Irreversibility Entropic Theorems

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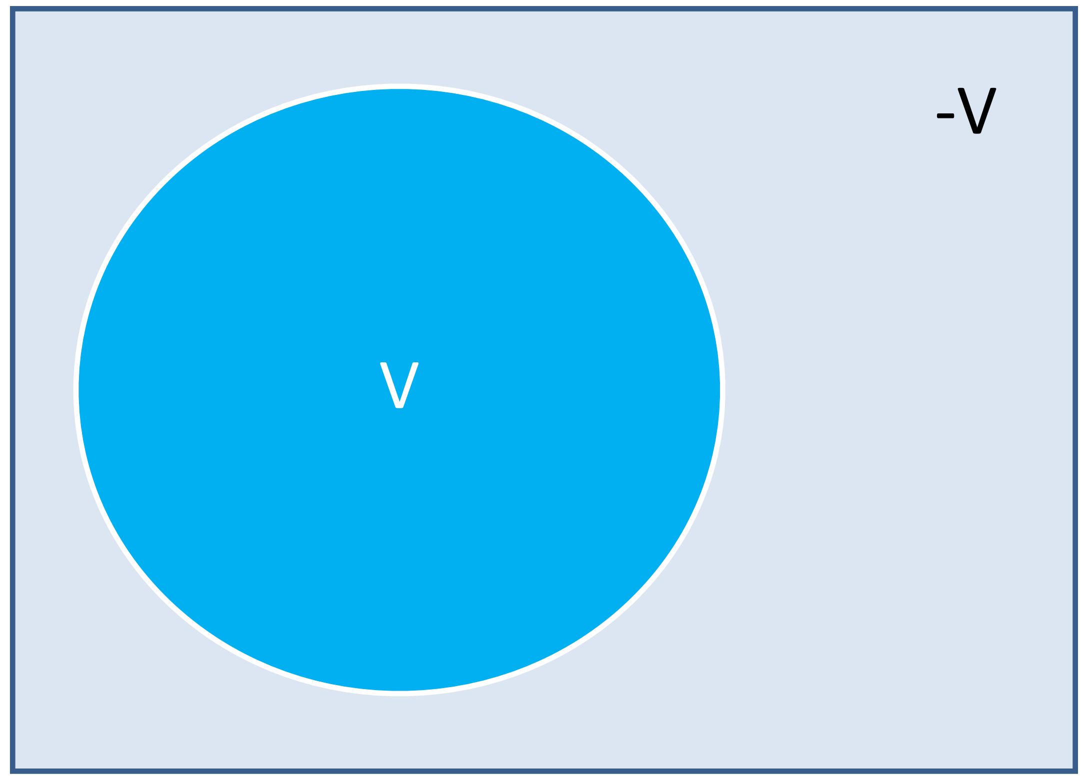
Fifth Mandelstam Theoretical Physics School and Workshop

Recent developments in Entanglement, Large N in QFT and String theory

Johannesburg, January 2023

Entanglement entropy for regions



Bipartition

$H_V \otimes H_{-V}$

$\rho = |0\rangle \langle 0|$

$\longrightarrow \rho_V = \mathrm{tr}_{(H_{-V})}\rho \longrightarrow S(V) = -\mathrm{tr}\rho_V \log \rho_V$

The entropy is divergent in the continuum but…. admits an expansion in powers of ϵ
In d dimensions

$$S(V) = \frac{g_{d-2}[\partial V]}{\epsilon^{d-2}} + \cdots + \frac{g_1[\partial V]}{\epsilon} + g_0[\partial V]\log(\epsilon) + S_0(V)$$

$$S \sim \frac{R^{d-2}}{\epsilon^{d-2}}$$

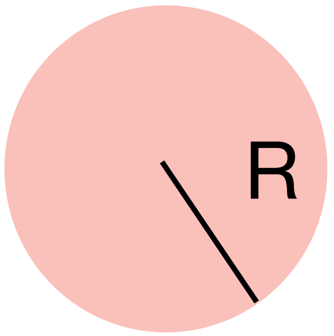
Area law

Non universal

Finite term
Depends on the region and the state

$S_0 = -\gamma$

Topological theory



d = 2 $\quad S = c/3 \log(r/\epsilon)$

d = 3 $\quad S = c_1 R/\epsilon - F$

d = 4 $\quad S(r) = c_2 \frac{r^2}{\epsilon^2} - 4A \log(R/\epsilon)$

The coefficients g_i depend on the regulazation. They are local and extensive on the boundary and independent of the state. In order to extract useful information you can introduce different regions or different states.

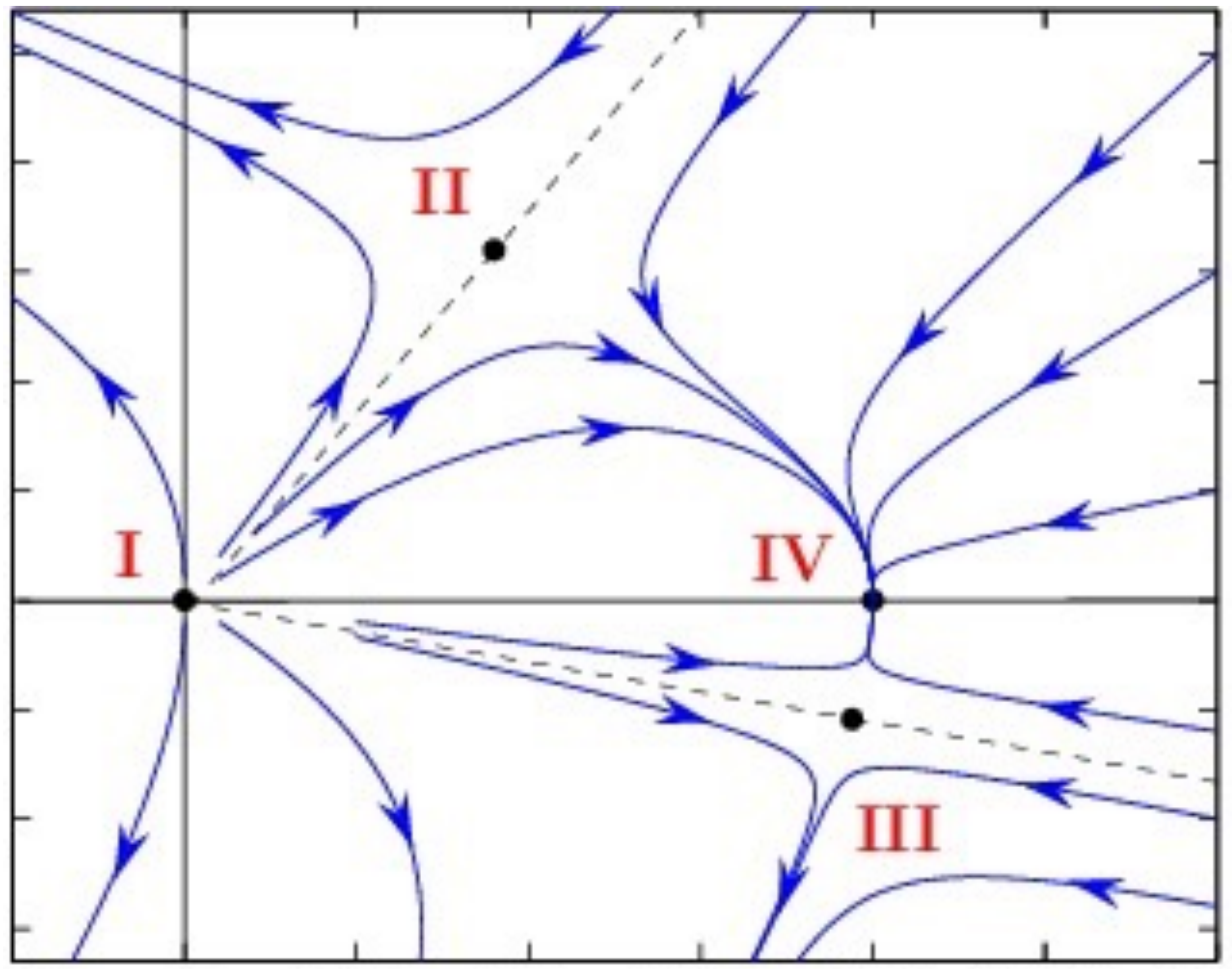
Irreversibility Entropic Theorems

H Casini, M H. (2004), H Casini, M. H. (2012), H Casini, E Teste, G Torroba (2017)

Renormalization Group in QFT

⟶ Mathematical tool that tells about changes in the physics with scale through the change of coupling constants $\{g_i\}$ with the RG flow.

$$\tau\frac{dg_i}{d\tau} = \beta_i(\{g(\tau)\})$$



At fixed points there is scale invariance: the theory “looks the same” at all scales.

The RG flow Interpolates between fixed points.

In this picture, the theories with mass scales, for example, are obtained by perturbing critical points and following the renormalization group (RG) trajectories. However, interestingly, in the flow not all critical points can be joined with each other: there are constraints provided by c-theorems.

C – Theorem : General constraint for the RG flow. Ordering of the fixed points.

In the flow, not all critical points can be joined with each other: there are constraints provided by c-theorems. These state that there is a universal c function (charge) that decreases from the UV to the IR fixed point. This implies irreversibility of the RG flow.

if $c_1 \geq c_2$ then $c_1 \to c_2$ but $c_2 \not\to c_1$

In two dimensions

There is a universal function c defined for any theory which is dimensionless, decreasing along the renormalization group trajectories and takes finite values at fixed points proportional to the Virasoro central charge.

A. B. Zamolodchikov, (1986)

C-Theorem: What is needed?

1) A regularization independent quantity C, well defined in the space of theories.

2) C dimensionless and finite at the fixed points.

3) C decreases along the renormalization group trajectories. In particular

$$C_{\mathrm{UV}} \geq C_{\mathrm{IR}}$$

small size $C(r) \to C_{UV}$

large size $C(r) \to C_{IR}$

- A universal dimensionless decreasing function $C(r, g(\tau), \tau)$ will do the job

From (1) $\tau \frac{\partial}{\partial \tau} C = -\sum_i \beta_i(g) \frac{\partial}{\partial g_i} C$

From (2) $\left(r \frac{\partial}{\partial r} - \tau \frac{\partial}{\partial \tau} \right) C = 0$

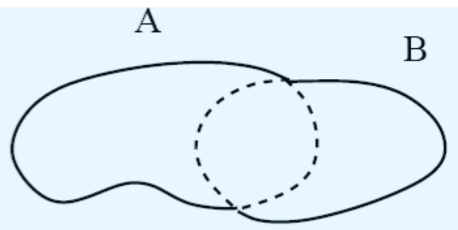
$$r \frac{dC(r)}{dr} = -\sum \beta_i(g) \frac{\partial}{\partial g_i} C$$

In more dimensions

Holographic c-theorems (Myers and Sinha, 2010)
In d=4 (Komargodski, Schwimmer 2011)

The entropic theorem

SSA

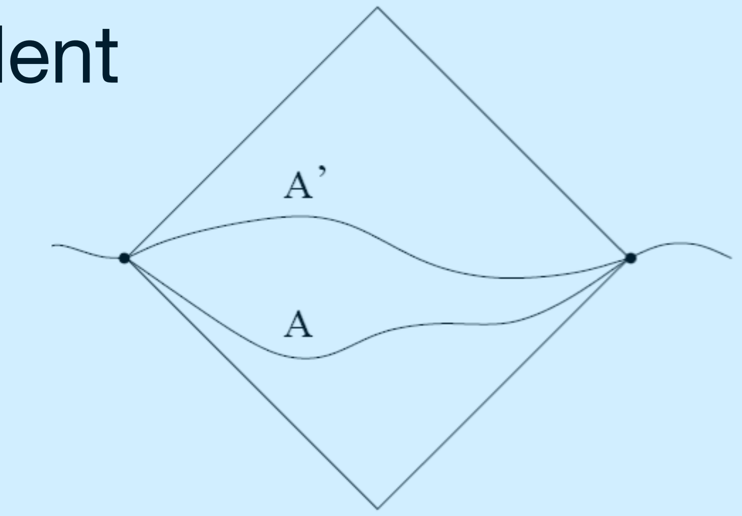


$$S(A)+S(B)\geq S(A\cap B)+S(A\cup B)$$

(Lieb, Ruskai (1973))

Cauchy surface independent

$$\boxed{S(A)=S(A') \quad (\rho_A=\rho_{A'})}$$



Regions:

Boosted spheres on the null cone

$$\Delta C\leq 0$$

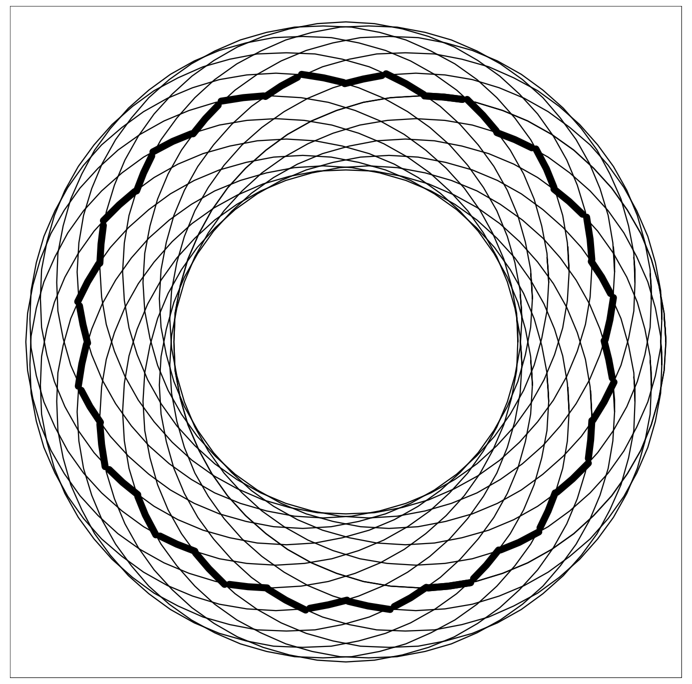
$$\Delta A\leq 0$$

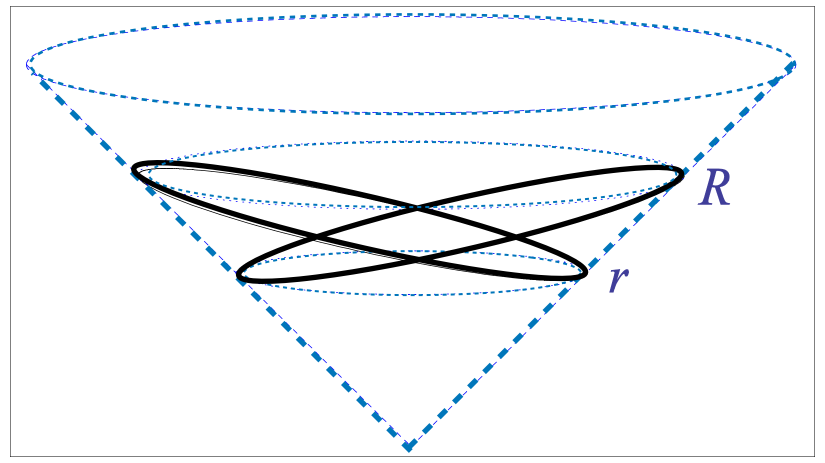
$$\Delta F\leq 0$$

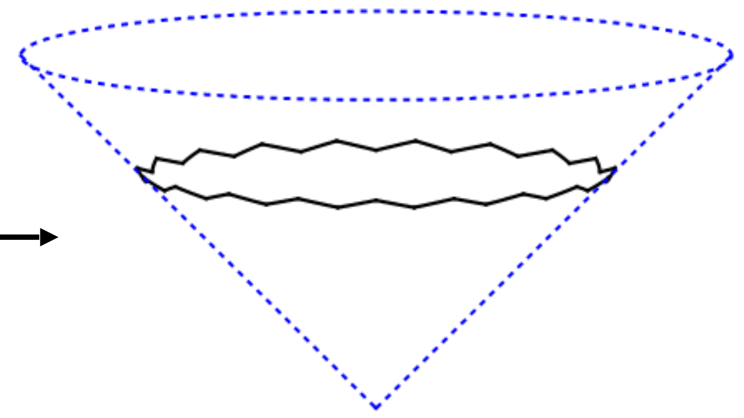
d = 2 and 4 (c and A theorems)

d = 3 (F theorem)

d = 3

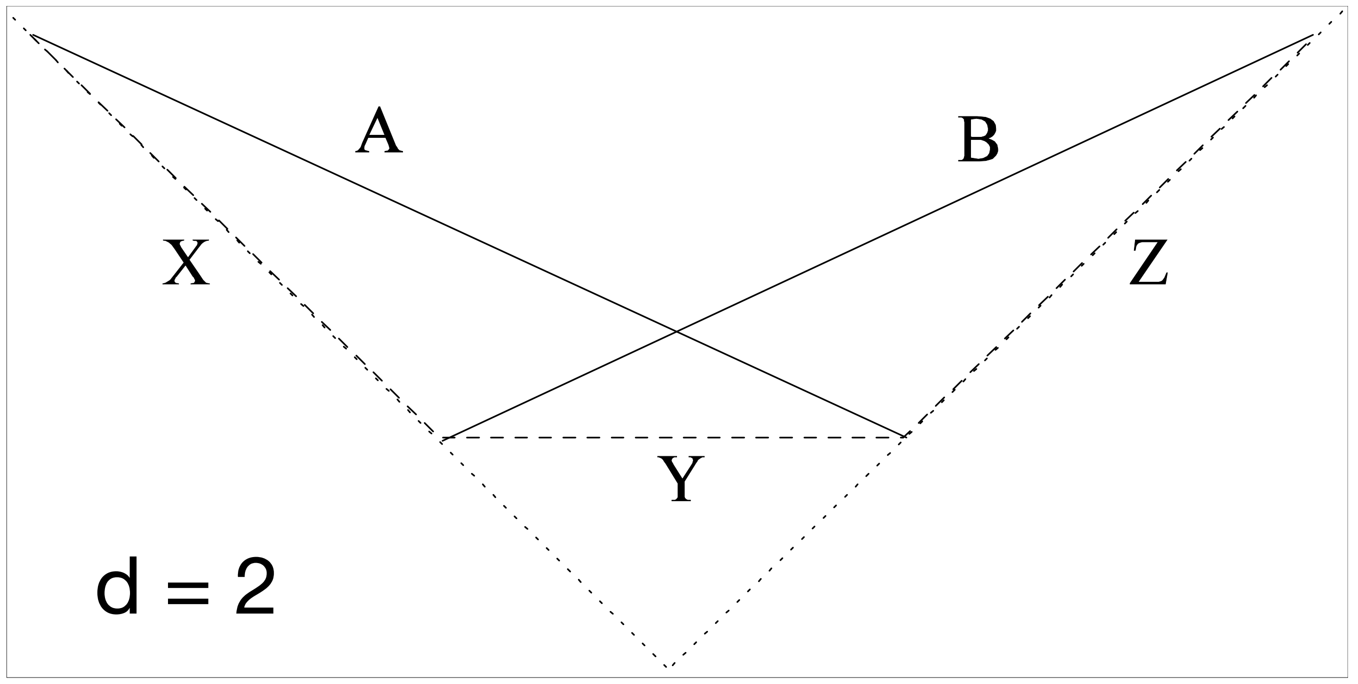






In d=4 new ingredient: Markovian property of the vacuum for CFT and regions in the null cone: there is no entanglement between A and B

$$S(A)+S(B)-S(A\cap B)-S(A\cup B)=0$$



d = 2

The theorem says the same in all dimensions

$$\Delta S''(a)\leq 0 \qquad a=r^{d-2}$$

It unifies all the irreversibility known theorems

In d = 3 there is only the entropic version!

Interpretation: what is lost with the size?

- The entropic theorems tells you the EE universal terms of the sphere decreces with the renormalization group flow.

- This leaves us closer to a thermodinamic interpretation: The entropic c function measures certain entanglement lost along the renormalization group trajectories

- This is significantly different from the usual interpretation in terms of degrees of freedom. For example, in odd dimensions the topological theories, without degrees of freedom, have a finite topological entropy.

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