_-} - \frac{1}{2} b_+ \cosh \sigma_+ \epsilon - \frac{1}{8} b_+^2 \cosh^2 \sigma_+ \epsilon^2 - \frac{b_-^2}{16} \epsilon^2 + \cdots$$

- The result :

$$A_4 = \frac{1}{4} \ln^2 \left(\frac{r_c^2}{-8\pi^2 s} \right) + \frac{1}{4} \ln^2 \left(\frac{r_c^2}{-8\pi^2 t} \right) - \frac{1}{4} \ln^2 \left(\frac{s}{t} \right) - \frac{\pi^2}{3}$$

- The result agrees in form with the BDS ansatz; [BDS '05]
- This gives a strong coupling evaluation of four-point YM scattering amplitudes;
- One would like to generalize the four-point calculation to more general cases with more points.

2. General method

- String sigma model : conformal gauge

$$\partial\bar{\partial}Y - (\partial Y \cdot \bar{\partial}Y)Y = 0$$
$$\partial Y \cdot \partial Y = \bar{\partial}Y \cdot \bar{\partial}Y = 0$$

- Pohlmeyer reduction :

$$\begin{aligned} e^{2\alpha(z,\bar{z})} &= \frac{1}{2}\partial Y \cdot \bar{\partial}Y, \\ N_a &= \frac{e^{-2\alpha}}{2}\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d, \\ p &= \frac{1}{2}N \cdot \partial^2 Y, \quad \bar{p} = -\frac{1}{2}N \cdot \bar{\partial}^2 Y, \end{aligned}$$

- EOM of the scalar field :

$$\partial\bar{\partial}\alpha(z,\bar{z}) - e^{2\alpha} + p(z)\bar{p}(\bar{z})e^{-2\alpha} = 0.$$

- Two SL(2) scattering problems :

$$\partial\psi^L_\alpha + (B^L_z)^\beta_\alpha \psi^L_\beta = 0, \qquad \bar\partial\psi^L_\alpha + (B^L_{\bar z})^\beta_\alpha \psi^L_\beta = 0$$
$$\partial\psi^R_{\dot\alpha} + (B^R_z)^{\dot\beta}_{\dot\alpha} \psi^R_{\dot\beta} = 0, \qquad \bar\partial\psi^R_{\dot\alpha} + (B^R_{\bar z})^{\dot\beta}_{\dot\alpha} \psi^R_{\dot\beta} = 0$$

- The scattering potentials :

$$B^L_z = \begin{pmatrix} \frac{1}{2}\partial\alpha & -e^{\alpha} \\ -e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^L_{\bar z} = \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & -e^{-\alpha}\bar p(\bar z) \\ -e^{\alpha} & \frac{1}{2}\bar\partial\alpha \end{pmatrix}$$
$$B^R_z = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha}p(z) \\ -e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^R_{\bar z} = \begin{pmatrix} \frac{1}{2}\bar\partial\alpha & -e^{\alpha} \\ e^{-\alpha}\bar p(\bar z) & -\frac{1}{2}\bar\partial\alpha \end{pmatrix}$$

- Introduction of a spectral parameter :

$$B_z(\zeta) = \begin{pmatrix} \frac{1}{2}\partial\alpha & -\frac{1}{\zeta}e^{\alpha} \\ -\frac{1}{\zeta}e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha \end{pmatrix}, \quad B_{\bar z}(\zeta) = \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & -\zeta e^{-\alpha}\bar p(\bar z) \\ -\zeta e^{\alpha} & \frac{1}{2}\bar\partial\alpha \end{pmatrix}$$

$$\Rightarrow B^L_z = B_z(1), \qquad B^R_z = UB_z(i)U^{-1}$$

Hitchin equations

- Decomposition :

$$B_z(\zeta) = A_z + \frac{1}{\zeta}\Phi_z, \qquad B_{\bar{z}}(\zeta) = A_{\bar{z}} + \zeta\Phi_{\bar{z}}$$

- The Hitchin equations :

$$D_{\bar{z}}\Phi_z = D_z\Phi_{\bar{z}} = 0, \qquad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

  where the covariant derivatives and field strengths are defined as

$$D_\mu\Phi = \partial_\mu\Phi_z + [A_\mu, \Phi_z]$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- The string solution :

$$Y_{a\dot{a}} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix}_{a\dot{a}} = \psi^L_{\alpha,a} M_1^{\alpha\dot{\beta}} \psi^R_{\dot{\beta},\dot{a}}, \quad M_1^{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Evaluation of the area

- Regularization of the area:

$$
\begin{aligned}
A &= \int_{r\geq\mu} d^2z\ e^{2\alpha} = \int_{\Sigma} d^2w\ e^{2\hat{\alpha}} \\
&= \int_{\Sigma} d^2w + \int d^2w(e^{2\hat{\alpha}} - 1)
\end{aligned}
$$

where we introduced the change of variables

$$
\alpha(z,\bar{z}) = \hat{\alpha}(z,\bar{z}) + \frac{1}{4}\ln[p(z)\bar{p}(\bar{z})]
$$

$$
w = \int \sqrt{p(z)}dz, \quad \bar{w} = \int \sqrt{\bar{p}(\bar{z})}d\bar{z}
$$

- The first piece should be regularized using the boundary polygon

$$
\int_{\Sigma} d^2w = A_{cutoff} = A_{div} + A_{BDS-like-even} + A_{extra}
$$

- The second piece is finite

$$
\int d^2w(e^{2\hat{\alpha}} - 1) = A_{sinh}
$$

Wavefunctions

- The asymptotic wavefunctions :

$$\eta^L_+ = \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix}, \qquad \eta^L_- = \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix},$$
$$\eta^R_+ = \begin{pmatrix} e^{(w-\bar{w})/i} \\ 0 \end{pmatrix}, \qquad \eta^R_- = \begin{pmatrix} 0 \\ e^{-(w-\bar{w})/i} \end{pmatrix}$$

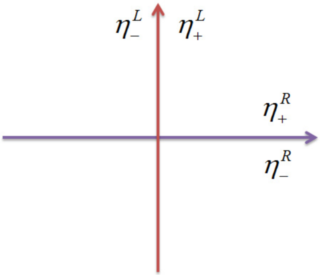


Figure: Dominating solutions at each sector.

Stokes phenomena

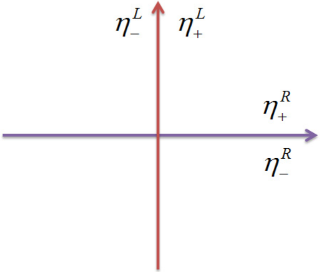
- Approximate solution at large z :

$$\psi = b^{+}\eta_{+} + b^{-}\eta_{-}$$

- Stokes matrices:

$$S_p = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad S_n = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

Generally, γ is different for each line.



Identification of cusps

- The locations of cusps are :

$$(i,i): \quad \frac{1}{r} = b_1^+ \tilde{b}_1^+ e^u, \quad x_i^+ = \frac{b_2^+}{b_1^+}, \quad x_i^- = \frac{\tilde{b}_2^+}{\tilde{b}_1^+},$$

$$(i+1,i): \quad \frac{1}{r} = -b_1^- \tilde{b}_1^+ e^v, \quad x_{i+1}^+ = \frac{b_2^-}{b_1^-}, \quad x_i^- = \frac{\tilde{b}_2^+}{\tilde{b}_1^+},$$

$$(i+1,i+1): \quad \frac{1}{r} = b_1^- \tilde{b}_1^- e^{-u}, \quad x_{i+1}^+ = \frac{b_2^-}{b_1^-}, \quad x_{i+1}^- = \frac{\tilde{b}_2^-}{\tilde{b}_1^-},$$

where $u = (w+\bar{w}) + (w-\bar{w})/i, v = -(w+\bar{w}) + (w-\bar{w})/i$.

- The kinematic variables :

$$x_{i+1}^+ - x_i^+ = \frac{1}{b_1^+ b_1^-}, \qquad x_{i+2}^+ - x_i^+ = \frac{\gamma_{i+1}^L}{(b_1^+ + \gamma_{i+1}^L b_1^-) b_1^+}$$

$$x_{i+1}^- - x_i^- = \frac{1}{\tilde{b}_1^+ \tilde{b}_1^-}, \qquad x_{i+2}^- - x_i^- = \frac{\gamma_{i+1}^R}{(\tilde{b}_1^+ + \gamma_{i+1}^R \tilde{b}_1^-) \tilde{b}_1^+}$$

Evaluation of A_{cutoff}

- Variations around cutoff : $r = \mu$

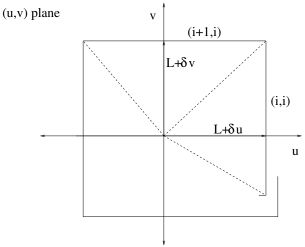
$$\begin{aligned} (i,i): \qquad & u = -\ln\mu + \delta u_i, \quad \delta u_i = -\ln(b_1^+ \tilde{b}_1^+) \\ (i+1,i+1): \qquad & u = -(-\ln\mu + \delta u_{i+1}), \quad \delta u_{i+1} = -\ln(b_1^- \tilde{b}_1^-) \end{aligned}$$

one finds

$$\delta u_i + \delta u_{i+1} = l_i^+ + l_i^-, \quad l_i^\pm = \ln(x_{i+1}^\pm - x_i^\pm)$$

- Similarly

$$\delta v_i + \delta v_{i+1} = l_{i+1}^+ + l_i^-$$



The result of A_{cutoff}

The result is given in [AM '09]

$$
\begin{aligned}
A_{cutoff} &= \frac{1}{4}\Big[\sum_{i=1}^{n}(L+\delta v_i)(2L+\delta u_{i+1}+\delta u_i)+\sum_{i=2}^{n}(L+\delta u_{i+1}) \\
&\quad \times(2L+\delta v_i+\delta v_{i-1})+(L+\delta u_1)(L+\delta v_1)+(L+\delta u_{n+1}) \\
&\quad \times(L+\delta v_n)+2(L+\delta u_{n+1})v_s-u_s v_s\Big] \\
&\equiv A_{div}+A_{BDS-like-even}+A_{extra}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{div} &= \sum_{J=1}^{2n}\frac{1}{8}\left(\ln^2\frac{d^2_{J,J+2}}{\mu^2}\right)+g\ln\frac{d^2_{J,J+2}}{\mu^2} \\
A_{BDS-like-even} &= \sum_{i,j} l_i^+\hat{M}^{(1)}_{ij}l_j^- - \frac{1}{2}\left(\frac{w_s-\bar{w}_s}{i}\right)l_1^+ + \frac{1}{2}(w_s+\bar{w}_s)\hat{l}_1^- \\
&\quad -\left(\sum_{i=1}^{n}(-1)^i l_i^+\right)^2-\left(\sum_{i=1}^{n}(-1)^i l_i^-\right)^2 \\
A_{extra} &= -\frac{1}{2}(w_s+\bar{w}_s)\ln\gamma_1^R+\frac{1}{2}\frac{(w_s-\bar{w}_s)}{i}\ln\gamma_1^L
\end{aligned}
$$

3. Evaluation of A_{sinh}

- Now we need to calculate the finite piece A_{sinh}

$$A_{sinh} = \int d^2 w (e^{2\hat{\alpha}} - 1)$$

- Pohlmeyer reduction : [Jevicki & Jin '09]

$$\alpha(z, \bar{z}) \equiv \ln[\partial Y \cdot \bar{\partial} Y]$$

$$\Rightarrow \partial\bar{\partial}\alpha - e^{\alpha} + p(z)\bar{p}(\bar{z})e^{-\alpha} = 0$$

- The change of variables :

$$\alpha(z, \bar{z}) = \hat{\alpha}(z, \bar{z}) + \frac{1}{2}\ln[p(z)\bar{p}(\bar{z})]$$

$$w = \int \sqrt{p(z)}dz, \quad \bar{w} = \int \sqrt{\bar{p}(\bar{z})}d\bar{z}$$

- Standard sinh-Gordon equation :

$$\partial_w \bar{\partial}_{\bar{w}} \hat{\alpha}(w, \bar{w}) - 2\sinh\hat{\alpha} = 0$$

Regular polygon

- The area :

$$A_{sinh} = \int d^2w(e^{\hat{\alpha}} - 1)$$

- Regular polygon :

$$p(z) = z^{n-2}, \qquad \bar{p}(\bar{z}) = \bar{z}^{n-2}$$

- We are interested in rotationally invariant solutions where

$$\partial_w \bar{\partial}_{\bar{w}} = \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}$$

- Boundary conditions: we expect α to be regular everywhere.
  Large ρ: $\hat{\alpha} \to 0$
  Small ρ: $\hat{\alpha} \sim -\frac{n-2}{2} \ln z\bar{z} \sim -\frac{n-2}{n} \ln w\bar{w} \sim -2\frac{n-2}{n} \ln \rho$

Exact solution to the sinh-Gordon equation

- Monopole solutions in self-dual YM theory;
- Matrix model integrals;
- Exact integral representation :

$$\hat{\alpha}(\rho|\lambda) = W(\rho|\lambda) - W(\rho| - \lambda)$$

$$W(\rho|\lambda) = 2\sum_{k=1}^{\infty} \frac{\lambda^k}{k} W_k(\rho) = 2\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \int_0^{\infty} \prod_{i=1}^{k} \frac{2e^{-\sqrt{2}\rho \cosh[\ln x_i]}}{x_i + x_{i+1}} dx_i$$

Exact result of the area

- Large ρ expansion : $2\pi\lambda = \sin(\pi\zeta/2)$

$$W(\rho|\lambda) = 4\lambda K_0(\sqrt{2}\rho) + \cdots$$

- Small ρ expansion : $\zeta = (n-2)/n$

$$W(\rho|\lambda) = -\frac{\zeta(\zeta+2)}{2}\ln\rho + \cdots$$

- Exact result :

$$A_{sinh} = \frac{\pi n}{2}\int_0^\infty \rho d\rho (e^{\hat{\alpha}} - 1) = \frac{\pi n}{2}\left(\rho\frac{dW}{d\rho}\right)\Big|_{\rho=0}^{\rho=\infty} = \frac{\pi}{4n}(3n^2 - 8n + 4)$$

4. Approximate method

- One would like to calculate gluon scattering amplitudes in more general backgrounds like $AdS_{4,5}$.
- The integrable models for these cases are the generalized Toda equations.
- It is highly nontrivial to find instanton-type solution of the Toda system.
- There are no similar exact multi-integral solutions to our best knowledge.
- We develop an approximate method by expanding the solution using a series at large ρ with a nontrivial match of the boundary conditions at small ρ. [Jevicki & Jin '09]
- This method was used in the case of sine-Gordon solitons in [Manton '79].

4.1 A toy model: the kink case

- The static Lagrangian of the φ^4 theory:

$$L = \int dx \left(-\frac{1}{2}\varphi_x^2 - \frac{1}{4}(1-\varphi^2)^2 \right)$$

- Equation of motion:

$$\varphi_{xx} + \varphi(1-\varphi^2) = 0$$

- Series expansion at $x = -\infty$:

$$\varphi(x) = -1 + \sum_{n=1}^{\infty} c_n e^{n\sqrt{2}x}$$

- Recursion relation:

$$2\sum_n c_n n^2 e^{n\sqrt{2}x} - 2\sum_n c_n e^{n\sqrt{2}x} + 3\sum_{m,l} c_m c_l e^{(m+l)\sqrt{2}x} - \sum_{m,l,k} c_m c_l c_k e^{(m+l+k)\sqrt{2}x} = 0$$

Matching the boundary condition

- The first coefficient is undetermined.
  The others are related to the first one as

$$c_2 = -\frac{1}{2}c_1^2, \quad c_3 = \frac{1}{4}c_1^3, \quad c_4 = -\frac{1}{8}c_1^4, \quad \cdots$$

- Summing up:

$$\varphi = -1 + c_1 e^{\sqrt{2}x} - \frac{1}{2}c_1^2 e^{2\sqrt{2}x} + \frac{1}{4}c_1^3 e^{3\sqrt{2}x} + \cdots = \frac{\frac{1}{2}c_1 e^{\sqrt{2}x} - 1}{\frac{1}{2}c_1 e^{\sqrt{2}x} + 1}$$

- Impose the boundary condition at origin

$$\varphi(0) = 0 \Rightarrow c_1 = 2$$

- Exact solution:

$$\varphi = \frac{e^{\sqrt{2}x} - 1}{e^{\sqrt{2}x} + 1} = \tanh\frac{x}{\sqrt{2}}$$

4.2 Sinh-Gordon equation revisited

- The sinh-Gordon equation :

$$\partial_w \bar{\partial}_{\bar{w}} \hat{\alpha}(w, \bar{w}) - 2 \sinh \hat{\alpha} = 0$$

- Series expansion at $\rho = \infty$:

$$\hat{\alpha} = \sum_{odd \ \ n} \alpha_n$$

- The first order equation :

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} \alpha_1 - 2\alpha_1 = 0$$

The solution satisfying the boundary condition :

$$\alpha_1 = a K_0(\tilde{\rho})$$

where $\tilde{\rho} \equiv \sqrt{2}\rho$.

Higher orders

- The third order :

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\alpha_3 - 2\alpha_3 - \frac{1}{3}\alpha_1^3 = 0$$

The solution can be written as a double integral :

$$\alpha_3 = K_0(\tilde{\rho})\int_{\tilde{\rho}}^{\infty}\frac{d\tilde{\rho}'}{\tilde{\rho}' K_0^2(\tilde{\rho}')}\int_{\tilde{\rho}'}^{\infty}\frac{1}{6}a^3 K_0^4(\tilde{\rho}'')\tilde{\rho}'' d\tilde{\rho}''$$

- The fifth order equation :

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\alpha_5 - 2\alpha_5 - \alpha_1^2\alpha_3 - \frac{1}{60}\alpha_1^5 = 0$$

The solution :

$$\alpha_5 = K_0(\tilde{\rho})\int_{\tilde{\rho}}^{\infty}\frac{d\tilde{\rho}'}{\tilde{\rho}' K_0^2(\tilde{\rho}')}\int_{\tilde{\rho}'}^{\infty} S_5^{\alpha}(\tilde{\rho}'')K_0(\tilde{\rho}'')\tilde{\rho}'' d\tilde{\rho}''$$

where the source term is

$$S_5^{\alpha}(\tilde{\rho}'') = \frac{1}{2}\alpha_1^2\alpha_3 + \frac{1}{120}\alpha_1^5\Big|_{\tilde{\rho}''}$$

Matching of boundary condition

- Matching with the singularity at small ρ : $K_0(\rho) \sim -\ln\rho$

$$\frac{\hat{\alpha}}{-\ln\rho} \sim a + \frac{\pi^2}{96}a^3 + \frac{3\pi^4 a^5}{10240} + \cdots = 2\frac{n-2}{n}$$

- The area :

$$\begin{aligned} A_{sinh} &= \frac{\pi n}{4}\int_0^\infty \tilde{\rho} d\tilde{\rho}(e^{\hat{\alpha}} - 1) \\ &= \frac{\pi n}{4}\int_0^\infty \tilde{\rho} d\tilde{\rho}(\alpha_1 + \frac{1}{2}\alpha_1^2 + \alpha_3 + \frac{1}{6}\alpha_1^3 + \alpha_1\alpha_3 + \frac{1}{24}\alpha_1^4 + \\ &\qquad + \alpha_5 + \frac{1}{2}\alpha_1^2\alpha_3 + \frac{1}{120}\alpha_1^5 + \cdots) \\ &= \frac{\pi n}{4}(a + \frac{1}{4}a^2 + 0.102808a^3 + 0.0514042a^4 \\ &\qquad + 0.0285378a^5 + \cdots) \end{aligned}$$

Comparison to the exact result

	A_{exact}	$A^{(3)}_{sinh}$	Error$^{(3)}$	$A^{(5)}_{sinh}$	Error$^{(5)}$
$2n = 6$	1.83260	1.81188	1.13%	1.82983	0.15%
$2n = 8$	3.92699	3.80629	3.07%	3.89526	0.81%
$2n = 10$	6.12611	5.84269	4.63%	6.02938	1.58%
$2n = 12$	8.37758	7.89331	5.78%	8.18848	2.26%

Table: Comparison of the estimation with the exact results.

- For small n, just a few orders in the expansion that we generate are capable of producing a solution giving the area close to the exact one;
- The error increases when n gets larger.

4.3 Pohlmeyer reduction in d dimensions

- The string sigma model :

$$\partial\bar{\partial}Y - (\partial Y \cdot \bar{\partial}Y)Y = 0$$
$$\partial Y \cdot \partial Y = \bar{\partial}Y \cdot \bar{\partial}Y = 0$$

- Basis of the string coordinates :

$$e_i = (Y, \partial Y, \bar{\partial}Y, B_4, \cdots, B_{d+1}), \qquad i = 1, 2, \cdots, d+1$$

- Defining the scalar fields :

$$\alpha(z, \bar{z}) \equiv \ln[\partial Y \cdot \bar{\partial}Y]$$
$$u_i \equiv B_i \cdot \partial^2 Y, \quad v_i \equiv B_i \cdot \bar{\partial}^2 Y$$

- Equations of motion :

$$\partial\bar{\partial}\alpha - e^{\alpha} - e^{-\alpha}\sum_{i=4}^{d+1} u_i v_i = 0$$
$$\bar{\partial}u_i = \sum_{j\neq i}(B_j \cdot \bar{\partial}B_i)u_j, \quad \partial v_i = \sum_{j\neq i}(B_j \cdot \partial B_i)v_j$$

AdS$_4$ in detail

- The auxiliary fields

$$u_4 = +p(z)\cos\bar{\gamma}(z,\bar{z}), \qquad v_4 = -\bar{p}(\bar{z})\cos\gamma(z,\bar{z})$$
$$u_5 = -p(z)\sin\bar{\gamma}(z,\bar{z}), \qquad v_5 = -\bar{p}(\bar{z})\sin\gamma(z,\bar{z})$$

- Defining a new field $\beta(z,\bar{z}) \equiv \gamma(z,\bar{z}) + \bar{\gamma}(z,\bar{z})$, the EOMs are

$$\partial\bar{\partial}\alpha - e^{\alpha} + p(z)\bar{p}(\bar{z})e^{-\alpha}\cos\beta = 0$$
$$\partial\bar{\partial}\beta - p(z)\bar{p}(\bar{z})e^{-\alpha}\sin\beta = 0$$

- Change of variables :

$$\alpha(z,\bar{z}) = \hat{\alpha}(z,\bar{z}) + \frac{1}{2}\ln[p(z)\bar{p}(\bar{z})]$$
$$dw = \sqrt{p(z)}dz, \quad d\bar{w} = \sqrt{\bar{p}(\bar{z})}d\bar{z}$$

one finds

$$\partial_w\bar{\partial}_{\bar{w}}\hat{\alpha} - e^{\hat{\alpha}} + e^{-\hat{\alpha}}\cos\beta = 0$$
$$\partial_w\bar{\partial}_{\bar{w}}\beta - e^{-\hat{\alpha}}\sin\beta = 0$$

Series expansion

- Series expansion at $\rho = \infty$:

$$\hat{\alpha} = \sum_{n=1}^{\infty} \alpha_n, \qquad \beta = \sum_{n=1}^{\infty} \beta_n$$

- The first order equations are decoupled :

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\alpha_1 - 2\alpha_1 = 0$$
$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\beta_1 - \beta_1 = 0$$

with the solution satisfying the boundary condition as

$$\alpha_1(\tilde{\rho}) = aK_0(\tilde{\rho})$$
$$\beta_1(\rho) = bK_0(\rho)$$

where $\tilde{\rho} \equiv \sqrt{2}\rho$.

Second order

- The second order equations are

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\alpha_2 - 2\alpha_2 - \frac{1}{2}\beta_1^2 = 0$$
$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\beta_2 - \beta_2 + \alpha_1\beta_1 = 0$$

The solutions can be written in terms of double integrals

$$\alpha_2(\tilde{\rho}) = +K_0(\tilde{\rho})\int_{\tilde{\rho}}^{\infty}\frac{d\tilde{\rho}'}{\tilde{\rho}' K_0^2(\tilde{\rho}')}\int_{\tilde{\rho}'}^{\infty}\frac{1}{4}b^2 K_0^2(\tilde{\rho}''/\sqrt{2})K_0(\tilde{\rho}'')\tilde{\rho}'' d\tilde{\rho}''$$
$$\beta_2(\rho) = -K_0(\rho)\int_{\rho}^{\infty}\frac{d\rho'}{\rho' K_0^2(\rho')}\int_{\rho'}^{\infty} abK_0(\sqrt{2}\rho'')K_0^2(\rho'')\rho'' d\rho''$$

Third order

- The third order equations are

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\alpha_3 - 2\alpha_3 - \frac{1}{3}\alpha_1^3 + \frac{1}{2}\alpha_1\beta_1^2 - \beta_1\beta_2 = 0$$
$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\beta_3 - \beta_3 + \frac{1}{6}\beta_1^3 - \frac{1}{2}\alpha_1^2\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 = 0$$

The solutions are

$$\alpha_3(\tilde{\rho}) = +K_0(\tilde{\rho})\int_{\tilde{\rho}}^{\infty}\frac{d\tilde{\rho}'}{\tilde{\rho}'K_0^2(\tilde{\rho}')}\int_{\tilde{\rho}'}^{\infty}S_3^{\alpha}(\tilde{\rho}'')K_0(\tilde{\rho}'')\tilde{\rho}''d\tilde{\rho}''$$
$$\beta_3(\rho) = -K_0(\rho)\int_{\rho}^{\infty}\frac{d\rho'}{\rho'K_0^2(\rho')}\int_{\rho'}^{\infty}S_3^{\beta}(\rho'')K_0(\rho'')\rho''d\rho''$$

where the source terms are

$$S_3^{\alpha}(\tilde{\rho}'') = \frac{1}{6}\alpha_1^3 - \frac{1}{4}\alpha_1\beta_1^2 + \frac{1}{2}\beta_1\beta_2\Big|_{\tilde{\rho}''}$$
$$S_3^{\beta}(\rho'') = \frac{1}{6}\beta_1^3 - \frac{1}{2}\alpha_1^2\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1\Big|_{\rho''}$$

Matching of the boundary conditions

In the case of $(1,3)$ signature, there is no singularity for the β field near the origin. The coefficients a, b are determined by the following equations

$$\begin{aligned} a + \frac{\pi}{8}b^2 + \frac{\pi^2}{96}a^3 - \frac{\pi^2}{32}ab^2 &= 2\frac{n-2}{n} \\ b - \frac{\pi}{8}ab - \frac{7\pi^2}{384}b^3 + \frac{3\pi^2}{128}a^2 b &= 0 \end{aligned}$$

The finite piece of the area up to third order can be written as

$$\begin{aligned} A_{Toda} &= \frac{\pi n}{4}\int_0^\infty \tilde{\rho} d\tilde{\rho}(\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_1^2 + \alpha_3 + \alpha_1\alpha_2 + \frac{1}{6}\alpha_1^3 + \cdots) \\ &= \frac{\pi n}{4}(a + \frac{1}{4}a^2 + 0.142699b^2 + 0.102808a^3 + 0.0842739ab^2 + \cdots) \end{aligned}$$

which is minimized at

$$a + \frac{\pi^2}{96}a^3 = 2\frac{n-2}{n}, \qquad b = 0$$

for small $n \leq n_c$.

5. Scattering amplitudes in AdS_5

- Very recently, Alday, Gaiotto and Maldacena [AGM '09] considered minimal area surfaces in AdS_5 (with null polygonal boundary conditions) which can be mapped into a SU(4) Hitchin system;
- For the hexagon, the area is determined by integral equations which are identical to those of Thermodynamics Bethe Ansatz equations;
- Moreover, the area is given by the free energy of the TBA system.
- Regularization of the area :

$$A = A_{div} + A_{BDS} - R$$

where the remainder function can be written as

$$R = \sum_{i=1}^{3} \Big(\frac{1}{8} \ln^2 u_i + \frac{1}{4} Li_2(1 - u_i) \Big) - A_{periods} - A_{free} + constant$$

where u_i are cross ratios.

The free energy

$$A_{free} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\Big(2|Z|\cosh\theta\ln(1+e^{-\epsilon}\mu)(1+\frac{e^{-\epsilon}}{\mu}) + 2\sqrt{2}|Z|\cosh\theta\ln(1+e^{-\tilde{\epsilon}})\Big)$$

where ϵ and $\tilde{\epsilon}$ are determined by the integral equations

$$\begin{aligned}
\epsilon(\theta) &= 2|Z|\cosh\theta + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\ln(1+e^{-\tilde{\epsilon}}) \\
&\quad + \frac{1}{2\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\ln(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu}), \\
\tilde{\epsilon}(\theta) &= 2\sqrt{2}|Z|\cosh\theta + \frac{1}{\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\ln(1+e^{-\tilde{\epsilon}}) \\
&\quad + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\ln(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu}).
\end{aligned}$$

High temperature limit

- In this limit, $u_1 = u_2 = u_3 = u$.
- The integral equations can be solved and one gets the final result of the remainder function

$$R(u,u,u) = -\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 + \frac{3}{8}(\ln^2 u + 2Li_2(1-u)),$$

where

$$u = \frac{1}{4\cos^2(\phi/3)}.$$

Analytic continuation

- One can continue the $(1,3)$ signature to $(2,2)$ signature which will allow extra singularities for the β field at the origin.
- For the hexagon, choose the boundary conditions as

$$\hat{\alpha}(\rho) = -\frac{2}{3}\ln\rho + c_\alpha + \cdots\ , \quad \beta(\rho) = -\frac{4}{3}\ln\rho + c_\beta + \cdots$$

The approximate area calculated numerically as

$$A^{++}_{hexagon} \sim 2.85$$

6. Conclusions and Extensions

- We reviewed the calculation of four- and eight-gluon scattering amplitudes in AdS_3;
- For amplitudes in more general dimensions like $AdS_{4,5}$, we have described an approximate method for construction of Euclidean instanton type solutions of the associated Toda equations;
- The method is based on a series solution at large distance with a nontrivial matching at short distance boundary conditions;
- In the case of sinh-Gordon, the finite piece of area calculated up to a few terms is seen to be very close to the exact result;
- The procedure is then extended and demonstrated to be applicable for the generalized Toda systems;
- Our calculation only concerns with the finite piece of area which is independent of the cross ratios;
- It remains to be a challenge to evaluate the complete amplitudes for any points.