Modular factorization of superconformal indices

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Based on 2210.17551
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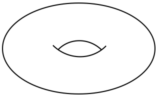
What is the nature of the **microstates** counted by $S_{BH} = \frac{A}{4G_N}$?

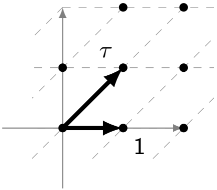
Central question in **quantum gravity**.

Within string theory, **AdS/CFT** provides a (partial) answer.

Historically, **CFT_2** plays a special role.



- Z_{CFT_2} in/covariant under $SL(2,\mathbb{Z})$: $\tau \to \frac{k\tau+l}{m\tau+n}$.
- Powerful symmetry; e.g. $S : \tau \to -\frac{1}{\tau}$ leads to S_{Cardy}.
- More generally: $Z_{\text{CFT}}(\tau) = \sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} (\cdots)$.



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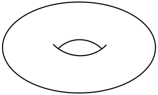

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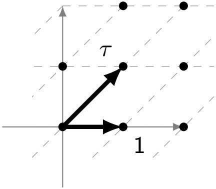

Quantum gravity interpretation:

* S_{Cardy} computes S_{BH} for a special class of black holes. [Strominger-Vafa]
* $\Gamma_\infty \backslash SL(2,\mathbb{Z})$ labels black hole saddles in (Euclidean) AdS_3.

  $\Rightarrow Z_{\mathrm{CFT}}(\tau)$ as grav. path integral. [Dijkgraaf-Maldacena-Moore-Verlinde,(Maloney-)Witten]

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Type IIb SUGRA on $AdS_5 \times S^5$*: Kerr-Newman BPS black holes*

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Study BHs through protected CFT quantity: **superconformal index**.

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New insight: exact S_{BH} from index!

[Cabo-Bizet-Cassani-Martelli-Murthy,Choi-Kim-Kim-Nahmgoong,Benini-Milan]

Every derivation relies on a **modular property**:

$$\Gamma(z;\tau,\sigma) = e^{-i\pi Q(z;\tau,\sigma)}\Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)\Gamma\left(\frac{z}{\tau};\frac{\sigma}{\tau},-\frac{1}{\tau}\right)$$

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Unexpected: path integral on $S^3 \times S^1$ should **not** have modular symmetries.

Relatedly:

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2. and two "modular group" actions: $SL(3,\mathbb{Z})$ and $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$.

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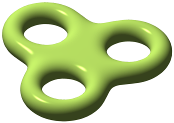
Aim: a **physical interpretation** akin to 2d. [Gadde]

$\Rightarrow$ **Modular factorization** of superconformal indices.

Overview

1. Motivation
2. Heegaard splitting of Hopf surfaces
3. Review of holomorphic block factorization
4. Modular factorization of superconformal indices
5. $SL(3, \mathbb{Z})$ 1-cocycle condition
6. Summary and future directions

Heegaard splitting of a three-manifold:



$$M_3 \cong H_g \overset{f}{\sqcup} H'_g \, .$$

* H_g is genus g handlebody: $\Sigma_g = \partial H_g$.

* Σ_g is identified with Σ'_g up to an orientation reversing diffeo f.

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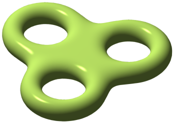

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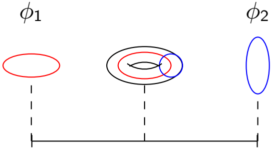
Example of **genus 1** Heegaard splitting:

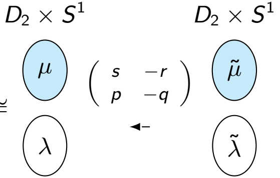


* $S^3 \subset \mathbb{C}^2$: $|z_1|^2 + |z_2|^2 = 1$.

* $H_1 = D_2 \times S^1$.

* $f = S\,\mathcal{O}$, where $S \in SL(2,\mathbb{Z}) = \mathrm{MCG}[\Sigma_1]$.

M_f is a **lens space** for a general $SL(2,\mathbb{Z})$ element:

$L(p,q) \cong$ 

where: $S^3 \supset L(p,q) = \{(z_1, z_2) \sim (e^{\frac{2\pi i q}{p}} z_1, e^{-\frac{2\pi i}{p}} z_2) \mid \gcd(p,q) = 1\}\,.$

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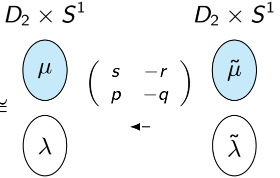

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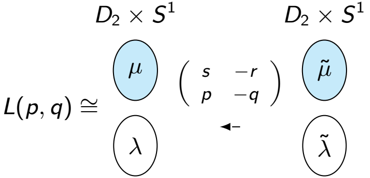

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$(\lambda, \tilde{\lambda})$ are defined modulo $(\mu, \tilde{\mu}) \Rightarrow$ **redundancy** in f.

Γ_∞: $(\mu, \lambda) \to (\mu, \lambda + k\mu)$. Then for $\gamma, \tilde{\gamma} \in \Gamma_\infty$:

$$M_{f'} \cong M_f \quad \text{when} \quad f' = \gamma\, f\, \tilde{\gamma}^{-1}\,.$$

Implies $L(p,q) \cong L(p, q+p)$, consistent with quotient defn.

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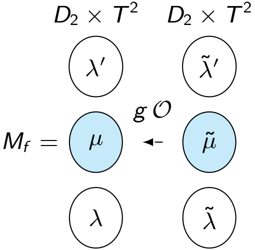
How is **topology** of M_f determined by $g \in SL(3, \mathbb{Z})$?



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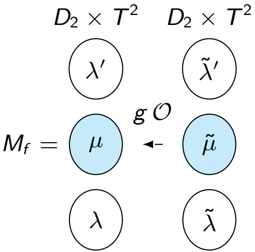

* $h = \begin{pmatrix} * & 0 & * \\ * & 1 & * \\ * & 0 & * \end{pmatrix}$ preserves μ.

* $H \equiv SL(2, \mathbb{Z})_{\lambda'\lambda} \ltimes \mathbb{Z}^2_\mu \subset SL(3, \mathbb{Z})$.

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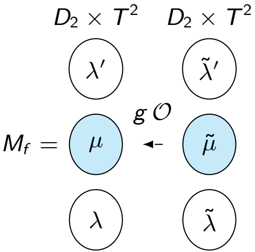

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For any $g \in SL(3,\mathbb{Z})$ there exist $h, \tilde{h} \in H$ for some (p,q) such that:

$$f_{p,q} = h\, f\, \tilde{h}^{-1}\,,\ f_{p,q} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & -r \\ 0 & p & -q \end{pmatrix} \quad \Rightarrow \quad M_f \cong L(p,q) \times S^1\,.$$

Extension to include **complex structure moduli**:

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) \cong M_{f_{p,q}}(\boldsymbol{\rho})\,.$$

On LHS:

* A (secondary) **Hopf surface** with topology $L(p,q) \times S^1$.
* $\hat{\boldsymbol{\rho}} \equiv (\hat{z}_a; \hat{\tau}, \hat{\sigma}) \in \mathbb{C} \times \mathbb{H}^2$. $(\hat{\tau}, \hat{\sigma})$ squash and fiber, $\hat{z}_a$ holonomies along S^1.

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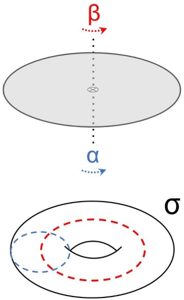
On RHS:

* $M_{f_{p,q}}(\boldsymbol{\rho}) \equiv D_2 \times T^2(\boldsymbol{\rho}) \overset{f_{p,q}}{\sqcup} D_2 \times T^2(f_{p,q}^{-1}\boldsymbol{\rho})$
* $\boldsymbol{\rho} \equiv (z_a; \tau, \sigma) = \left(\frac{Z_a}{x_1}; \frac{x_2}{x_1}, \frac{x_3}{x_1}\right)\,, \quad \tau = \alpha\sigma - \beta\,.$
* $SL(3,\mathbb{Z})$ acts on $\vec{x}$ by left multiplication.
* $\boldsymbol{\rho}$ is related linearly to $\hat{\boldsymbol{\rho}}$.



Consider the **combined action** on the Heegaard splitting:

$$M_{f_{p,q}}(\boldsymbol{\rho}) \to M_{f'}(\boldsymbol{\rho}')\,, \quad \text{with} \quad f' = h\, f_{p,q} \tilde{h}^{-1}\,, \qquad \boldsymbol{\rho}' = h\boldsymbol{\rho}\,.$$

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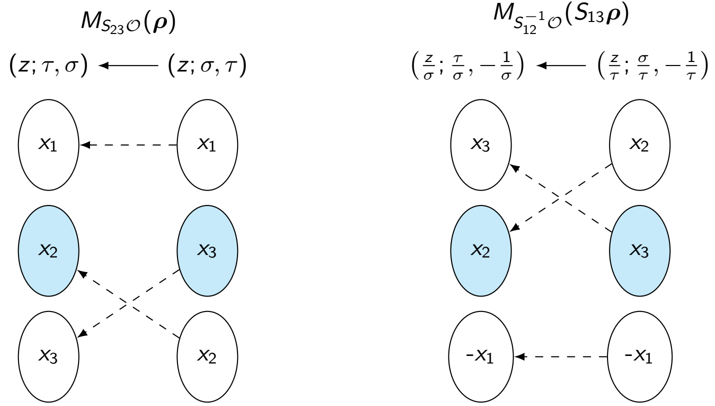
Claim: a given Hopf surface can be split into any such $M_{f'}(\boldsymbol{\rho}')$:

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) = D_2 \times T^2(h\boldsymbol{\rho}) \quad \overset{h\, f_{p,q}\, \tilde{h}^{-1}}{\sqcup} \quad D_2 \times T^2(\tilde{h}\, f_{p,q}^{-1} \boldsymbol{\rho}) \,.$$

$H \times H$ action reflect **ambiguities in Heegaard splitting**.

Example: two Heegaard splittings of $\mathcal{M}_{(1,0)}(\hat{\boldsymbol{\rho}}) \cong S^3 \times S^1$.

Take $f_{1,0} = S_{23}\,\mathcal{O}$ and $h = \tilde{h} = S_{13} \in SL(2,\mathbb{Z})_{\lambda'\lambda}$ such that $f' = S_{12}^{-1}\mathcal{O}$.



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4d $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry formulate on $\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}})$. [Festuccia-Seiberg]

Susy partition function depends only on $\hat{\boldsymbol{\rho}}$. [Closset-Dumitrescu-Festuccia-Komargodski]

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Path integral on Hopf surface $\Leftrightarrow$ lens **index**:

$$\mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}}) \equiv \mathrm{tr}_{\mathcal{H}_{(p,q)}}(-1)^F \hat{p}^{j_1+j_2-\frac{r}{2}} \hat{q}^{j_1-j_2-\frac{r}{2}} \hat{x}_a^{q_a} e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}$$

- $\mathcal{H}_{(p,q)}$ is Hilbert space on $L(p,q)$.
- $j_{1,2}$ angular momenta and q_a global symmetry charges.
- Fugacities in index are exponentiated moduli $\hat{\boldsymbol{\rho}}$.
- Counts BPS states with $\{\mathcal{Q},\mathcal{Q}^\dagger\} = \Delta - 2j_1 + \frac{3}{2}r = 0$; RG invariant.

$S^3 \times S^1$ index for general $\mathcal{N} = 1$ gauge theory:

[Romelsberger,Kinney-Maldacena-Minwalla-Raju,Dolan-Osborn]

$$\mathcal{I}_{(1,0)}(\hat{\boldsymbol{\rho}}) = \frac{1}{|W|} \oint_{|v_i|=1} \prod_{i=1}^{r} \frac{dv_i}{2\pi i v_i} \, \Delta_G(\vec{u}) \, I_{V_G}(\vec{u}; \hat{\tau}, \hat{\sigma}) \prod_i I^{R_i}_{(\boldsymbol{r}_i, \boldsymbol{r}'_i)}(\vec{u}, \vec{z}; \hat{\tau}, \hat{\sigma}) \, .$$

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Chiral multiplets with R-charge R_i in reps $(\mathbf{r}_i, \mathbf{r}'_i)$ of $G \times F$:

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Defined in terms of **elliptic Γ function**:

$$\Gamma(z; \tau, \sigma) = \exp\left(\sum_{\ell=1}^{\infty} \frac{x^\ell - (x^{-1} p q)^\ell}{\ell (1 - p^\ell)(1 - q^\ell)} \right) = \prod_{m,n=0}^{\infty} \frac{1 - x^{-1} p^{m+1} q^{n+1}}{1 - x p^m q^n} \, .$$

Similar expressions for $L(p,1) \times S^1$ and $S^2 \times T^2$ index.

Formula follows from:

1. Explicit evaluation trace:
    * Construct (gauge variant) BPS Hilbert space at UV free fixed point.
    * $\prod_{m,n}$ results from two BPS derivatives.
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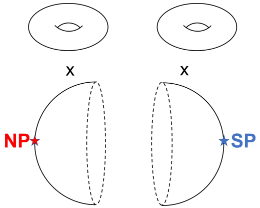
   * $\oint$ projects onto gauge invariant states.
2. Supersymmetric path integral: [Peelaers]
   * Equivariant localization on "north and south poles".
   * $\prod_{m,n}$ from 1-loop determinants of T^2 mode expansion.
   * $\oint$ represents a remaining integral over flat connections.



The index **factorizes**, manifesting localization on NP and SP: [Nieri-Pasquetti]

$$\mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}}) = e^{-i\pi\mathcal{P}_f(\boldsymbol{\rho})}\sum_{\alpha}\mathcal{B}^{\alpha}_S(\boldsymbol{\rho})\mathcal{B}^{\alpha}_S(f^{-1}_{p,q}\boldsymbol{\rho})\,.$$

Here $\mathcal{B}^{\alpha}(\boldsymbol{\rho})$ is the $D_2 \times T^2(\boldsymbol{\rho})$ partition function and: [Longhi-Nieri-Pittelli]

$$\mathcal{B}^{\alpha}_S(\boldsymbol{\rho}) \equiv \mathcal{B}^{\alpha}(S_{13}\boldsymbol{\rho}) = \mathcal{B}^{\alpha}\left(\tfrac{\vec{z}}{\sigma};\tfrac{\tau}{\sigma},-\tfrac{1}{\sigma}\right)\,, \qquad S_{13} \in SL(2,\mathbb{Z})_{\lambda'\lambda} \subset H\,.$$

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$$\mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}}) = e^{-i\pi\mathcal{P}_f(\boldsymbol{\rho})} \sum_\alpha \mathcal{B}_S^\alpha(\boldsymbol{\rho})\mathcal{B}_S^\alpha(f_{p,q}^{-1}\boldsymbol{\rho})\,.$$

Here $\mathcal{B}^\alpha(\boldsymbol{\rho})$ is the $D_2 \times T^2(\boldsymbol{\rho})$ partition function and: [Longhi-Nieri-Pittelli]

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This provides a complete physical interpretation of:

$$\underbrace{\Gamma(z;\tau,\sigma)}_{\mathcal{I}^{\chi_0}_{(1,0)}(\boldsymbol{\rho})} = e^{-i\pi Q(z;\tau,\sigma)} \underbrace{\Gamma\left(\tfrac{z}{\sigma}; \tfrac{\tau}{\sigma}, -\tfrac{1}{\sigma}\right)}_{\mathcal{B}^{\chi_0}_{S}(\boldsymbol{\rho})} \underbrace{\Gamma\left(\tfrac{z}{\tau}; \tfrac{\sigma}{\tau}, -\tfrac{1}{\tau}\right)}_{\mathcal{B}^{\chi_0}_{S}(f^{-1}_{(1,0)}\boldsymbol{\rho})} \,.$$

Overview

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2. Heegaard splitting of Hopf surfaces

3. Review of holomorphic block factorization

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Define: $\mathcal{Z}_{f_{p,q}}(\boldsymbol{\rho}) \equiv \mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}})$. We have:

$$\mathcal{Z}_{f'}(\boldsymbol{\rho}') = \mathcal{Z}_{f_{p,q}}(\boldsymbol{\rho})\,, \qquad f' = h\, f_{p,q}\, \tilde{h}^{-1}\,, \quad \boldsymbol{\rho}' = h\boldsymbol{\rho}\,,$$

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since any such Heegaard splitting leads to an **identical** Hopf surface.

Non-trivial at the level of the **factorized expression**:

$$e^{i\pi\mathcal{P}'} \sum_{\alpha} \mathcal{B}^{\alpha}_{h}(\boldsymbol{\rho})\mathcal{B}^{\alpha}_{\tilde{h}}(f^{-1}_{p,q}\boldsymbol{\rho}) = e^{i\pi\mathcal{P}} \sum_{\alpha} \mathcal{B}^{\alpha}(\boldsymbol{\rho})\mathcal{B}^{\alpha}(f^{-1}_{p,q}\boldsymbol{\rho}) \,,$$

where: $\mathcal{B}^{\alpha}_{h}(\boldsymbol{\rho}) \equiv \mathcal{B}^{\alpha}\left(\frac{\vec{z}}{m\sigma+n}; \frac{\tau+a\sigma+b}{m\sigma+n}, \frac{k\sigma+l}{m\sigma+n}\right)$ and $\mathcal{B}^{\alpha}(\boldsymbol{\rho}) = \mathcal{B}^{\alpha}(\vec{z}; \tau, \sigma)$.

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* $\sum_\alpha$ ensures index does not depend on α.
* For each α, there are two BPS b.c.'s on $\mathcal{N}=1$ multiplets. [Longhi-Nieri-Pittelli]
* Denote blocks respectively by $\mathcal{B}^\alpha(\boldsymbol{\rho})$ and $\mathcal{C}^\alpha(\boldsymbol{\rho})$.
* The condition becomes:

$$\mathcal{B}^\alpha_h(\boldsymbol{\rho})\mathcal{B}^\alpha_{\tilde{h}}(f_{p,q}^{-1}\boldsymbol{\rho}) \cong \mathcal{C}^\alpha_h(\boldsymbol{\rho})\mathcal{C}^\alpha_{\tilde{h}}(f_{p,q}^{-1}\boldsymbol{\rho}),$$

  up to a phase independent of α.

We first solve for pairs $(h, \tilde{h})$ for fixed $f_{p,q}$.

Result: any Heegaard splitting $M_{f'}(h\boldsymbol{\rho})$ allowed if:

$$f' \equiv h\, f_{p,q}\, \tilde{h}^{-1} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \in F\,\mathcal{O} \subset SL(3,\mathbb{Z})\,\mathcal{O}\,,$$

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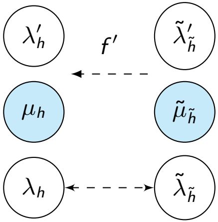
* $F = SL(2,\mathbb{Z})_{\lambda'\mu} \ltimes \mathbb{Z}^2_\lambda$ preserves $\lambda_h = \tilde{\lambda}_{\tilde{h}}$.



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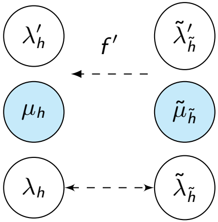

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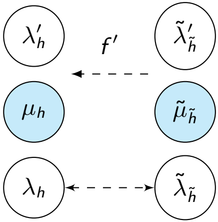
- Fixed cycle naturally identified with $S^1_\beta \subset \mathcal{M}_{(p,q)}$.
- $(h, \tilde{h})$ parametrizes ambiguities in embedding S^1_β inside the $D_2 \times T^2$ geometries for fixed $f_{p,q}$.



Explicit solution: $(h, \tilde{h}) \in S_{f_{p,q}} \subset H \times H$:

$$h = \begin{pmatrix} n & 0 & m \\ -cl & 1 & -ck \\ l & 0 & k \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} -sn + pc & 0 & m \\ (qc - rn)\tilde{l} & 1 & (qc - rn)\tilde{k} \\ \tilde{l} & 0 & \tilde{k} \end{pmatrix}.$$

Here, $c \in \mathbb{Z}$ and $kn - lm = 1 = \tilde{k}(-sn + pc) - \tilde{l}m$.

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$\Rightarrow$ **Modular factorization conjecture**:

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Example: Consider $L(p,-1)\times S^1$ corresponding to gluing element:

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Modular factorization in this case:

$$e^{-i\pi\mathcal{P}^h_{f_p}(\boldsymbol{\rho})}\mathcal{B}^\alpha(h\boldsymbol{\rho})\,\mathcal{B}^\alpha(hf_p^{-1}\boldsymbol{\rho}) = e^{-i\pi\mathcal{P}^1_{f_p}(\boldsymbol{\rho})}\mathcal{B}^\alpha(\boldsymbol{\rho})\,\mathcal{B}^\alpha(f_p^{-1}\boldsymbol{\rho})\,,$$

Covariance under the combined $SL(2,\mathbb{Z})$ action:

$$\boldsymbol{\rho}\to h\boldsymbol{\rho}\,,\quad f_p \to h\,f_p\,h^{-1}\,.$$

Condition can also be solved for fixed $(h, \tilde{h}) \in H \times H$:

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We conclude:

* $\mathcal{B}^\alpha_h(\boldsymbol{\rho})$ and $\mathcal{B}^\alpha_{\tilde{h}}(\boldsymbol{\rho})$ are **universal blocks** for indices $\mathcal{Z}_f(\boldsymbol{\rho})$.
* $F_{h,\tilde{h}_{\mathcal{O}}} \subset SL(3,\mathbb{Z})$ is the **maximal** such subset.
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Example: for $h = \tilde{h} = S_{13}$ the universal block is $\mathcal{B}^\alpha_S(\boldsymbol{\rho})$ for:

$$F_S \equiv S_{13}^{-1}\, F\, S_{13} \ni g_{p,q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & r \\ 0 & p & q \end{pmatrix}\,.$$

This explains previous use of $\mathcal{B}^\alpha_S(\boldsymbol{\rho})$. [Nieri-Pasquetti]

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Main ingredient: (independently derived) properties of $\Gamma(z;\tau,\sigma)$. E.g.:

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3-integer parameter generalization of the case $(m;n,\tilde{n})=(1,0,0)$.

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3-integer parameter generalization of the case $(m;n,\tilde{n})=(1,0,0)$.

Agrees precisely with compatible Heegaard splittings of $\mathcal{M}_{(1,0)}(\hat{\boldsymbol{\rho}})$:

$$\mathcal{M}_{(1,0)}(\hat{\boldsymbol{\rho}})\cong D_2\times T^2(h\boldsymbol{\rho})\overset{f'}{\sqcup}D_2\times T^2(\tilde{h}\,f_{1,0}^{-1}\boldsymbol{\rho})\,,\quad(h,\tilde{h})\in S_{f_{1,0}}\,.$$

Similar results for general $f_{p,q}$.

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If X is a **non-trivial** class, there is no trivialization for all g.

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$$X_{g_1 g_2}(\boldsymbol{\rho}) = X_{g_1}(\boldsymbol{\rho}) X_{g_2}(g_1^{-1}\boldsymbol{\rho}) \mod M \quad \text{(closed)}\,.$$

A (local) **trivialization** of X corresponds to:

$$X_g(\boldsymbol{\rho}) = \frac{B(\boldsymbol{\rho})}{B(g^{-1}\boldsymbol{\rho})} \mod M \quad \text{(exact)}\,.$$

This trivially satisfies the 1-cocycle (closedness) condition:

$$\frac{B(\boldsymbol{\rho})}{B(g_2^{-1}g_1^{-1}\boldsymbol{\rho})} = \frac{B(\boldsymbol{\rho})}{B(g_1^{-1}\boldsymbol{\rho})} \frac{B(g_1^{-1}\boldsymbol{\rho})}{B(g_2^{-1}g_1^{-1}\boldsymbol{\rho})}\,.$$

If X is a **non-trivial** class, there is no trivialization for all g.

This construction generalizes automorphic forms as classes in $H^0(G, N/M)$.

Consider "normalized part of lens index", defined for $g \in SL(3,\mathbb{Z})$ and $\hat{\mathcal{Z}}_1^{\alpha} = 1$:

$$\hat{\mathcal{Z}}_g^{\alpha}(\boldsymbol{\rho}) \equiv \frac{\mathcal{Z}_f^{\alpha}(\boldsymbol{\rho})}{\mathcal{Z}_{\mathcal{O}}^{\alpha}(g^{-1}\boldsymbol{\rho})}\,, \qquad f = g\,\mathcal{O}\,,$$

Proposal: this defines a non-trivial cohomology class in $H^1(SL(3,\mathbb{Z}), N/M)$. [Gadde]

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Our proof relies on:

- Ambiguities in the Heegaard splitting:

$$\hat{\mathcal{Z}}^\alpha_{hg\tilde{h}_{\mathcal{O}}^{-1}}(h\boldsymbol{\rho}) = \hat{\mathcal{Z}}_g^\alpha(\boldsymbol{\rho}) \mod M \quad \Rightarrow \quad \hat{\mathcal{Z}}_h^\alpha(\boldsymbol{\rho}) = 1\,.$$

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* Modular factorization. For any $g \in F_h \subset SL(3,\mathbb{Z})$:

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Factorization of indices in terms of $\mathcal{B}_h^\alpha(\boldsymbol{\rho}) \Leftrightarrow$ **trivialization** of $\hat{\mathcal{Z}}_g^\alpha(\boldsymbol{\rho})$ on F_h.

Since $F_h \subset SL(3,\mathbb{Z})$ maximal, $\hat{\mathcal{Z}}_g^\alpha(\boldsymbol{\rho})$ is **non-trivial** in cohomology.

Overview

1. Motivation
2. Heegaard splitting of Hopf surfaces
3. Review of holomorphic block factorization
4. Modular factorization of superconformal indices
5. $SL(3,\mathbb{Z})$ 1-cocycle condition
6. Summary and future directions

Summary:

* Ambiguities in Heegaard splitting lead to modular factorization of indices.
* Compatible blocks parametrized by possible embeddings of S^1_β.
* When $S_f \subset H \times H$ embeds as a subgroup, modular factorization can be interpreted as a covariance under a combined action on f and $\boldsymbol{\rho}$.
* Restricted compatibility of blocks relates to non-triviality of cohomology class.

Future directions (gravitational dual):

* Bulk modular factorization through gravitational blocks?

* Original modular property connects with BPS AdS_5 back hole.

  Can we view $\Gamma'_\infty \times \Gamma'_\infty \backslash S_{f_{1,0}}$ as a classification of bulk solutions?

* Analogue of the Poincaré series expansion for modular factorization?

Future directions (SCFT):

* Better matching of physically and mathematically natural objects?

* $SL(2, \mathbb{Z})$ property Schur index as remnant of modular factorization?

Extra slides

Hopf surfaces

Hopf surface: $\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}})$, $\hat{\boldsymbol{\rho}} = (\hat{z}_a; \hat{\tau}, \hat{\sigma})$ holonomies and moduli.

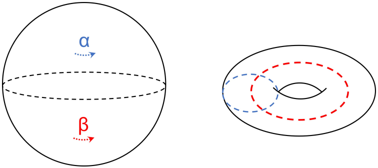
Primary: $\mathcal{M}_{(1,0)}(\hat{\boldsymbol{\rho}})$ is quotient of $(z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$:

$$(z_1, z_2) \sim (\hat{p} z_1, \hat{q} z_2)\,, \quad 0 < |\hat{p}| \leq |\hat{q}| < 1\,, \quad \hat{p} = e^{2\pi i \hat{\tau}}\,,\ \hat{q} = e^{2\pi i \hat{\sigma}}\,.$$

Topologically $S^3 \times S^1$.

Secondary: $\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}})$ from additional lens quotient; topologically $L(p,q) \times S^1$.

Also: $\mathcal{M}_{(0,-1)}(\hat{\boldsymbol{\rho}}) \equiv S^2 \times T^2$ with $\hat{\tau} = \hat{\alpha}\hat{\sigma} - \hat{\beta}$ and $\hat{\sigma}$ complex structure T^2.

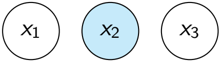


$D_2 \times T^2$

Endow $D_2 \times T^2$ with complex structure moduli:

$$\boldsymbol{\rho} \equiv (z_a; \tau, \sigma) = \left(\frac{Z_a}{x_1}; \frac{x_2}{x_1}, \frac{x_3}{x_1} \right) , \qquad \tau = \alpha\sigma - \beta \, .$$

x_i (complexified) **cycle lengths** of T^3:



Gluing group consists of large diffeos and gauge transformations of T^3:

$$\mathcal{G} \equiv SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^{3r} \, .$$

Action on $\boldsymbol{\rho}$:

$$\begin{aligned} g \in SL(3,\mathbb{Z}) : & \qquad \mathbf{x} \mapsto g \cdot \mathbf{x} \, , \\ t_i^{(a)} \in \mathbb{Z}_a^3 : & \qquad Z_a \mapsto Z_a + x_i \, . \end{aligned}$$

Heegaard splitting

Heegaard splitting of Hopf surface:

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) \cong M_f(\boldsymbol{\rho}) \equiv D_2 \times T^2(\boldsymbol{\rho}) \overset{f}{\sqcup} D_2 \times T^2(f^{-1}\boldsymbol{\rho}) \,,$$

where for $g_{(p,q)} \in SL(2,\mathbb{Z})_{\mu\lambda}$, $\hat{\boldsymbol{\rho}}$ is related to $\boldsymbol{\rho}$ through:

$$(z_a;\tau,\sigma) = \begin{cases} (\hat{z}_a;\hat{\tau},\hat{\sigma})\,, & \text{for } p = r = 0\,,\ q = s = -1\,, \\ (\hat{z}_a;\hat{\tau}+s\hat{\sigma},p\hat{\sigma})\,, & \text{for } p \neq 0\,. \end{cases}$$

Ambiguities in the Heegaard splitting:

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) \cong M_{f'}(\boldsymbol{\rho}')\,, \quad f' = h\,f\,\tilde{h}^{-1}\,,\ \boldsymbol{\rho}' = h\boldsymbol{\rho} \quad \text{for any} \quad h,\tilde{h} \in H\,,$$

Note that: $f^{-1}\boldsymbol{\rho} \to (f')^{-1}\boldsymbol{\rho}' = \tilde{h}\,(f^{-1}\boldsymbol{\rho})$.

Combined action in general **not** a diffeomorphism!

Higgs branch formula

To make factorization manifest, one first performs the contour integral:

[Yoshida,Peelaers,Chen-Chen,Nieri-Pasquetti]

$$\mathcal{I}_{(1,0)}(\hat{\boldsymbol{\rho}}) = \sum_{\alpha} \mathcal{Z}^{\alpha}_{\text{1-loop}}(\hat{\boldsymbol{\rho}}) \mathcal{Z}^{\alpha}_{\text{v}}(\hat{z}, \hat{\tau}; \hat{\sigma}) \mathcal{Z}^{\alpha}_{\text{v}}(\hat{z}, \hat{\sigma}; \hat{\tau}) \, .$$

Known as Higgs branch formula. In this language:

* α labels a finite set of Higgs branch vacua.
* $\mathcal{Z}^{\alpha}_{\text{v}}$ captures vortices localized at NP and SP respectively.
* $\mathcal{Z}^{\alpha}_{\text{1-loop}}(\hat{\boldsymbol{\rho}})$ captures 1-loop fluctuations around the vortices.

Observation: $\mathcal{Z}^{\alpha}_{\text{1-loop}}(\hat{\boldsymbol{\rho}})$ can be factorized: [Nieri-Pasquetti]

$$\mathcal{Z}^{\alpha}_{\text{1-loop}}(\hat{\boldsymbol{\rho}}) = e^{-i\pi \mathcal{P}(\hat{\boldsymbol{\rho}})} b^{\alpha} \left(\tfrac{\hat{z}}{\hat{\sigma}}; \tfrac{\hat{\tau}}{\hat{\sigma}}, -\tfrac{1}{\hat{\sigma}} \right) b^{\alpha} \left(\tfrac{\hat{z}}{\hat{\tau}}; \tfrac{\hat{\sigma}}{\hat{\tau}}, -\tfrac{1}{\hat{\tau}} \right) \, ,$$

Follows from:

$$\Gamma(z; \tau, \sigma) = e^{-i\pi Q(z;\tau,\sigma)} \Gamma\left(\tfrac{z}{\sigma}; \tfrac{\tau}{\sigma}, -\tfrac{1}{\sigma} \right) \Gamma\left(\tfrac{z}{\tau}; \tfrac{\sigma}{\tau}, -\tfrac{1}{\tau} \right)$$

BPS boundary conditions

A vector and chiral multiplet admit two $\frac{1}{2}$-BPS boundary conditions on $\partial D_2 \times T^2$.

We assume Neumann boundary conditions for the vector multiplet.

For the (anti-)chiral multiplets, we assume Dirichlet (Robin-like) or vice versa.

Denote the full blocks by $\mathcal{B}^\alpha(\boldsymbol{\rho})$ and $\mathcal{C}^\alpha(\boldsymbol{\rho})$ respectively.

Coupling to a boundary theory changes the boundary conditions:

$$\mathcal{B}^\alpha(\boldsymbol{\rho}) = Z_\partial^\alpha(\vec{z};\tau)\mathcal{C}^\alpha(\boldsymbol{\rho}),$$

$Z_\partial^\alpha(\vec{z};\tau)$ transforms simply under $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2r}$:

$$Z_\partial^\alpha\left(\tfrac{\vec{z}}{m\tau+n};\tfrac{k\tau+l}{m\tau+n}\right) \cong Z_\partial^\alpha(\vec{z};\tau), \quad Z_\partial^\alpha\left(\vec{z}+\vec{a}\tau+\vec{b};\tau\right) \cong Z_\partial^\alpha(\vec{z};\tau).$$

Consistency condition

The consistency condition:

$$\mathcal{B}^{\alpha}_{h}(\boldsymbol{\rho})\mathcal{B}^{\alpha}_{\tilde{h}}(f^{-1}\boldsymbol{\rho}) \cong \mathcal{C}^{\alpha}_{h}(\boldsymbol{\rho})\mathcal{C}^{\alpha}_{\tilde{h}}(f^{-1}\boldsymbol{\rho})\,,$$

is satisfied when:

$$Z^{\alpha}_{\partial}(z'_a;\tau')\tilde{Z}^{\alpha}_{\partial}(\tilde{z}'_a;\tilde{\tau}') \cong 1\,,$$

where:

$$\boldsymbol{\rho}' \equiv (z'_a;\tau',\sigma') = h\boldsymbol{\rho}\,, \qquad \tilde{\boldsymbol{\rho}}' \equiv (\tilde{z}'_a;\tilde{\tau}',\tilde{\sigma}') = \tilde{h}f^{-1}\boldsymbol{\rho}\,.$$

Due to properties of $Z^{\alpha}_{\partial}(\vec{z};\tau)$, this follows from:

$$\tilde{z}'_a + \tilde{\mu}_a\tilde{\tau}' + \tilde{\nu}_a = \frac{z'_a + \mu_a\tau' + \nu_a}{\gamma\tau' + \delta}\,, \qquad \tilde{\tau}' = -\frac{\alpha\tau' + \beta}{\gamma\tau' + \delta}\,, \qquad \alpha\delta - \beta\gamma = 1\,,$$

Constrains $(h,\tilde{h})$ for some integers α, β, γ, δ, μ_a, $\tilde{\mu}_a$, ν_a and $\tilde{\nu}_a$.

Group cohomology

Abstraction of automorphic forms: group cohomology.

Introduce G-cochains $C^k(G,A)$ consisting of functions $\alpha : G^k \to A$ such that $\alpha_{g_1\ldots,g_k} = 1$ if $g_i = 1$ for some i. Define $C^0(G,A) \equiv A$.

A is the (multiplicative) abelian group of functions on a complex manifold X with G-action, and the induced action for $g \cdot \alpha(\mathbf{x}) = \alpha(g^{-1}\mathbf{x})$ for $\alpha \in A$ and $\mathbf{x} \in X$.

Define a coboundary operator δ as follows:

$$(\delta\alpha)_{g_1,\ldots,g_{k+1}}(\boldsymbol{\rho}) = \alpha_{g_1,\ldots,g_k}(\boldsymbol{\rho}) \left(\alpha_{g_2,\ldots,g_{k+1}}(g_1^{-1}\boldsymbol{\rho}) \prod_{j=1}^{k} \alpha_{g_1,\ldots,g_j g_{j+1},\ldots,g_{k+1}}(\boldsymbol{\rho})^{(-1)^j}\right)^{(-1)^{k+1}} ,$$

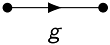
Furthermore, for $k=0$:

$$(\delta\alpha)_g(\boldsymbol{\rho}) = \frac{\alpha(\boldsymbol{\rho})}{\alpha(g^{-1}\boldsymbol{\rho})} .$$

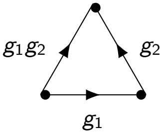
$\delta^2 = 1$, and cohomology is defined as usual:

$$H^k(G,A) = \frac{\ker \delta_k}{\text{im } \delta_{k-1}} , \; k \geq 1 , \quad H^0(G,A) = \ker \delta_0 .$$

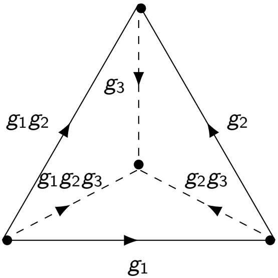
Depiction of action δ



(a) $(\delta\alpha)_g(\boldsymbol{\rho}) = \frac{\alpha(\boldsymbol{\rho})}{\alpha(g^{-1}\boldsymbol{\rho})}$



(b) $(\delta\alpha)_{g_1,g_2}(\boldsymbol{\rho}) = \frac{\alpha_{g_1}(\boldsymbol{\rho})\alpha_{g_2}(g_1^{-1}\boldsymbol{\rho})}{\alpha_{g_1 g_2}(\boldsymbol{\rho})}$



(c) $(\delta\alpha)_{g_1,g_2,g_3}(\boldsymbol{\rho}) = \frac{\alpha_{g_1,g_2}(\boldsymbol{\rho})\alpha_{g_1 g_2,g_3}(\boldsymbol{\rho})}{\alpha_{g_1,g_2 g_3}(\boldsymbol{\rho})\alpha_{g_2,g_3}(g_1^{-1}\boldsymbol{\rho})}$

Example: automorphic form

Consider $\mathcal{J} = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ and let X be parametrized by $\boldsymbol{\tau} = (z;\tau)$.

There is a standard action of $\mathcal{J}$ on X:

$$(z;\tau) \to \left(\tfrac{\vec{z}}{m\tau+n}; \tfrac{k\tau+l}{m\tau+n}\right)\,, \qquad (z;\tau) \to \left(\vec{z} + \vec{a}\tau + \vec{b}; \tau\right)\,.$$

Let N denote meromorphic functions and M holomorphic, non-zero functions.

A (weak) Jacobi form is an element of $H^0(\mathcal{J}, N/M)$ with:

$$\delta\left(\chi(\boldsymbol{\tau})\right)_g \equiv \frac{\chi(\boldsymbol{\tau})}{\chi(g^{-1}\boldsymbol{\tau})} = \phi_g(\boldsymbol{\tau}) \quad \chi(\boldsymbol{\tau}) \in N\,,\, \phi_g(\boldsymbol{\tau}) \in M\,.$$

Because of the cohomological structure, we have:

$$(\delta\phi)_{g_1,g_2} = 1 \quad \Rightarrow \quad \frac{\phi_{g_1}(\boldsymbol{\rho})\phi_{g_2}(g_1^{-1}\boldsymbol{\rho})}{\phi_{g_1 g_2}(\boldsymbol{\rho})} = 1\,.$$

which corresponds to the group homomorphism condition for Jacobi forms.

Example: degree 1 automorphic form

Consider now $\mathcal{G} = SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^{3r}$ and let Y be parametrized by $\boldsymbol{\rho} = (z;\tau,\sigma)$.

An element in $H^1(\mathcal{G}, N/M)$ satisfies:

$$\delta(X)_{g_1,g_2}(\boldsymbol{\rho}) = \frac{X_{g_1}(\boldsymbol{\rho})X_{g_2}(g_1^{-1}\boldsymbol{\rho})}{X_{g_1g_2}(\boldsymbol{\rho})} = \xi_{g_1,g_2}(\boldsymbol{\rho})\,, \quad \xi_{g_1,g_2}(\boldsymbol{\rho}) \in C^2(\mathcal{G}, M)\,.$$

Because of the cohomological structure, we now have:

$$(\delta\xi)_{g_1,g_2,g_3} = 1 \quad \Rightarrow \quad \frac{\xi_{g_1,g_2}(\boldsymbol{\rho})\xi_{g_1g_2,g_3}(\boldsymbol{\rho})}{\xi_{g_1,g_2g_3}(\boldsymbol{\rho})\xi_{g_2,g_3}(g_1^{-1}\boldsymbol{\rho})} = 1\,.$$

Finally, there is a notion of trivializable elements:

$$X_g(\boldsymbol{\rho}) = (\delta B)_g(\boldsymbol{\rho}) \mod M = \frac{B(\boldsymbol{\rho})}{B(g^{-1}\boldsymbol{\rho})} \mod M\,.$$

Strategy proof 1-cocycle condition

Given a function $X_g(\boldsymbol{\rho})$ for $g \in \mathcal{G}$ satisfying $X_1(\boldsymbol{\rho}) = 1$, this defines a 1-cocycle for the free group of generators of $\mathcal{G}$.

It descends to a 1-cocycle for $\mathcal{G}$ if and only if it satisfies the relations in $\mathcal{G}$:

$$X_{T_{ij}}(\boldsymbol{\rho})\, X_{T_{kl}}\left(T_{ij}^{-1}\boldsymbol{\rho}\right) \cong X_{T_{kl}}(\boldsymbol{\rho})\, X_{T_{ij}}\left(T_{kl}^{-1}\boldsymbol{\rho}\right), \qquad i \neq l, \quad j \neq k,$$
$$X_{T_{ij}}(\boldsymbol{\rho})\, X_{T_{jk}}\left(T_{ij}^{-1}\boldsymbol{\rho}\right) \cong X_{T_{ik}}(\boldsymbol{\rho})\, X_{T_{jk}}\left(T_{ik}^{-1}\boldsymbol{\rho}\right) X_{T_{ij}}\left(T_{jk}^{-1}T_{ik}^{-1}\boldsymbol{\rho}\right),$$
$$X_{S_{ij}}(\boldsymbol{\rho})\, X_{S_{ij}}\left(S_{ij}^{-1}\boldsymbol{\rho}\right) X_{S_{ij}}\left(S_{ij}^{-2}\boldsymbol{\rho}\right) X_{S_{ij}}\left(S_{ij}^{-3}\boldsymbol{\rho}\right) \cong 1,$$

and

$$X_{T_{ij}}(\boldsymbol{\rho})\, X_{t_k}\left(T_{ij}^{-1}\boldsymbol{\rho}\right) \cong X_{t_k}(\boldsymbol{\rho})\, X_{T_{ij}}\left(t_k^{-1}\boldsymbol{\rho}\right), \qquad i \neq k,$$
$$X_{T_{ij}}(\boldsymbol{\rho})\, X_{t_i}\left(T_{ij}^{-1}\boldsymbol{\rho}\right) \cong X_{t_i}(\boldsymbol{\rho})\, X_{t_j^{-1}}\left(t_i^{-1}\boldsymbol{\rho}\right) X_{T_{ij}}\left(t_j t_i^{-1}\boldsymbol{\rho}\right),$$
$$X_{t_i}(\boldsymbol{\rho})\, X_{t_j}\left(t_i^{-1}\boldsymbol{\rho}\right) \cong X_{t_j}(\boldsymbol{\rho})\, X_{t_i}\left(t_j^{-1}\boldsymbol{\rho}\right).$$

Here, T_{ij} for $1 \leq i, j \leq 3$ are the generators of $SL(3,\mathbb{Z})$ and $S_{ij} = T_{ij}T_{ji}^{-1}T_{ij}$.

Finally, t_i for $1 \leq i \leq 3$ generate the $\mathbb{Z}^3$ factor.

$SL(3,\mathbb{Z})$ relations and modular properties

Modular properties of elements in $H^1(\mathcal{G}, N/M)$ are labeled by relations in $\mathcal{G}$.

This is in contrast with automorphic forms, whose properties are labeled by elements.

Modular factorization corresponds to:

$$h\, g_{p,q}\, \tilde{h}_{\mathcal{O}}^{-1} = S_{23}\, h'\, S_{23}^{-1}\,, \qquad \tilde{h}_{\mathcal{O}} \equiv \mathcal{O}\, \tilde{h}\, \mathcal{O}\,, \qquad (h, \tilde{h}) \in S_{f_{p,q}}\,.$$

Through the 1-cocycle condition, this leads to:

$$\hat{\mathcal{Z}}^{\alpha}_{g_{(p,q)}}(\boldsymbol{\rho}) \cong \frac{\hat{\mathcal{Z}}^{\alpha}_{S_{23}}(h\boldsymbol{\rho})}{\hat{\mathcal{Z}}^{\alpha}_{S_{23}}(\tilde{h}_{\mathcal{O}}\, g^{-1}\boldsymbol{\rho})}\,.$$

$S_{f_{p,q}}$ family of "modular properties" relating (normalized part of) lens index to (normalized part of) $S^3 \times S^1$ indices.

Technically and conceptually, connection to modular factorization is not obvious.