Towards an understanding of 3d Newton-Cartan gravity

Yang Lei

University of Witwatersrand

Work with Simon Ross (1504.07252), Jelle Hartong, Niels Obers and Gerben Oling (1604.08054, 1712.05794)
Introduction

Newton-Cartan gravity

Conclusion and future work
Plan of the talk

1. Motivations
   - Holography is the most powerful tool to study quantum effects of gravity. How general the idea of holography can be?
   - Puzzles in Lifshitz higher spin theory

2. Examples of Newton-Cartan Chern-Simons theory
   - Torsionless Chern-Simons NC gravity
   - Twistless torsional Chern-Simons NC gravity = Schrödinger gravity
   - pseudo-Newton-Hooke gravity

3. Discussion
Introduction: holography based on symmetry

It is believed the principle for quantum gravity theory is holographic principle. In the past decades, people are trying answer the question: how general the holography principle can be? The theories dual to each other have the same symmetry group. In above case, $SO(d, 2)$. In the special case

- $d = 2$, we have $SO(2, 2) \sim SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. This is $AdS_3/CFT_2$ duality.
- Higher spin generalization: $SL(N, \mathbb{R})$ or more complicated $hs[\lambda]$;
- Taking non-relativistic limit: this is found to be Newton-Cartan/NCFT duality
Figure 1:

The theory cube

Galilean
Quantum
Gravity

Newtonian
Gravity

General
Relativity

Quantum
Gravity

Classical
Mechanics

Special
Relativity

Quantum
Mechanics

Quantum
Field Theory

Figure 1:
Why non-relativistic higher spin theory

There are wide classes of strongly correlated systems which scale anisotropically in time and space directions. These are called quantum Lifshitz fixed points.

\[ t \rightarrow \lambda^z t; \quad x \rightarrow \lambda x \]

Many body field theories describing anisotropic fixed points were proposed to be holographically dual to gravity in the background of Lifshitz geometries, where time and space scale asymptotically with the same ratio \( z \). (Kachru, Liu, Mulligan, 08)

The question is: \textbf{whether one can have conformal symmetry for anisotropic system?}
Non-relativistic spacetimes

Non-relativistic spacetime solutions are found in Einstein gravity theory with gauge matter fields. Lifshitz spacetimes: (Kachru, Liu, Mulligan, 08)

\[ ds^2 = -r^{2z} \, dt^2 + \frac{dr^2}{r^2} + r^2 \, dx_i dx_i \]

Schrödinger spacetime (Son, 08)

\[ ds^2 = -r^{2z} \, dt^2 - 2r^2 \, dt d\xi + \frac{dr^2}{r^2} + r^2 \, dx_i dx_i \]

\( z > 1 \) generically for Null Energy condition. Schrödinger algebra contains special conformal transformation at \( z = 2 \).
Higher spin Lifshitz solution

A Lifshitz gravity theory with conformal symmetry was constructed in 3D using Chern-Simons higher spin theory later. \( z = 2 \) Lifshitz solution is (Gary, Grumiller, Rashkov 1201.0013):

\[
\begin{align*}
  a &= cW_2 dt + kL_1 dx \\
  \bar{a} &= pW_{-2} dt + qL_{-1} dx
\end{align*}
\]

whose metric is

\[
ds^2 = -cpe^{4\rho} dt^2 + d\rho^2 + kqe^{2\rho} dx^2
\]

Schrödinger solution \( z = 2 \):

\[
\begin{align*}
  a &= (cW_2 + kL_1) dt \\
  \bar{a} &= pW_{-2} dt + qL_{-1} d\xi
\end{align*}
\]

results in \( z = 2 \) Schrödinger metric

\[
ds^2 = -cpe^{4\rho} dt^2 + kqe^{2\rho} dt d\xi + d\rho^2
\]
Lifshitz spacetime is not vacuum solution to Einstein equation. In fact, the Lifshitz spacetime is a solution to Einstein gravity with massive gauge fields

\[ ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 (dx^2 + dy^2) \]

\[ A = \sqrt{\frac{2(z - 1)}{z}} r^z dt; \quad m_A^2 = 4z \]

From Einstein equation point of view, non-AdS spacetime solutions in higher spin Chern-Simons theory are supported by some matter fields. In this case, spin-3 matter.

\[ \phi = -\frac{1}{4} (k^2 p - cq^2) e^{4\rho} dt dx^2 \]

But one can turn off spin-3 matter fields in equation above without affecting Lifshitz metric. How can one have Lifshitz spacetime without matter to support it?
Frame like fields have 24 degrees of freedom. Local Lorentz invariance fixes 8 gauge redundancies, leaving $16 = 10 + 6$ physical degrees of freedom.

In spin-2 gravity, non-equivalence would happen if coordinate singularity is present or $\det(e_a^\mu) = 0$.

In spin-3 case, map between frame and metric is a 16 to 16 variable map. Map works if and only if frame is invertible or metric is invertible. The idea is to ask whether spin-connection can be uniquely determined by frame? I.e.

$$de + \omega \wedge e + e \wedge \omega = 0$$

This is called non-degeneracy condition.

It turns out Lifshitz does not satisfy non-degeneracy condition but Schödinger solution satisfies the condition.

$$\omega_{\text{Lif}} = \frac{1}{2}(A + \bar{A}) + \lambda_1 \hat{A} + \lambda_2 W_0 dx$$
The Schrödinger solution

\[ a_t = L_1 + \sigma W_2; \quad a_{x^-} = 0 \]
\[ \bar{a}_t = \sigma W_{-2}; \quad \bar{a}_{x^-} = 2L_{-1} \]

which corresponds to metric

\[ ds^2 = -\sigma^2 r^4 dt^2 + \frac{dr^2}{r^2} + 2r^2 dtdx^- \]
\[ \phi_{t--} = \frac{\sigma}{3} r^4; \quad \phi_{ttt} = -\frac{\sigma}{4} r^4 \]

They are the solutions of action

\[ \mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_F, \]

The spin-3 fields part is up to quadratic order

\[ \mathcal{L}_F(\phi^2) = \phi^{\mu\nu\rho}(\mathcal{F}_{\mu\nu\rho} - \frac{3}{2} g_{\mu\nu} \mathcal{F}_{\rho}) + m_1 \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} + m_2 \phi_\mu \phi^\mu 
+ 3R_{\rho\sigma}(k_1 \phi_\rho^{\mu\nu} \phi^{\sigma\mu\nu} + k_2 \phi_\mu^{\rho\sigma} \phi^\mu + k_3 \phi^{\rho} \phi^{\sigma}) 
+ 3R(k_4 \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} + k_5 \phi_\mu \phi^\mu) \]
Introduction

Newton-Cartan gravity

Conclusion and future work
Why Newton-Cartan plays the role?

▶ Dynamical Newton-Cartan gravity is equivalent to Horava-Lifshitz gravity. (Hartong, Obers, 1504.07461)
▶ There is no need to discuss non-relativistic gravity by starting from relativistic gravity. The role of extra matter field needs explanation.
▶ Horava-Lifshitz gravity allows Lifshitz spacetime as vacuum solution without matter to support.
▶ In Newton-Cartan gravity, metric can be degenerate.
Classifications

Based on the features of torsion, Newton-Cartan gravity is classified into three kinds:

- **Torsionless Newton-Cartan gravity**: $d\tau = 0$. This corresponds to projectable Horava-Lifshitz gravity.

- **Twistless torsional Newton-Cartan gravity**: $\tau \wedge d\tau = 0$. This means there is a field $b$, such that $d\tau - zb \wedge \tau = 0$. This means $b$ is the field coupled to scaling generator $D$ and $[H, D] = zH$. Therefore, we should expect once we introduce scaling symmetry into Galilean algebra, (Schrödinger algebra), one should expect we obtain this kind of theory. This also corresponds to non-projectable Horava Lifshitz gravity.

- **Torsional Newton-Cartan gravity**: no constraints on $\tau$ (Never seen examples yet..)
The gravity theory is classified with respect to its isometry group. We are interested in several examples:

- **Galilean/Bargmann algebra** contains \( \{H, G_a, P_a, J, (N)\} \)

\[
\begin{align*}
[J, P_a] &= \epsilon_{ab} P_b, \\
[J, G_a] &= \epsilon_{ab} G_b, \\
[H, G_a] &= P_a, \\
[P_a, G_b] &= N \delta_{ab} \quad \text{(Bargmann central extension)}
\end{align*}
\]

\( a = 1, 2. \) Without non-relativistic boost, Galilean reduces to Lifshitz-like algebra.

- **Newton-Hooke algebra**, it includes Galilean algebra with extra.

\[
[H, P_a] = -\Lambda G_a
\]

- **Carrollian algebra** (\( c \to 0 \) limit)
Schrödinger algebra (introducing dilatation $D$)

\[
[H, D] = zH, \quad [P_a, D] = P_a, \quad [G_a, D] = (1 - z)G_a
\]

\[
[D, N] = (z - 2)N
\]

At $z = 2$, symmetry can be enhanced to include special conformal transformation $K$, so that $H, D, K$ form a $SL(2, \mathbb{R})$ and $[P_a, K] = G_a$

Galilean conformal algebra (Bagchi, Gopakumar, 0902.1385) (too many commutators...)

BMS algebra, with identification $BMS_3$ is identical to $GCA_2$

\[
[L_n, L_m] = (n - m)L_{m+n} + \frac{c_L}{12}n(n^2 - 1)
\]

\[
[L_n, M_m] = (n - m)M_{m+n} + \frac{c_M}{12}n(n^2 - 1)
\]
Einstein Gravity by gauging Poincare groups

In general dimensions, the Poincare algebra is

\[
[M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a \\
[M_{ab}, M_{cd}] = \eta_{ac} M_{bd} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad} + \eta_{bd} M_{ac}
\]

Then gauge field

\[ A_\mu = \frac{1}{2} M_{ab} \omega^a_\mu + P_a e^a_\mu \]

transforms under gauge transformation as \( \delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda] \)

where

\[ \Lambda = \frac{1}{2} M_{ab} \sigma^{ab} + P_a \zeta^a \]

we can then derive

\[ \delta e^a_\mu = \partial_\mu \zeta^a + e^c_\mu \sigma^a_c + \omega^a_{\mu c} \zeta_c \]
To identify gauge transformation $\zeta^a$ with diffeomorphism by $\zeta^a = \xi^\rho e^a_\rho$, we need to study

$$\mathcal{L}_\xi e^a_\mu = \xi^\rho (\partial_\rho e^a_\mu - \partial_\mu e^a_\rho) + \partial_\mu (\xi^\rho e^a_\rho)$$

$$= -\xi^\rho R_{\rho\mu}(P_a) + \xi^\rho (e^c_\rho \omega^a_{\mu} + e^a_\mu \omega^c_{\rho}) + \partial_\mu (\xi^\rho e^a_\rho)$$

where

$$F = dA + A \wedge A = P_a R^a(P) + M_{ab} R^{ab}(M)$$

we see we can make the identification as long as curvature constraint $R_{\rho\mu}(P_a) = 0$ is imposed. This is the torsion free condition for Einstein gravity which is used to solve spin-connection uniquely from veilbein.

**Invariant field**

$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ is the geometric field invariant under local Lorentz transformation.
First let me explain how to understand Newton-Cartan gravity in general dimensions.

\[ A = H\tau + P_a e^a + G_a \omega^a + J\omega + Nm \]

\( m \) is the central extended gauge field, interpreted as mass current. Then it is easy to calculate

\[ F = dA + A \wedge A \]

\[ = HR(H) + R^a(P)P_a + R^a(G)G_a + R(J)J + R(N)N \]

We would like to impose the curvature constraints so that spin-connection is uniquely solved in terms of smaller set of fields. In this case \( \tau, e^a \) and the gauge field \( m \). Then we need to impose \( R(H) = R^a(P) = R(N) = 0 \). (See Bergshoef et. 1011.1145)
After imposing curvature constraints, one can check

\[
\begin{align*}
\delta \tau_\mu & = \partial_\mu (\xi^\rho \tau_\rho) - \xi^\rho R_{\rho \mu} (H) \\
\delta e^a_\mu & = \partial_\mu (\xi^\rho e^a_\rho) - \xi^\rho R^a_{\rho \mu} (P_a) + (\xi^\rho \omega^a_\rho) e^b_\mu + (\xi^\rho \omega^a_\rho) \tau_\mu \\
& = \partial_\mu \xi^a + \lambda e^b_\mu + \lambda^a \tau_\mu
\end{align*}
\]

where \( \xi^\rho = \{ \xi^0 = \xi^\rho \tau_\rho, \xi^a = \xi^\rho e^a_\rho \} \). Therefore, gauge transformation is equivalent to diffeomorphism up to local Galilean transformation! The fields invariant under local Galilean transformations are \( \tau_\mu \) and \( h^{\mu \nu} = e^a_\mu e^b_\nu \delta^{ab} \). One can always define the Lorentzian metric to be

\[
g_{\mu \nu} = -\tau_\mu \tau_\nu + h_{\mu \nu}
\]

**Torsion constraints**

One can show \( d\tau \) is proportional to \( \Gamma^\rho_{[\mu \nu]} \), i.e. torsion tensor. So Bargmann theory with \( d\tau = 0 \) is called torsionless Newton-Cartan gravity.
Chern-Simons theory

In $d = 3$ dimensional spacetime, one usually uses $J_a = \frac{1}{2} \epsilon_{abc} M^{bc}$ as the algebra generator. For

$$A = P_a e^a + J_a \omega^a$$

we can write down Chern-Simons action

$$S_{cs} = \text{Tr}_R \int \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Tr is actually an invariant bilinear product on Lie algebra, which maps two generators to a number. In case of $\text{ISO}(2,1)$, this bilinear product is (Witten 88')

$$< P_a, J_b > = \eta_{ab}$$

resulting in Einstein-Hilbert action

$$S = \int e^a \wedge R_a(\omega)$$
Chern-Simons Newton-Cartan gravity

The Galilean/Bargmann algebra are not semi-simple. The Cartan-Killing metric is degenerate. It is pointed out by Witten that for non-semisimple Lie algebra, non-degenerate invariant bilinear product can exist sometimes. For Lie algebra \([T_A, T_B] = f_{AB}^C T_C\), such invariant bilinear product is defined by solving

\[ f_{AB}^D \Omega_{CD} + f_{AC}^D \Omega_{BD} = 0 \]

In (Witten, Nappi, 9310112), they consider algebra

\(T_A = \{J, P_1, P_2, T\}\),

\([J, P_a] = \epsilon_{ab} P_b, \quad [P_a, P_b] = \epsilon_{ab} T\),

which is known as central extended Euclidean group \(E_2^c\). They find bilinear product

\(< P_a, P_b > = < J, T > \delta_{ab} \)

Such theory thus allows a WZW model to be built.
However, even by solving the equations above, one cannot have non-degenerate bilinear product for Bargmann algebra. This makes it hard to build Chern-Simons theory. A resolution suggested in (Papageorgiou, Schroers, 0907.2880) is to introduce more generators in the Lie algebra so that bilinear product can be non-degenerate. They find we only need to extend

\[ [G_a, G_b] = S \epsilon_{ab} \]

Then \{H, P_a, G_a, J, N, S\} form an NDBP, which is given by

\[ \langle H, S \rangle = - \langle J, N \rangle = - \langle P_1, G_2 \rangle = \langle P_2, G_1 \rangle \]

Therefore, we can write down a Chern-Simons gravity action!
Chern-Simons action

Let’s take $A = H\tau + P_a e^a + G_a \Omega^a + J\Omega + Nm + S\zeta$, then

$$\mathcal{L} = \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$= -\epsilon_{ab} R^a(G) \wedge e^b + \frac{1}{2} \epsilon_{ab} \tau \wedge \Omega^a \wedge \Omega^b - \Omega \wedge dm$$

$$+ \zeta \wedge d\tau$$

where

$$R^a(G) = d\Omega^a - \epsilon^{ab} \Omega \wedge \Omega^b$$

As one can see $\zeta$ works as Lagrangian multiplier, used to impose torsionless condition $d\tau = 0$! The extra generators are necessary.
Projectable Hořava-Lifshitz action

The action in terms of the metric field is written as

\[ \mathcal{L} = e \left( h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - (h^{\mu\nu} K_{\mu\nu})^2 - \tilde{\Phi} R \right), \]

where

\[ \tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu. \]

This is exactly the projectable Hořava-Lifshitz gravity with \( \lambda = 1 \). Recall the generic Hořava Lifshitz action has two independent kinematic terms of \( h_{\mu\nu} \) and is of the form

\[ \mathcal{L} = e \left( h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda (h^{\mu\nu} K_{\mu\nu})^2 + \text{potential} \right), \]
TTNC = Schrödinger gravity

For $z = 2$ Schrödinger algebra, we need to introduce three more generators to make Jacobi identity satisfied and bilinear product non-degenerate. (Niels Obers, Jelle Hartong, YL, 1604.08054)

$$A = H_T + P_a e^a + G_a \omega^a + J \omega + N m + D b + K f + S \zeta + Y \alpha + Z \beta.$$

where $[P_a, P_b] = \epsilon_{ab} Z$ and $[P_a, G_b] = N \delta_{ab} - Y \epsilon_{ab}$. The whole extended Schrödinger algebra is

$$[H, D] = 2H, \quad [H, K] = D, \quad [D, K] = 2K,$$

$$[H, G_a] = P_a, \quad [D, P_a] = -P_a, \quad [D, G_a] = G_a,$$

$$[K, P_a] = -G_a. \quad [J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b,$$

$$[P_a, G_b] = N \delta_{ab} - Y \epsilon_{ab}, \quad [P_a, P_b] = \epsilon_{ab} Z, \quad [G_a, G_b] = \epsilon_{ab} S,$$


$$[K, Z] = 2Y, \quad [D, S] = 2S, \quad [D, Z] = -2Z.$$
Action

The bilinear product is

\[ B(H, S) = B(D, Y) = B(K, Z) = -B(J, N) = c_1, \]
\[ B(P_a, G_b) = c_1 \epsilon_{ab}, \quad B(H, K) = -c_2, \]
\[ B(D, D) = 2c_2, \quad B(J, J) = c_3, \]

The action is

\[ \mathcal{L} = 2c_1 \left[ \tilde{\mathcal{R}}^2(G) \wedge e^1 - \tilde{\mathcal{R}}^1(G) \wedge e^2 + \tau \wedge \omega^1 \wedge \omega^2 \\
- m \wedge d\omega - f \wedge e^1 \wedge e^2 + \zeta \wedge (d\tau - 2b \wedge \tau) \\
+ \alpha \wedge (db - f \wedge \tau) + \beta \wedge (df + 2b \wedge f) \right] \]

One can check \( z = 2 \) Lifshitz gauge field is indeed solution to flatness condition

\[ A = H \frac{dt}{r^2} + P_1 \frac{dr}{r} + P_2 \frac{dx}{r} - D \frac{dr}{r} - Z \frac{dx}{r} \]
Non-Projectable Hořava-Lifshitz action

The action in terms of the metric field is written as

\[
\mathcal{L} = e \left[ \left( h^{\alpha \nu} h^{\beta \mu} - h^{\alpha \mu} h^{\beta \nu} \right) K_{\alpha \mu} K_{\beta \nu} + 2 \hat{v}^{\mu} b_{\mu} h^{\nu \rho} K_{\nu \rho} \\
- 2 \left( \hat{v}^{\mu} b_{\mu} \right)^2 - \Phi \tilde{R} - 2 \hat{v}^{\mu} f_{\mu} + 2 \epsilon^{\mu \nu \rho \tau} \hat{v}^{\sigma} \beta_{\nu} R_{\mu \sigma}(K) \right].
\]

where

\[
\epsilon^{\mu \nu \rho \tau} \hat{v}^{\sigma} \beta_{\nu} R_{\mu \sigma}(K) = -\frac{1}{4} \epsilon^{\mu \nu \rho \beta_{\nu} \tau_{\rho} (\partial_{\mu} + 2 a_{\mu}) I,
\]

for

\[
I = \left( \hat{v}^{\mu} N^{-1} \partial_{\mu} N \right)^2 - 4 \left( \hat{v}^{\mu} b_{\mu} \right)^2 + 2 \hat{v}^{\nu} \partial_{\nu} \left( \hat{v}^{\mu} N^{-1} \partial_{\mu} N - 2 \hat{v}^{\mu} b_{\mu} \right) - 4 \hat{v}^{\mu} f_{\mu},
\]

The \( N \) is the factor in time-like frame \( \tau_{\mu} = N \partial_{\mu} \tau \).
Infinite extended symmetry

Take \( \hat{L}_m = \{H, D, K\} \), \( \hat{M}_m = \{S, Y, Z\} \), \( Y^i_r = \{P_a, G_a\} \)

\[
\begin{align*}
[\hat{L}_m, \hat{L}_n] &= (m - n)\hat{L}_{m+n} + \frac{c_L}{2}(m^3 - m)\delta_{m+n,0} \\
[\hat{L}_m, \hat{M}_n] &= (m - n)\hat{M}_{m+n} + \frac{c}{2}(m^3 - m)\delta_{m+n,0} \\
[\hat{L}_m, Y^i_r] &= (\frac{m}{2} - r)Y^i_{m+r}, \quad [\hat{L}_m, J_n] = -nJ_{m+n}, \\
[\hat{L}_m, N_n] &= -nN_{m+n}, \quad [Y^i_r, Y^i_s] = (r - s)N_{s+r}, \\
[Y^1_r, Y^2_s] &= -\hat{M}_{r+s} + c\left(s^2 - \frac{1}{4}\right)\delta_{r+s,0}, \quad [J_n, Y^i_r] = Y^j_{r+n}\epsilon^{ij} \\
[J_m, N_n] &= cn\delta_{m+n,0} \quad [J_n, \hat{M}_m] = -2nN_{m+n}, \\
[J_m, J_n] &= c_Jn\delta_{m+n,0}
\end{align*}
\]

"Super" GCA algebra (Bagchi, Gopakumar) with central extensions.

\[
Y^1_r = \epsilon_1 Q^1_r + \epsilon_2 Q^2_r
\]
Newton-Hooke gravity

For Newton-Hooke algebra (extended Bargmann with cosmological constant), exact analogous to AdS, there exist two parameters of non-degenerate bilinear product. This implies an isomorphism like

\[
so(2, 2) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})
\]

This turns out to be surprisingly:

\[
NH_3 = P_2^c \times P_2^c
\]

Each copy of \( P_2^c \) is defined to be

\[
[\mathcal{L}_{-1}, \mathcal{L}_0] = -\mathcal{L}_{-1}, \quad [\mathcal{L}_{-1}, \mathcal{N}_1] = -\mathcal{N}_0, \quad [\mathcal{L}_0, \mathcal{N}_1] = -\mathcal{N}_1.
\]

The bilinear product is

\[
\langle \mathcal{L}_0, \mathcal{L}_0 \rangle = \frac{1}{2} \gamma_1, \quad \langle \mathcal{L}_0, \mathcal{N}_0 \rangle = -\langle \mathcal{L}_{-1}, \mathcal{N}_1 \rangle = \gamma_2,
\]
Gravity action

If we write the gauge field as

$$A = \tau D + e^a T_a + m N + \omega M + \omega^a R_a + \zeta S.$$ 

One can write down Chern-Simons gravity action

$$\text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$= \frac{\gamma_2 + \bar{\gamma}_2}{2l} \left( 2\tau \wedge d\zeta + 2\epsilon_{ab} e^a \wedge d\omega^b + 2m \wedge d\omega \\
- \frac{1}{l^2} \tau \wedge \epsilon_{ab} e^a \wedge e^b + \tau \wedge \epsilon_{ab} \omega^a \wedge \omega^b + 2\eta_{ab} e^a \wedge \omega^b \wedge \omega \right)$$

$$+ \frac{\gamma_2 - \bar{\gamma}_2}{2} \left( \eta_{ab} \omega^a \wedge d\omega^b - \frac{1}{l^2} \eta_{ab} e^a \wedge de^b + \frac{2}{l^2} m \wedge d\tau + 2\omega \wedge d\zeta \\
+ \frac{2}{l^2} \eta_{ab} \tau \wedge e^a \wedge \omega^b - \frac{1}{l^2} \epsilon_{ab} e^a \wedge e^b \wedge \omega + \omega \wedge \epsilon_{ab} \omega^a \wedge \omega^b \right)$$

$$+ \frac{1}{l} (\gamma_1 + \bar{\gamma}_1) \tau \wedge d\omega + \frac{1}{2} (\gamma_1 - \bar{\gamma}_1) \left( \frac{1}{l^2} \tau \wedge d\tau + \omega \wedge d\omega \right)$$
Reduction from Einstein gravity

We can obtain the theory above by starting from Einstein gravity with two \( u(1) \) fields, and take non-relativistic limit.

\[
A = E^A T_A + \Omega^A J_A + Z^1 Q_1 + Z^2 Q_2.
\]

The action is

\[
\mathcal{L}_{CS} = \frac{\gamma_s + \bar{\gamma}_s}{2l} \left( 2E^A \wedge d\Omega^B \eta_{AB} + \epsilon_{ABC} E^A \wedge \Omega^B \wedge \Omega^C \\
+ \frac{1}{3l^2} \epsilon_{ABC} E^A \wedge E^B \wedge E^C \right) \\
+ \frac{\gamma_s - \bar{\gamma}_s}{2} \left( \Omega^A \wedge d\Omega^B \eta_{AB} + \frac{1}{3} \epsilon_{ABC} \Omega^A \wedge \Omega^B \wedge \Omega^C \\
+ \frac{1}{l^2} E^A \wedge dE^B \eta_{AB} + \frac{1}{l^2} \epsilon_{ABC} E^A \wedge E^B \wedge \Omega^C \right) \\
+ \frac{\gamma_u - \bar{\gamma}_u}{2} \left( \frac{1}{l^2} Z^1 \wedge dZ^1 + Z^2 \wedge dZ^2 \right) + \frac{\gamma_u + \bar{\gamma}_u}{l} Z^1 \wedge dZ^2.
\]
Reduction from Einstein gravity

By making identification of gauge fields according to

\[ E^2 = \tau + \frac{m}{2\alpha^2}, \quad E^a = \frac{1}{\alpha} \epsilon_{b}^a e^b, \quad Z^1 = -\tau + \frac{m}{2\alpha^2}, \]
\[ \Omega^2 = \omega + \frac{\zeta}{2\alpha^2}, \quad \Omega^a = \frac{1}{\alpha} \omega^a, \quad Z^2 = -\omega + \frac{\zeta}{2\alpha^2}. \]

and set \( \alpha \to \infty \), we can recover the pseudo-NH from Einstein. From algebra perspective,

\[ P_a = \alpha P_a, \quad K_a = \alpha K_a, \]
\[ D = \frac{1}{2} D + \alpha^2 N, \quad Q_1 = -\frac{1}{2} D + \alpha^2 N, \]
\[ M = \frac{1}{2} M + \alpha^2 S, \quad Q_2 = -\frac{1}{2} M + \alpha^2 S. \]

For each chiral copy:

\[ L_{-1} = \alpha L_{-1}, \quad L_0 = \frac{L_0}{2} + \frac{\alpha^2}{2} N_0, \quad L_1 = \alpha N_1, \quad N_0 = -\frac{L_0}{2} + \frac{\alpha^2}{2} N_0. \]
The vacuum solution can be written in AdS form

\[ \tau = d\rho , \quad h_{MN} dx^M dx^N = - \cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 , \quad m = 0 , \]

This is AdS$_3$ spacetime if written in the Lorentz metric

\[ g_{\mu\nu} = -\tau_\mu \tau_\nu + h_{\mu\nu} \]
Asymptotic symmetry

For any finite $\alpha$, $\mathcal{L}, \mathcal{N}$ is just another equivalent expression for $sl_2 \oplus u(1)$ algebra.

$$a_+ = \mathcal{N}_+ + \mathcal{F}^0 \mathcal{L}_0 + \mathcal{F}^- \mathcal{L}_- + \mathcal{F}^N \mathcal{N}_0.$$ 

Solving $\delta a = d\lambda + [a, \lambda]$, we get

$$\lambda^0 = \left( \mathcal{F}^0 - \frac{\mathcal{F}^N}{\alpha^2} \right) \lambda^+ - \partial \lambda^+ - \frac{\lambda^N}{\alpha^2}.$$ 

To make infinitesimal charge integrable $\delta Q_\lambda = -2 \oint d\varphi \langle \lambda, \delta a_+ \rangle$, we need to redefine

$$\lambda^+ = \lambda^+, \quad \lambda^N = \overline{\lambda}^N + \mathcal{F}^N \lambda^+.$$
Asymptotic symmetry

The boundary charges can then be written as
\[ Q_\lambda = \oint d\varphi \left( \lambda^+ \mathcal{T} + \bar{\lambda}^N \mathcal{J} \right) \] with currents

\[ \mathcal{T} = \gamma_1 \left( \frac{\mathcal{F}^-}{\alpha^2} - \frac{1}{2} (\mathcal{F}^0)^2 - \partial \mathcal{F}^0 - \frac{1}{2\alpha^4} (\mathcal{F}^N)^2 \right) \]
\[ + 2\gamma_2 \left( \mathcal{F}^- - \mathcal{F}^N \mathcal{F}^0 - \partial \mathcal{F}^N \right), \]
\[ \mathcal{J} = \left( 2\gamma_2 - \frac{\gamma_1}{\alpha^2} \right) \left( \mathcal{F}^0 - \frac{\mathcal{F}^N}{\alpha^2} \right). \]

\[ \delta \mathcal{T} = \lambda^+ \partial \mathcal{T} + 2\mathcal{T} \partial \lambda^+ + \gamma_1 \partial^3 \lambda^+ + \mathcal{J} \partial \bar{\lambda}^N - 2 \left( \gamma_2 - \frac{\gamma_1}{2\alpha^2} \right) \partial^2 \bar{\lambda}^N, \]
\[ \delta \mathcal{J} = \lambda^+ \partial \mathcal{J} + \mathcal{J} \partial \lambda^+ + \left( 2\gamma_2 - \frac{\gamma_1}{\alpha^2} \right) \left( \partial^2 \lambda^+ + \frac{2\partial \bar{\lambda}^N}{\alpha^2} \right). \]
In terms of Fourier modes,

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + 2\pi \gamma_1 m^3 \delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{N}_n] &= -n\mathcal{N}_{m+n} - 2\pi i m^2 \left(2\gamma_2 - \frac{\gamma_1}{\alpha^2}\right) \delta_{m+n,0}, \\
[\mathcal{N}_m, \mathcal{N}_n] &= -\frac{4\pi m}{\alpha^2} \left(2\gamma_2 - \frac{\gamma_1}{\alpha^2}\right) \delta_{m+n,0}.
\end{align*}
\]

As we set \(\alpha \to \infty\), we have so-called twisted Warped Virasoro algebra

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + 2\pi \gamma_1 m(m^2 - 1) \delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{N}_n] &= -n\mathcal{N}_{m+n} - 4\pi i \gamma_2 m(m + 1) \delta_{m+n,0}.
\end{align*}
\]
Therefore, we have analogy with AdS/CFT.

<table>
<thead>
<tr>
<th>AdS/CFT holography</th>
<th>novel holography</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk gravity in $AdS_3$</td>
<td>Bulk gravity in pNC 3D</td>
</tr>
<tr>
<td>Topological massive gravity</td>
<td>pNC gravity with dynamics</td>
</tr>
<tr>
<td>isometry $SL(2,R) \times SL(2,R)$</td>
<td>isometry $P^c_2 \times P^c_2$</td>
</tr>
<tr>
<td>isometry of $AdS_2$ is $SL(2,R)$</td>
<td>isometry of pNC$_2$ is $P^c_2$</td>
</tr>
<tr>
<td>$Coset \frac{so(2,2)}{su(2)}$</td>
<td>$Coset \frac{P^c_2 \times P^c_2}{P^c_2 \times U(1)}$</td>
</tr>
<tr>
<td>$Vir \times Vir$</td>
<td>$U(1)_t Vir \times U(1)_t Vir$</td>
</tr>
<tr>
<td>CFT</td>
<td>twisted WCFT (maybe)</td>
</tr>
<tr>
<td>WZW Exist, Liouville theory</td>
<td>WZW Exist, (Nappi-Witten)</td>
</tr>
<tr>
<td>$d$-dimension $SO(d + 1,2)$</td>
<td>extension $SO(d,1) \times U(1)$</td>
</tr>
<tr>
<td>BTZ black hole</td>
<td>???</td>
</tr>
<tr>
<td>Higher spin generalization $hs[\lambda]$</td>
<td>???</td>
</tr>
<tr>
<td>EE as length of geodesics</td>
<td>??</td>
</tr>
</tbody>
</table>
Discussion

- The $u(1)$ gauge fields in Einstein gravity can come from $R$-symmetry current of $\mathcal{N} = (2, 2)$ supergravity.
- Define a coupling constant $g = \alpha^{-2}$ and identify the energy $E$ and charge $J$ as $E = \alpha^{-2}D = \mathcal{N} + \frac{g}{2}D$, $J = -\alpha^{-2}Q_1 = \mathcal{N} - \frac{g}{2}D$. The limit $\alpha \to \infty$ zooms in on states close to $E = J$ which is the lowest lying state in the spectrum, so by unitarity we have $E \geq J$. The Spin Matrix theory (1409.4417) limit corresponds to sending $g \to 0$ while keeping $(E - J)/g$ fixed, which is precisely what happens in our limit for the operators $D$ and $Q_1$ along with the limit $\alpha \to \infty$.
- Recall Schrödinger gravity is also mysteriously related to super Lie-algebra.
- Simpler holography because foliation is the radial direction.
Compare to other models

<table>
<thead>
<tr>
<th></th>
<th>Rel Bulk</th>
<th>Non-Rel Bulk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rel FT</td>
<td>AdS/CFT</td>
<td><strong>pseudo-NH holography</strong></td>
</tr>
<tr>
<td>Non-Rel FT</td>
<td>Massive gauge fields/GB gravity</td>
<td>Newton-Cartan holography</td>
</tr>
</tbody>
</table>
Introduction

Newton-Cartan gravity

Conclusion and future work
Conclusion

- We have successfully given an interpretation of \( z = 2 \) Lifshitz solution in Chern-Simons theory in terms of Newton-Cartan gravity rather than higher spin theory. However, it is not clear how to interpret Lifshitz solutions with other dynamical exponent \( z \).

- We pave the way to build up a non-relativistic gravity theory which is completely analogous to AdS\(_3\) gravity. It is interesting to talk about any generalizations about it.
Future work

We have many future work can do:

▶ Chern-Simons Lifshitz solution for other dynamical exponent.
▶ String embedding from AdS reduction
▶ What is the dual field theory?
▶ Higher spin
▶ BTZ black hole analogue
▶ Everything else you did in $AdS_3/CFT_2$