The gravitational path integral for N=4 BPS black holes from black hole microstate counting

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XII Joburg Workshop on String Theory

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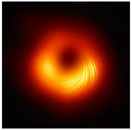
arXiv:2007.10302, 2112.10023, 2211.06873

BPS black holes in string theory

Black holes: gravitational solutions to Einstein's field equations.



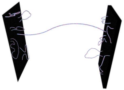
Behave as thermodynamic systems w/ entropy:
(Bekenstein+Hawking in the 70's)

$$S_{BH} = \frac{A_H}{4} + c_1 \log A_H + \frac{c_2}{A_H} + \dots \quad \text{(area of event horizon)}$$

Boltzmann: black hole microstates $S_{BH} = \log d \quad , \quad d \in \mathbb{N}$

Central question in quantum gravity: microstates ? d?

Four-dimensional BPS black holes in $N = 4$ heterotic string theory:



$$d(m, n, \ell) \, , \quad m, n, \ell \in \mathbb{Z} \quad , \quad \Delta \equiv 4mn - \ell^2 > 0$$

$$\log \, d(m, n, \ell) = \pi \sqrt{\Delta} + \dots = \frac{A_H}{4} + \dots$$

BPS black holes in string theory

- Heterotic string theory compactified on a six-torus:

  $\frac{1}{4}$ BPS black holes with unit torsion.

- BPS: supersymmetric. Asymptotically flat black holes.
- Single-centre black holes. Near horizon geometry is $AdS_2 \times S^2$,

$$ds_4^2 = v_* \left(-r^2 - 1)dt^2 + \frac{dr^2}{(r^2 - 1)} + d\Omega_2^2 \right)$$

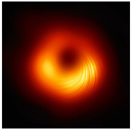
- Dyonic black holes:

  electric-magnetic charges (q_I, p^I) (several Maxwell fields F^I),

  charge bilinears $\quad m = p \cdot p, \quad n = q \cdot q, \quad \ell = q \cdot p$

- Supported by complex scalar fields Y^I.

  Attractor mechanism: near horizon geometry supported by constant scalar fields $Y^I(q,p)$. $\quad S_{BH}(q,p)$.

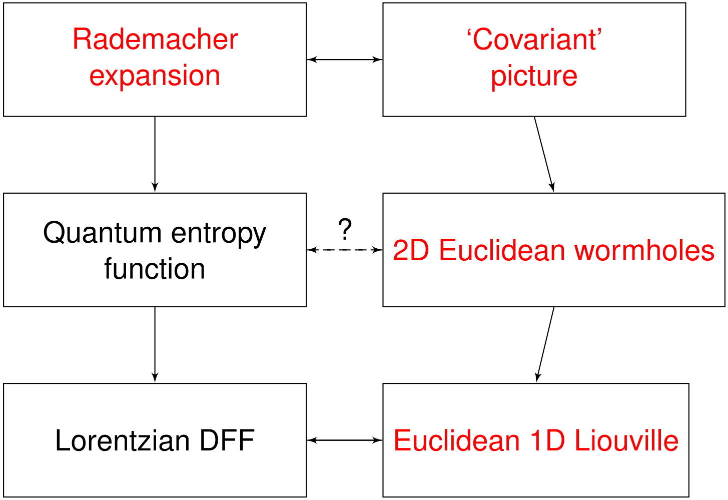


1. Number theory:
   $d(m, n, \ell)$: meromorphic Siegel modular form.
   Exact expression as a Rademacher type expansion. C, Nampuri, Rosselló, arXiv: 2112.10023

2. Quantum entropy function: Ashoke Sen, arXiv:0805.0095
   $d(m, n, \ell)$ from a quantum gravity path integral: sum over space-time geometries that asymptote to a product geometry $AdS_2 \times S^2$.

3. Conformal quantum mechanics:
   AdS_2/CFT_1 correspondence: Maldacena, arXiv:9711200
   $d(m, n, \ell)$ from a conformal quantum mechanics model (DFF model). de Alfaro, Fubini, Furlan, 1976

Three approaches to BPS black hole entropy



The 'covariant' picture suggests a space-time interpretation involving 2D wormholes in AdS_2. Lin, Maldacena, Rozenberg, Shan, arXiv:2207.00408

Number theory: meromorphic Siegel modular form

Heterotic string theory on T^6: $\frac{1}{4}$ BPS states with unit torsion.

Microstate degeneracies $d(m,n,\ell)$ given in terms of the Fourier coefficients of $1/\Phi_{10}$. Φ_{10} Igusa cusp form of weight 10.

Dijkgraaf, Verlinde, Verlinde, arXiv: 9607026

$$d(m,n,\ell) = \int_C d\sigma\, dv\, d\rho\, \frac{e^{-2\pi i(m\rho + n\sigma + \ell v)}}{\Phi_{10}(\rho,\sigma,v)}$$

Three contour integrations. Since $1/\Phi_{10}$ is meromorphic Siegel modular form, $d(m,n,\ell)$ depends on the choice of the integration contour C. $\Delta = 4mn - \ell^2$.

$1/\Phi_{10}$ captures degeneracies of single-centre $(\Delta > 0)$ as well as of two-centre black holes $(\Delta < 0)$. Ashoke Sen, arXiv:0705.3874

Need to select a contour C that only captures single-centre degeneracies $d(m,n,\ell)$, with $\Delta > 0$.

Siegel modular form of degree 2

Siegel's upper half plane $\mathcal{H}_2$:

$$\mathcal{H}_2 = \{\Omega \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \Omega^T = \Omega \, , \; \mathrm{Im}\Omega > 0\}$$

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \quad , \quad \rho_2 > 0, \; \sigma_2 > 0, \; \det(\mathrm{Im}\Omega) > 0$$

Siegel modular group $Sp(4, \mathbb{Z})$ acts on $\mathcal{H}_2$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \quad , \quad A^T D - C^T D = I_2$$

$$\Omega \mapsto \Omega' = (A\Omega + B)(C\Omega + D)^{-1} \quad , \quad (\rho, \sigma, v) \mapsto (\rho', \sigma', v')$$

A Siegel modular form Φ_k of weight $k \in \mathbb{N}$ is a holomorphic function $\Phi_k : \mathcal{H}_2 \to \mathbb{C}$ s.t.

$$\Phi_k((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k \, \Phi_k(\Omega)$$

Poles of $1/\Phi_{10}$

Poles of $1/\Phi_{10}$ labelled by 5 integers (n_1, n_2, j, m_1, m_2) with $n_2 \geq 0$:

$$n_2(\rho\sigma - v^2) + jv + n_1\sigma - m_1\rho + m_2 = 0 \quad , \quad m_1 n_1 + m_2 n_2 = \frac{1}{4}(1 - j^2)$$

Parametrized in terms of two distinct $SL(2,\mathbb{Z})$ subgroups of $Sp(4,\mathbb{Z})$:

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \quad , \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

and $\Sigma \in \mathbb{Z}$,

$$\begin{aligned} n_2 &= -ac\gamma \quad , \quad n_1 = -bd\alpha - \gamma\Sigma \\ m_1 &= ac\delta \quad , \quad m_2 = -bd\beta - \delta\Sigma \quad , \quad j = ad + bc \, , \end{aligned}$$

Two types: quadratic ($n_2 > 0$) and linear ($n_2 = 0$) poles.

Linear poles: Decay of BH bound state when crossing $jv_2 + n_1\sigma_2 - m_1\rho_2 = 0$.

Poles of $1/\Phi_{10}$

Idea: Evaluate the first integral (over ρ) as a sum over residues associated with the quadratic poles $n_2 > 0$ located in the Siegel upper half plane $\mathcal{H}_2$.

This is achieved by mapping the quadratic poles to the linear pole $v' = 0$ using the two two distinct $SL(2, \mathbb{Z})$ subgroups.

Under this map $\frac{1}{\Phi_{10}(\rho,\sigma,v)} = (\gamma\sigma + \delta)^{10} \frac{1}{\Phi_{10}(\rho',\sigma',v')}$

At the linear pole $v' = 0$,

$$\frac{1}{\Phi_{10}}(\rho', \sigma', v') = -\frac{1}{4\pi^2} \frac{1}{v'^2} \frac{1}{\eta^{24}(\rho')} \frac{1}{\eta^{24}(\sigma')} + \mathcal{O}(v'^0)$$

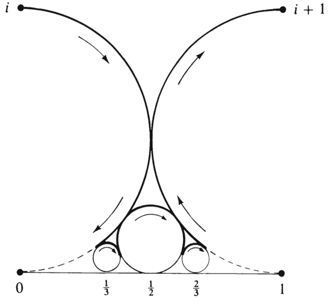
The linear pole $v' = 0$ has to lie the Siegel upper half plane, hence $\mathrm{Im}\rho' > 0, \mathrm{Im}\sigma' > 0$.

Rademacher path

Steps: 1) Evaluate the first integral (over ρ) as a sum over residues associated with the quadratic poles $n_2 > 0$.

2) The integration contour for v is determined by the conditions $\mathrm{Im}\rho' > 0, \mathrm{Im}\sigma' > 0$.

3) The locus of the quadratic poles ($n_2 > 0$) fixes the integration contour for σ to be a union of Ford circles, anchored at rational points $0 \leq -\delta/\gamma < 1$.



Replace horizontal segment by Rademacher path Γ_N composed of the upper arcs of Ford circles. Then $N \to \infty$: infinite sum of integrals over Ford circles.

Rademacher contour for $N = 3$.

Rademacher expansion: a classic example

Dedekind's eta function $\eta : \mathbb{D} \to \mathbb{C}$: $\quad \eta^{24}(q) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}$

Meromorphic modular form of weight $k = -12$: $\quad q = e^{2\pi i \tau}, \mathrm{Im}\, \tau > 0$

$$\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \tau} \quad , \quad d(n) = \int_z^{z+1} \frac{e^{-2\pi i n \tau}}{\eta^{24}(\tau)}\, d\tau \quad , \quad n > 0$$

Polar coefficient: $d(-1) = 1$. For $n > 0$, modular properties:

$$\eta^{-24}(\tau) = (c\tau + d)^{-12}\, \eta^{-24}\left(\frac{a\tau + b}{c\tau + d}\right)$$

$$d(n) = \int_z^{z+1} \frac{e^{-2\pi i n \tau}}{\eta^{24}(\tau)}\, d\tau = d(-1) \frac{2\pi}{n^{13/2}} \sum_{c>0} \frac{K(n,-1,c)}{c}\; I_{13}\left(\frac{4\pi\sqrt{n}}{c}\right)$$

Rademacher expansion: polar coefficients, classical Kloosterman sums K, Bessel functions I_{13}

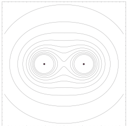
Uses: modular symmetry $SL(2,\mathbb{Z})$, Ford circles.

Exact expression for $d(m,n,\ell)$ with $\Delta = 4mn - \ell^2 > 0$

Theorem: $\quad d(m,n,\ell) \in \mathbb{N}, \quad \tilde{\Delta} = 4m\tilde{n} - \tilde{\ell}^2 < 0$

$$
\begin{aligned}
d(m,n,\ell) = (-1)^{\ell+1} \sum_{\gamma=1}^{+\infty} \sum_{\tilde{\ell} \in \mathbb{Z}/2m\mathbb{Z}} \Bigg(& 2\pi \sum_{\substack{\tilde{n} \geq -1, \\ \tilde{\Delta} < 0}} c_m^F(\tilde{n}, \tilde{\ell}) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}, \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde{\Delta}|}\right) \\
& - \delta_{\tilde{\ell},0} \sqrt{2m}\, d(m) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1; \gamma, \psi)_{\ell 0}}{\sqrt{\gamma}} \left(\frac{4m}{\Delta}\right)^6 I_{12}\left(\frac{2\pi}{\gamma}\sqrt{\frac{\Delta}{m}}\right) \\
& + \frac{1}{2\pi} d(m) \sum_{\substack{g \in \mathbb{Z}/2m\gamma\mathbb{Z} \\ g = \tilde{\ell} \bmod 2m}} \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1 - \frac{g^2}{4m}; \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma^2} \\
& \left(\frac{4m}{\Delta}\right)^{25/4} \int_{-1/\sqrt{m}}^{1/\sqrt{m}} dx' f_{\gamma,g,m}(x') (1 - mx'^2)^{25/4} I_{25/2}\left(\frac{2\pi}{\gamma\sqrt{m}}\sqrt{\Delta(1 - mx'^2)}\right) \Bigg),
\end{aligned}
$$

with



$$
c_m^F(\tilde{n}, \tilde{\ell}) = \sum_{\substack{a>0,\, c<0 \\ b \in \mathbb{Z}/a\mathbb{Z},\ ad-bc=1 \\ 0 \leq \frac{b}{a} + \frac{\tilde{\ell}}{2m} < -\frac{1}{ac}}} \Big((ad+bc)\tilde{\ell} + 2ac\tilde{n} + 2bdm\Big)\, d(c^2\tilde{n} + d^2 m + cd\tilde{\ell})\, d(a^2\tilde{n} + b^2 m + ab\tilde{\ell})
$$

$$
\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i \tau n} \quad , \quad \text{two } SL(2,\mathbb{Z})
$$

Exact Rademacher type expansion for $1/\Phi_{10}$

Area law:

$\gamma = 1,\ \tilde{n} = -1,\ \tilde{\ell} = m: \qquad d(m,n,\ell) \approx e^{\pi\sqrt{\Delta}} = e^{\frac{1}{4}A_H}$

Rademacher type expansion that we obtained arises as:

- A sum over residues of the quadratic poles of $1/\Phi_{10}$
- Expansion uses two $SL(2,\mathbb{Z})$ subgroups
- Integration contour: uses Ford circles (integral over σ)
- Expansion encoded in degeneracies of the perturbative $\frac{1}{2}$ BPS states! $\quad c_m^F(\tilde{n},\tilde{\ell}) = \sum L\, d(M)\, d(N)$ $\quad$ bound state degeneracy
- Contributions I_{12} and $I_{25/2}$: reflect underlying Mock modular behaviour that is encoded in the Fourier-Jacobi decomposition of $1/\Phi_{10}$,

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m=-1}^{\infty} \psi_m(\sigma,v)\, e^{2\pi i m\rho}$$

Dabholkar + Murthy+ Zagier, arXiv:1208.4074; Ferrari + Reys, arXiv:1702.02755

'Covariant picture'

Integrate $1/\Phi_{10}$ over ρ , add total derivative term that vanishes on integration contour, change of variables $(\sigma, v) \rightarrow (\tau_1, \tau_2)$:

$$d(m,n,\ell)_{\Delta>0} = \sum_{SL^2(2,\mathbb{Z}),\,\Sigma} \frac{e^{i\pi\varphi}}{\gamma} \frac{1}{(ac)^{13}} \int_{\Gamma_2} \frac{d\tau_2}{\tau_2^2} \left(\int_{\Gamma_1} d\tau_1 \, f(\tau_1, \tau_2) \right)$$

$$f(\tau_1,\tau_2) = \left[26 + \frac{2\pi}{n_2} \frac{\left(m(\tau_1^2 + \tau_2^2) + n - \ell\tau_1 \right)}{\tau_2} \right] \frac{e^{\frac{\pi}{n_2} \frac{m(\tau_1^2+\tau_2^2)+n-\ell\tau_1}{\tau_2}}}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^2}$$

$$\begin{aligned} \rho'_* &= -\frac{a}{c}\,\frac{\tau_1 + i\tau_2}{\gamma} - \frac{b}{c}\,\frac{\alpha}{\gamma} - \frac{a}{c}\,\Sigma \\ \varphi &= \frac{2}{n_2}\left(-\tfrac{1}{2} j\ell - m_1 n + n_1 m\right) \\ \Gamma_1: \quad \tau_1 &= \frac{\ell}{2m} + i\tau_2\,(-1 + 2y) \quad , \quad 0 < y < 1 \\ \Gamma_2: \quad \tau_2 &= \frac{\sqrt{\Delta}}{2m} + i\,t \quad , \quad -\infty < t < \infty \end{aligned}$$

S-duality invariance:

invariance under the T-generator of $SL(2,\mathbb{Z})$ manifest:

$$T = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad , \quad s \in \mathbb{Z}$$

$$\begin{aligned} m = p \cdot p &\rightarrow m, \\ n = q \cdot q &\rightarrow n - s\,\ell + s^2\, m \\ \ell = q \cdot p &\rightarrow \ell - 2s\, m \end{aligned}$$

$$m_1 \rightarrow m_1 \quad , \quad n_1 \rightarrow n_1 - sj - s^2 m_1 \quad , \quad j \rightarrow j + 2sm_1$$
$$m_2 \rightarrow m_2 \quad , \quad n_2 \rightarrow n_2$$

Quantum entropy function

Reproduce $d(m,n,\ell)$ by a suitable quantum gravity path integral, quantum entropy function (QEF). Ashoke Sen, arXiv:0805.0095, 0809.3304

- Functional integral over all fields in string theory in an Euclidean background B that asymptotes to a specific Euclidean $AdS_2 \times S^2$ solution fixed by the attractor mechanism. $W = \sum_B W_B$
- Background B: supported by Abelian gauge potentials A^I, constant complex scalar fields Y^I
- Using supersymmetric localization: QEF is a finite-dimensional integral over $\{\phi^I\}$, where $Y^I = \frac{1}{2}(\phi^I + ip^I)$. Dabholkar, Gomes, Murthy, 2010
- QEF→ Rademacher picture. $c_m^F(\tilde{n},\tilde{\ell})$ in measure.

Macroscopic interpretation of 'covariant picture':

- Saddle point analysis: $\tau_* = (\tau_1 + i\tau_2)_* = \frac{\ell}{2m} + i\frac{\sqrt{\Delta}}{2m}$

$$\frac{\pi}{n_2}\frac{\left(m(\tau_1^2+\tau_2^2)+n-\ell\tau_1\right)}{\tau_2}|_* + \pi i\varphi = \frac{1}{n_2}\left[\frac{A_H}{4} + 2\pi i\left(-\tfrac{1}{2}j\ell - m_1 n + n_1 m\right)\right]$$

Macroscopic interpretation of ‘covariant picture’

- Semi-classical interpretation in terms of sums over space-time backgrounds: $\mathbb{Z}_{n_2}$ orbifolds of Euclidean $AdS_2 \times S^2$

$$ds^2 = v_* \left((r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} + d\psi^2 + \sin^2\psi \, d\phi^2 \right)$$

$$0 \leq \theta < \frac{2\pi}{n_2} \quad , \quad 0 \leq \phi < \frac{2\pi}{n_2},$$

supported by gauge potentials A^I_θ that acquire a constant real part when orbifolding,

$$A^I_\theta = -i e^I_* \, (r - 1) \, d\theta + \mathrm{Re} A^I_\theta$$

$$\frac{1}{n_2} \left[\frac{A_H}{4} + \pi i \left(q \cdot \mathrm{Re} A_\theta - p \cdot \mathrm{Re} \tilde{A}_\theta \right) \right]$$

$(\mathrm{Re} A_\theta, \mathrm{Re} \tilde{A}_\theta)$ expressed in terms of $(q_I, p^I; m_1, n_1, j)$, symplectic vector under S-duality

Macroscopic interpretation of 'covariant picture'

- Measure:

$$\left[26 + \frac{2\pi}{n_2} \frac{\left(m(\tau_1^2 + \tau_2^2) + n - \ell\tau_1\right)}{\tau_2}\right]\Big|_* = 26 + 2\,\frac{A_H}{4\,n_2}$$

absence of $\log A_H$ term in $N = 4$ black hole entropy,

$S_{BH} = \frac{A_H}{4} + 0\,\log A_H + \ldots$ Banerjee, Jatkar, Sen, arXiv: 0810.3472

- Terms $\frac{1}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^{12}}$:

underlying bound state structure of Rademacher picture not manifest here $\quad (c_m^F = \sum L\, d(M)\, d(N))$

Consider $n_2 = 1$: $\quad \frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar{\tau})}\,\frac{1}{(\tau - \bar{\tau})^{12}}$

2D JT gravity point of view: how to account for this?

Macroscopic interpretation of ‘covariant picture’

Global Euclidean AdS_2: supported by constant dilaton field $\Phi_0 = v_*$

$$ds^2 = \frac{v_*}{\sin^2\sigma}\left(dT^2 + d\sigma^2\right) \quad , \quad -\pi < \sigma < 0 \quad , \quad T \cong T + 2\pi\tau_{2\,*} \, ,$$

Proposal:

Add 24 chiral + 24 antichiral periodic scalar fields (critical closed bosonic string), time-independent classical configuration: $T^{\rm cl}_{\mu\nu} = 0$.

1-loop partition function of periodic scalars:

$$Z^{1-\text{loop}} = \frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar\tau)}\,\frac{1}{(\tau - \bar\tau)^{12}}$$

$< T^{\rm quan}_{\mu\nu} > \neq 0$, backreacts on the dilaton

$$\Phi_0 + \Phi = \Phi_0 - 24\,\mathcal{E}\left[2\pi\tau_{2\,*}\right]\left(1 - \frac{\sigma + \frac{\pi}{2}}{\tan\sigma}\right) \quad , \quad -\pi < \sigma < 0$$

The resulting solution (trumpet + dilaton) is interpreted as an 2D Euclidean wormhole solution.

Garcia-Garcia, Godet, arXiv:2010.11633

Conclusions

Summarizing:

- QEF$\to$ Rademacher picture: $c_m^F(\tilde{n}, \tilde{\ell})$ in measure.

  $(c_m^F = \sum L\, d(M)\, d(N))$

- Macroscopic interpretation of 'covariant' picture: 2D picture.
  - $\frac{1}{4} A_H$: JT gravity on the disc.
  - η^{24} contributions: Euclidean wormhole on the trumpet.
- Euclidean 1D Liouville type action / Lorentzian DFF type action

$$
\begin{aligned}
S_{\rm Liouv} &= \int dt \left[\frac{1}{2} (l')^2 + 2 e^{-l} \right] \\
S_{\rm DFF} &= \int dt \left[\frac{(\nu')^2}{\alpha\, \nu} + \alpha \left(\frac{1}{\nu} + \nu \right) \right]
\end{aligned}
$$

by means of reparametrization $dt \to \alpha(t)\, dt$.

Thanks!