Matrix polar coordinates (and radial fermionization in higher dimensions)

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Emphasis of recent work (and in progress!), but mainly influenced by ideas going back to work done since 2005

Collaborators

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1. Multi matrix models

There are many reasons why multi-matrix models are of interest, particularly their large N limit.

By matrix models, we mean integrals over matrices or the quantum mechanics of matrix valued degrees of freedom

Some examples include

- Possible definition of M theory (BFSS)
- In the context of AdS/CFT duality, due to supersymmetry and conformal invariance, correlators of supergravity and ½ BPS states reduce to calculation of free matrix model overlaps or consideration of related matrix hamiltonians (e.g., Lee et al., Corley et al, Berenstein). Examples of other geometries discussed by e.g., R. de Mello Koch
- Similarly for stringy states, in the context of the BMN limit
- A plane-wave matrix theory (Kim, 2003) is related to the *N*=4 SYM dilatation operator
- In earlier works, it has been argued that QCD can be reduced to a finite number of matrices with quenched momenta. Alternatively, they can be associated with QCD zero modes on hyper-spheres.

Comments

- Very much work in progress
- Work moves away from the beautiful picture of YM gauge invariant states and their possible supergravity interpretation. More pedestrian, hopefully still interesting.

2 Matrix polar coordinates

We wish to consider the quantum mechanics of two $N \times N$ hermitean matrices X_1 and X_2 with hamiltonian

$$\begin{aligned} \hat{H} &= -\frac{1}{2}\left(\frac{\partial}{\partial(X_1)_{ij}}\frac{\partial}{\partial(X_1)_{ji}} + \frac{\partial}{\partial(X_2)_{ij}}\frac{\partial}{\partial(X_2)_{ji}}\right) + V(X_1, X_2) \\ &= -\frac{1}{2}\nabla^2 + V(X_1, X_2) \end{aligned} \tag{1}$$

We introduce matrix valued polar coordinates

$$X_1 + iX_2 = Z = RU \quad , \quad Z^\dagger = U^\dagger R \tag{2}$$

with R hermitean and U unitary. Since R is hermitean, it can be diagonalized as $R = V^\dagger r V$, with r a diagonal matrix and V unitary, and we obtain

$$\begin{aligned} Z = V^\dagger r V U &= V^\dagger r W, \quad W \equiv VU \\ Z^\dagger = U^\dagger V^\dagger r V &= W^\dagger r V \end{aligned} \tag{3}$$

We will refer to the parametrization in terms of (r, V, U) as parametrization (I), and the parametrization in terms of (r, V, W) as parametrization (II). The number of degrees of freedom is preserved, since these matrix coordinates are defined up to $V \to DV$, $W \to DW$ with D a diagonal unitary matrix.

Introducing the anti-hermitean, Lie-algebra valued differential matrices

$$dX \equiv V dU U^\dagger V^\dagger \quad , \quad dS \equiv dV V^\dagger \quad , \quad dT \equiv dW W^\dagger \tag{4}$$

we obtain:

$$\begin{aligned} dZ &= V^\dagger(dr + [r, dS] + r dX)VU = V^\dagger(dr + r dT - dS r)W \\ dZ^\dagger &= U^\dagger V^\dagger(dr + [r, dS] - dX r)V = W^\dagger(dr + r dS - dT r)V \end{aligned} \tag{5}$$

The metric is defined from the infinitesimal length squared

$$\begin{aligned} \mathrm{Tr} dZ^\dagger dZ &= \mathrm{Tr}\left[(dr)^2 + [r, dS][r, dS] - r^2(dX)^2 + [r, dS][r, dX]\right] \\ &= \mathrm{Tr}\left[(dr)^2 - r^2(dS)^2 - r^2(dT)^2 + 2r dS r dT\right] \end{aligned} \tag{6}$$

Starting with parametrization (I), we note that the infinitesimal length squared has a 'local'(or "pointwise") form in terms of the double index (i,j), i.e.:

$$\begin{aligned}
\mathrm{Tr}dZ^{\dagger}dZ &= \sum_i dr_i^2 - \sum_{ij}(r_i - r_j)^2 dS_{ij}dS_{ji} \\
&- \frac{1}{2}\sum_{ij}(r_i - r_j)^2\{dS_{ij}dX_{ji} + dX_{ij}dS_{ji}\} \\
&- \frac{1}{2}\sum_{ij}(r_i^2 + r_j^2)dX_{ij}dX_{ji}
\end{aligned} \tag{7}$$

Recalling the antihermiticity of the differentials and writing $ds^2 = g_{A,B}d\bar{x}^A dx^B$ with $dx^A = (dr_i, dX_{ii}, dS_{ij(i<j)}, dX_{ij(i<j)}, dS^*_{ij(i<j)}, dX^*_{ij(i<j)})$, we obtain for $g_{A,B}$:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & r_i^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (r_i - r_j)^2 & \frac{1}{2}(r_i - r_j)^2 & 0 \\
0 & 0 & 0 & \frac{1}{2}(r_i - r_j)^2 & \frac{1}{2}(r_i^2 + r_j^2) & 0 \\
0 & 0 & 0 & 0 & (r_i - r_j)^2 & \frac{1}{2}(r_i - r_j)^2 \\
0 & 0 & 0 & 0 & \frac{1}{2}(r_i - r_j)^2 & \frac{1}{2}(r_i^2 + r_j^2)
\end{pmatrix}.$$

It follows that

$$\det g_{A,B} = \prod_i r_i^2 \left(\prod_{i<j} \frac{1}{4}(r_i^2 - r_j^2)^2 \right)^2 = (\prod_i r_i^2)(\Delta_{MR}^2)^2 \tag{8}$$

where we have defined

$$\Delta_{MR}^2(r_i) \equiv \prod_{i<j} \frac{1}{4}(r_i^2 - r_j^2)^2. \tag{9}$$

The laplacian is obtained in the standard way, and it takes the form:

$$\nabla^2_{(I)} = \frac{1}{\prod_k r_k}\frac{1}{\Delta^2_{MR}}\frac{\partial}{\partial r_i}\left(\prod_k r_k \Delta^2_{MR}\right)\frac{\partial}{\partial r_i} \tag{10}$$
$$+ \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} + \sum_{ij}\frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X^*_{ij}}$$
$$- \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left(\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X^*_{ij}} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S^*_{ij}}\right)$$

For parametrization (II), we note that the second expression for the infinitesimal length squared in equation (6) can be further diagonalized, for each i and j,by introducing:

$$dY^+ = \frac{1}{\sqrt{2}}(dT + dS) \qquad dY^- = \frac{1}{\sqrt{2}}(dT - dS)$$

Then

$$\mathrm{Tr} dZ^\dagger dZ = \mathrm{Tr}[dr^2 + \frac{1}{2}[r, dY^+][r, dY^+] - \frac{1}{2}\{r, dY^-\}\{r, dY^-\}] \tag{11}$$

Writing again $ds^2 = \mathrm{Tr} dZ^\dagger dZ = g_{A,B} dx^A d\bar{x}^B$, we obtain

$$\det g_{A,B} = \prod_i 2r_i^2 (\Delta^2_{MR})^2$$

and straightfowardly

$$\nabla^2_{(II)} = \frac{1}{\prod_k r_k}\frac{1}{\Delta^2_{MR}}\frac{\partial}{\partial r_i}\left(\prod_k r_k \Delta^2_{MR}\right)\frac{\partial}{\partial r_i} \tag{12}$$
$$+ \sum_i \frac{1}{2r_i^2}\frac{\partial}{\partial Y^-_{ii}}\frac{\partial}{\partial Y^{*-}_{ii}} + \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\frac{\partial}{\partial Y^-_{ij}}\frac{\partial}{\partial Y^{*-}_{ij}}$$
$$+ \sum_{i\neq j}\frac{2}{(r_i-r_j)^2}\frac{\partial}{\partial Y^+_{ij}}\frac{\partial}{\partial Y^{*+}_{ij}}$$

3 Invariant states

We are interested in the action of the above laplacians, or Hamiltonian, on invariant states, i.e., states obtained by tracing strings of Z's and $Z^\dagger$'s :

$$\mathrm{Tr}(...Z^{n_p}Z^{\dagger^{m_p}}...Z^{n_q}Z^{\dagger^{m_q}}...)$$

These states depend only on the eigenvalues r_i of the radial matrix R and on the unitary matrix

$$Q \equiv VUV^\dagger = WV^\dagger$$

After some algebra one obtains the final form of the laplacian:

$$\begin{aligned}\nabla^2 &= \frac{1}{\Delta_{MR}^2}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i}\left(r_i\Delta_{MR}^2\right)\frac{\partial}{\partial r_i} - \sum_i \frac{1}{r_i^2}E_{ii}^{(L)}E_{ii}^{(L)} \\ &- \sum_{i\neq j}\left(\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}(E_{ij}^{(L)}E_{ji}^{(L)}+E_{ij}^{(R)}E_{ji}^{(R)}) - \frac{4r_ir_j}{(r_i^2-r_j^2)^2}(E_{ij}^{(L)}E_{ji}^{(R)}+E_{ij}^{(R)}E_{ji}^{(L)}\right)\end{aligned} \quad (1\text{:}$$

where we have introduced the generators of left and right $U(N)$ "rotations":

$$E_{ji}^{(L)} = Q_{jb}\frac{\partial}{\partial Q_{ib}} \quad E_{ji}^{(R)} = Q_{ai}\frac{\partial}{\partial Q_{aj}}$$

The states are subject to the constraint

$$E_{ii}^{(L)} = E_{ii}^{(R)} \quad (14)$$

4 Radial fermionization

For the hamiltonian system, it is a well known result that the singlet sector of a $N \times N$ single hermitean matrix hamiltonian with a potential depending only on its eigenvalues is equivalent to a system of N non-interacting fermions. This is a result of the anti-symmetry under the exchange of any two coordinates of the Van der Monde determinant

$$\Delta(x_k) = \prod_{i<j}(x_i - x_j)$$

Returning to two matrices (or a single complex matrix), We consider the case of a potential that depends only on the eigenvalues of the radial matrix U. An example would a potential of the form:

$$V(X_1, X_2) = \mathrm{Tr} v(ZZ^\dagger) = \mathrm{Tr} v(Z^\dagger Z)$$

with $v(x)$ a polynomial. Then, on "s-states" (independent of the "angular" degrees of freedom Q), and letting $\rho_i = r_i^2$,

$$-\frac{1}{2}\nabla^2 = -\frac{1}{2}\frac{1}{\Delta_{MR}^2}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i}\left(r_i \Delta_{MR}^2\right)\frac{\partial}{\partial r_i} = -\frac{2}{\Delta^2(\rho)}\sum_i \frac{\partial}{\partial \rho_i}\left(\rho_i \Delta^2(\rho)\right)\frac{\partial}{\partial \rho_i}$$

This kinetic energy operator acts on symmetric wavefunctions Φ. Defining

$$\Psi = \Delta\ \Phi \quad , \tag{15}$$

its action on Ψ takes the form:

$$\begin{aligned}
&- \frac{2}{\Delta}\sum_i \frac{\partial}{\partial \rho_i}\rho_i \Delta^2 \frac{\partial}{\partial \rho_i}\frac{1}{\Delta} = -2\sum_i \frac{1}{\Delta}\frac{\partial}{\partial \rho_i}\Delta\ \rho_i\ \Delta \frac{\partial}{\partial \rho_i}\frac{1}{\Delta} \\
&= - 2\sum_i \left(\frac{\partial}{\partial \rho_i} + \sum_{k\neq i}\frac{1}{\rho_i - \rho_k}\right)\rho_i\left(\frac{\partial}{\partial \rho_i} - \sum_{j\neq i}\frac{1}{\rho_i - \rho_j}\right) \\
&= - 2\sum_i \left(\frac{\partial}{\partial \rho_i}\rho_i\frac{\partial}{\partial \rho_i} - \sum_{j\neq i}\frac{1}{\rho_i - \rho_j} + \sum_{j\neq i}\frac{\rho_i}{(\rho_i - \rho_j)^2} - \sum_{j\neq i}\sum_{k\neq i}\frac{\rho_i}{\rho_i - \rho_k}\frac{1}{\rho_i - \rho_j}\right)
\end{aligned} \tag{16}$$

The second term clearly vanishes, and the last two vanish due to the identity

$$\sum_{\substack{i,j,k\\ i\neq j\\ i\neq k\\ j\neq k}} \frac{\rho_i}{(\rho_i-\rho_k)(\rho_i-\rho_j)} = 0$$

which generalizes the identity applicable to the single hermitean case. It is easily proven by choosing any three distinct eigenvalues.

Therefore, the eigenvalue equation for the energies of the system takes the form,

$$\left(-2\sum_i \frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i} + v(\rho_i)\right)\Psi = \left(-\frac{1}{2}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i}r_i\frac{\partial}{\partial r_i} + v'(r_i)\right)\Psi = E\Psi \tag{17}$$

This is the "s-state" Schroedinger equation for N non-interacting $2+1$ dimensional non-relativistic fermions.

5 Density description

In this section we describe the key features of the density descripton of radially symmetric fermions int terms of the density of radial eigenvalues. We use the collective field theory approach of Jevicki and Sakita.

The existence of such a descripton requires the identification of a suitable set of invariant operators which close under "'joining"' and "'splitting"', equivalent to the closure of underlying Schwinger-Dyson equations.

Remarkably, the following set can be identified as such:

$$\Phi_k = \mathrm{Tr} e^{ikZ^\dagger Z} = \sum_i e^{ikr_i^2} \quad ; \quad \Phi(x) = \int \frac{dk}{2\pi} e^{-ikx}\Phi_k = \sum_i \delta(x - r_i^2) \tag{18}$$

One has

$$\frac{\partial \Phi_k}{\partial Z^\dagger_{ij}} = ik\left(Z e^{ikZ^\dagger Z}\right)_{ji} \ , \quad \frac{\partial \Phi_k}{\partial Z_{ij}} = ik\left(e^{ikZ^\dagger Z}Z^\dagger\right)_{ji} \tag{19}$$

from which the "'joining"' operator Ω takes the form

$$\Omega_{kk'} = \frac{\partial\Phi_k}{\partial Z^\dagger_{ij}}\frac{\partial\Phi_{k'}}{\partial Z_{ji}} = -kk'\mathrm{Tr}Z^\dagger Z e^{i(k+k')Z^\dagger Z} \tag{20}$$

$$\Omega_{xx'} = \int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}e^{-ikx}e^{-ik'x}\Omega_{kk'} = \partial_x\partial_{x'}\left[x\Phi(x)\delta(x-x')\right]$$

For the "'splitting"' operator ω

$$\omega_k = \frac{\partial^2\Phi_k}{\partial Z^\dagger_{ij}\partial Z_{ji}} = -k\int_0^k dk'\Phi_k\mathrm{Tr}Z^\dagger Z e^{i(k-k')Z^\dagger Z} + ikN\Phi_k$$

$$= -k\sum_{ij}\int_0^k dk' e^{ik'r_i^2}e^{i(k-k')r_j^2}r_j^2 + ikN\sum_i e^{ik'r_i^2} \tag{21}$$

After careful manipulation and regularization:

$$w_x = -\partial_x\left[x\Phi(x)\left(2\int\frac{dy\Phi(y)}{x-y}\right)\right] \tag{22}$$

In the density variable descripton, the Jacobian J associated with the change of variables to the invariant operators satisfies:

$$\left(\int dx'\Omega_{xx'}\frac{\partial}{\partial\Phi(x')} + w_x + \int dx'\frac{\partial\Omega_{xx'}}{\partial\Phi(x')}\right)\ J = 0$$

Since $\int dx'\partial\Omega_{xx'}/\partial\Phi(x') = 0$, if follows from (20) and (22) that

$$\partial_x\frac{\partial}{\partial\Phi(x)}\ \ln J = 2\int\frac{dy\Phi(y)}{x-y}$$

The solution is

$$lnJ = \int dx\int dy\Phi(x)\Phi(y)\ln|x-y|\ ; \quad J = \prod_{i<j}\frac{1}{4}(r_i^2-r_j^2)^2 = \Delta^2_{MR}(r_i)$$

in precise agreement with the results of Section 2, e.g., (8). The prefactor in (8) is simply the result of the change of variables $x = r^2$.

How does the repulsion amongst the radial eigenvalues express itself as a contribution to the potential? This is given by:

$$\omega\Omega^{-1}\omega = \int_0^\infty dx\, x\Phi(x) \left(\int_0^\infty \frac{dy\ \Phi(y)}{x-y} \right)^2$$

Let us introduce a density of radial eigenvalues $\phi(r)$ such that

$$\int_0^\infty dx\Phi(x)\ f(x) = \int_0^\infty 2rdr\Phi(r^2)\ f(r^2) \equiv \int_0^\infty dr\phi(r)\ f(r^2)$$

and extend the domain of definition of $\phi(r)$ to the real line by requiring $\phi(-r) = \phi(r)$. Then

$$\begin{aligned} \omega\Omega^{-1}\omega &= \int_0^\infty dr\ r^2\phi(r) \left(\int_0^\infty \frac{ds\ \phi(s)}{r^2-s^2} \right)^2 \\ &= \frac{1}{2}\int_{-\infty}^\infty dr\phi(r) \left(\int_0^\infty \frac{ds\ r\ \phi(s)}{r^2-s^2} \right)^2 \\ &= \frac{1}{8}\int_{-\infty}^\infty dr\phi(r) \left(\int_{-\infty}^\infty \frac{ds\ \phi(s)}{r-s} \right)^2 \\ &= \frac{\pi^2}{24}\int_{-\infty}^\infty dr\phi^3(r) \end{aligned} \tag{23}$$

Remarkably, as is the case in the collective field descripton of the singlet sector of the single hermitean matrix [1], a local cubic potential is generated in the bosonized radially symmetric sector of the $2+1$ fermions.

6 More complex matrices

As is well known, the case of larger number of matrices is of great importance in the context of the AdS/CFT corespondence. Of particular importance is the case of 3 complex matrices, which are associated with the 3 Higgs of the bosonic sector of $N=4$ SYM.

Let us consider in general m complex matrices Z_A , $A = 1, ..., m$. Then

$$\sum_A Z_A^\dagger Z_A$$

[1]In this case the cubic potential is the Thomas-Fermi density term of 1 dimensional fermions

is an Hermitean, positive definite matrix. As in the previous subsections, we will denote its eigenvalues by r_i^2. The corresponding density variables are:

$$\Phi_k = \mathrm{Tr} e^{ik\sum_B Z_B^\dagger Z_B} = \sum_i e^{ikr_i^2} \quad ; \quad \Phi(x) = \int \frac{dk}{2\pi} e^{-ikx}\Phi_k = \sum_i \delta(x - r_i^2) \tag{24}$$

One has

$$\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} = ik\left(Z_A e^{ikZ_B^\dagger Z_B}\right)_{ji} \qquad \frac{\partial \Phi_k}{\partial (Z_A)_{ij}} = ik\left(e^{ikZ_B^\dagger Z_B} Z_A^\dagger\right)_{ji} \tag{25}$$

For Ω, the result is identical to the that of the previous section:

$$\begin{aligned}
\Omega_{kk'} &= \frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} = -kk' \mathrm{Tr} Z_A^\dagger Z_A e^{i(k+k')Z_B^\dagger Z_B} \\
\Omega_{xx'} &= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'x} \Omega_{kk'} = \partial_x \partial_{x'} \left[x\Phi(x)\delta(x - x')\right]
\end{aligned} \tag{26}$$

For ω, one obtains

$$\begin{aligned}
\omega_k &= \frac{\partial^2 \Phi_k}{\partial (Z_A)^\dagger_{ij} \partial (Z_A)_{ji}} = -k \int_0^k dk' \Phi_k \mathrm{Tr} Z^\dagger Z e^{i(k-k')Z^\dagger Z} + ik\; mN\; \Phi_k \\
&= -k \sum_{ij} \int_0^k dk' e^{ik' r_i^2} e^{i(k-k')r_j^2} r_j^2 + ik\; mN \sum_i e^{ik' r_i^2}
\end{aligned} \tag{27}$$

As described in the previous section, this yields

$$w_x = -\partial_x \left[x\Phi(x)\left(2\int \frac{dy\Phi(y)}{x-y} + \frac{N(m-1)}{x}\right)\right] \tag{28}$$

and hence the Jacobian satisfies

$$\partial_x \frac{\partial}{\partial \Phi(x)} \ln J = 2\int \frac{dy\Phi(y)}{x-y} + \frac{N(m-1)}{x}$$

A full treatment of the ensuing Jacobian is beyond the scope of this presentation, but the first term on the right hand of side shows it to be multiplied by the Van der Monde type determinant Δ_{MR}^2, associated with the radial inter-eigenvalue repulsion.

7 Integral and non-supersymmetric strong coupling

Suppose we now consider the Yang-Mills coupling potential

$$\begin{aligned} -g_{YM}^2 \mathrm{Tr}[X_1, X_2][X_1, X_2] &= 2g_{YM}^2 \left(\mathrm{Tr} R^4 - \mathrm{Tr} R^2 U R^2 U^\dagger\right) \\ &= 2g_{YM}^2 \left(\mathrm{Tr} r^4 - \mathrm{Tr} r^2 Q r^2 Q^\dagger\right) \end{aligned}$$

The (path) integral is then proportional to

$$\int dr_i \prod r_i \Delta_{MR}^2 e^{-(\frac{w^2}{2}\sum r_i^2 + 2g_{YM}^2 \sum_i r_i^4)} \int dU e^{2g_{YM}^2 \mathrm{Tr} R^2 U R^2 R^\dagger} = \int dr_i \prod r_i \Delta_{MR}^2 e^{-(\frac{w^2}{2}\sum r_i^2 + 2g_{YM}^2 \sum_i r_i^4)} \frac{\det_{ij} e^{2g_{YM}^2 r_i^2 r_j^2}}{\Delta_{MR}^2}$$

At weak coupling, it is the expression

$$\frac{\det_{ij} e^{2g_{YM}^2 r_i^2 r_j^2}}{\Delta_{MR}^2}$$

that has a large N weak coupling expansion.

For strong coupling, it may be possible that the radial inter-eigenvalue repulsion cancels out and a different mechanism is at work.