Nielsen complexity for superconformal primaries

Jaco van Zyl

based on [R de Mello Koch, M Kim, HJRvZ, 2108.10669], [P Rabambi, HJRvZ, 2208.05520]

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Talk Layout

1. Motivation
2. Background
3. BCH formulas
4. Circuits
5. Outlook

Motivation

- Complexity is related to the holographic description of black holes
- Growth of complexity = growth of black hole interiors
- Thermofield double is a famous example of this

[Chapman et al, 1810.05151]

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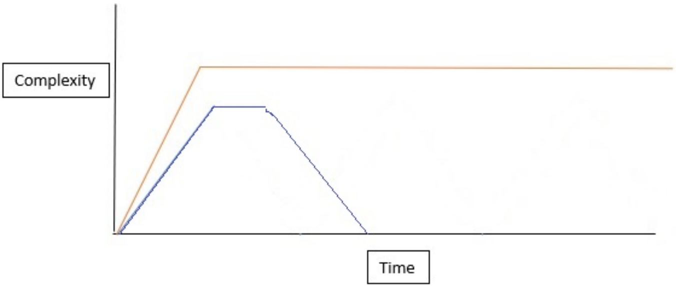
  [Chapman et al, 1810.05151]
- Complexity can be used as a diagnostic of quantum chaos

  [Chapman, Pelicastro, 2110.14672]
- Supplements diagnostics such as SFF, OTOC, Loschmidt echo...

Motivation



[Balasubramanian, DeCross, Kar, Li, Parrikar, 2101.02209]

Complexity

- Central question: How hard is it to synthesize a desired target state with the gates at your disposal?
- Need, $|\phi_r\rangle$, $|\phi_t\rangle$, $\{U_1, U_2, \cdots, U_n\}$, $g(U_1, U_2, \cdots, U_n)$

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- E.g. $U_1 U_2 U_1 U_3 (U_1)^3 U_2 |\phi_r\rangle = U_3 U_1 U_2 U_1 U_3 (U_1)^3 U_2 U_3 |\phi_r\rangle$, "complexity $= 8$"

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- Discrete notion of complexity closely related to quantum computational setups
- We will, however, be interested in a continuous notion of complexity

Nielsen Complexity

- Accessible gates are taken to be from some symmetry group

  [Nielsen, quant-ph/0502070]

- E.g. $SU(2)$: Gates $U = e^{i(s_1 J_1 + s_2 J_2 + s_3 J_3)}$
- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$

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- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$
- We have a manifold of target states on which one can define a metric
- Complexity = shortest distance connecting points
- Can introduce a circuit parameter $s_i = s_i(\sigma)$

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Nielsen Complexity

- Two examples of metrics (assuming all transformations equally hard)
- F_1 cost function: $\mathcal{F}_1 d\sigma = |\langle\phi_r|U^\dagger dU|\phi_r\rangle$

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- Group symmetries are encoded as metric isometries
- $\mathcal{F}_1 : \quad F_i = \partial_i \left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)|_{s'=s}$
- FS metric:
  $g_{ij} = \partial_i\partial'_j \log\left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)\Big|_{s'=s}$

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- Manifold of states $\Leftrightarrow$ group elements of G/H

BCH formulas

- Baker-Campbell-Hausdorff formulas can be used as powerful computational tools for the coherent state overlaps
- $SO(d,2)$, (Euclidean) conformal group, $P_\mu^\dagger = K_\mu, L_{\mu\nu}^\dagger = L_{\nu\mu}$,

$$[D, P_\mu] = P_\mu, \qquad [D, K_\mu] = -K_\mu$$

$$[L_{\mu\nu}, P_\rho] = \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \quad [L_{\mu\nu}, K_\rho] = \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2L_{\mu\nu}$$

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- Spinless conformal primary, $|\Delta\rangle$,
$D|\Delta\rangle = \Delta|\Delta\rangle \quad , \quad L_{\mu\nu}|\Delta\rangle = 0 \quad , \quad K_\mu|\Delta\rangle = 0$

[Chagnet, Chapman, De Boer, Zukowski, 2103.06920]

BCH formulas

- Consider the overlap $\langle \Delta | e^{\alpha^* \cdot K} e^{\alpha \cdot P} | \Delta \rangle$
- With BCH identities we can exchange

$$e^{\alpha^* \cdot K} e^{\alpha \cdot P} = e^{\beta(\alpha^*, \alpha) \cdot P} e^{d(\alpha^*, \alpha) D} e^{(\lambda(\alpha^*, \alpha))^{\nu\mu} L_{\mu\nu}} e^{\beta^*(\alpha^*, \alpha) \cdot K}$$

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- Thus $\langle\Delta|e^{\alpha^*\cdot K}e^{\alpha\cdot P}|\Delta\rangle=e^{d(\alpha^*,\alpha)\Delta}\langle\Delta|\Delta\rangle=e^{d(\alpha^*,\alpha)\Delta}$

BCH formulas

- Conjecture: $C_1 = [A, B] \quad , \quad C_{n+1} \equiv [[A, C_n], B]$
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- If $[A_i, A_j] = [B_i, B_j] = [C_i, C_j] = 0$ then
- $e^A e^B = \prod_{j=1}^{\infty} e^{\frac{1}{2^j-1} B_j} \prod_{j=1}^{\infty} e^{\frac{1}{2^j-1_j} C_j} \prod_{j=1}^{\infty} e^{\frac{1}{2^j-1} A_j}$
- Spoiler: We (roughly) have in mind reference kets annihilated by the A_j operators and that transform trivially under the C_j operators

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Spinor notation

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- $4d$ bi-spinor notation: $(\sigma^\mu)_{\alpha\dot{\alpha}} = (iI, \vec{\sigma}), (\bar{\sigma}^\mu)_{\alpha\dot{\alpha}} = (iI, \vec{\sigma})$
- Repackages the generators $P_{\alpha\dot{\alpha}}, K^{\dot{\alpha}\alpha}, L_\alpha^{\ \beta}, \bar{L}^{\dot{\alpha}}_{\ \dot{\beta}}$
- The rotations repackage as $SO(4) \to SU(2) \times SU(2)$

Supercharges

- Additionally, we have supercharges and conformal supercharges
- $\left\{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\right\} = \frac{1}{2}\delta^i_j P_{\alpha\dot{\alpha}} \quad , \quad \left\{\bar{S}^{i\dot{\alpha}}, S^\alpha_j\right\} = \frac{1}{2}\delta^i_j K^{\dot{\alpha}\alpha}$
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- $(Q^i_\alpha)^\dagger = S^\alpha_i \quad , \quad \bar{Q}_{i\dot{\alpha}} = \bar{S}^{i\dot{\alpha}} \quad , \quad (R^j_i)^\dagger = R^i_j$

Circuits

- We have $[D, L_\alpha^{\ \beta}] = 0 \quad , \quad [D, L^{\dot{\alpha}}_{\ \dot{\beta}}] = 0 \quad , \quad [D, R^i_j] = 0$
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- $D|\phi_r\rangle = \Delta|\phi_r\rangle \quad , \quad K^{\dot{\alpha}\alpha}|\phi_r\rangle = S^\alpha_j|\phi_r\rangle = \bar{S}^{i\dot{\alpha}}|\phi_r\rangle = 0$

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- The rotations are $SU(2)$'s,

$$
\begin{array}{cc}
L_2{}^2|\phi_r\rangle = h|\phi_r\rangle & \bar{L}^{\dot 1}{}_{\dot 1}|\phi_r\rangle = \bar{h}|\phi_r\rangle \\
L_2{}^1|\phi_r\rangle = 0 & \bar{L}^{\dot 2}{}_{\dot 1}|\phi_r\rangle = 0
\end{array}
$$

Circuit

- For this choice of reference state we obtain the following from a general group action of $SU(2,2|\mathcal{N})$
- $|\phi_t(\sigma)\rangle = N\ e^{p^{\alpha\dot{\alpha}}P_{\dot{\alpha}\alpha}} e^{q_i^{\alpha}Q^i_{\alpha}} e^{\bar{q}^{i\dot{\alpha}}\bar{Q}_{i\dot{\alpha}}} e^{l_2^1 L_1{}^2} e^{\bar{l}_{\dot{1}}^{\dot{2}}\bar{L}^{\dot{1}}{}_{\dot{2}}} e^{r_i^j R_j^i}|\phi_r\rangle$

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- The overlap we are interested in computing is
- $\langle\phi_r| e^{\bar r_i^j R_j^i} e^{\bar l^{\dot 1}_{\dot 2}\bar L^{\dot 2}{}_{\dot 1}} e^{l_1^2 L_2{}^1} e^{\bar s_{i\dot\alpha}\bar S^{i\dot\alpha}} e^{s^i_\alpha S_i^\alpha} e^{k_{\dot\alpha\alpha}K^{\alpha\dot\alpha}} e^{p^{\alpha\dot\alpha}P_{\dot\alpha\alpha}} e^{q_i^\alpha Q^i_\alpha} e^{\bar q^{i\dot\alpha}\bar Q_{i\dot\alpha}} e^{l_1^2 L_1{}^2} e^{\bar l^{\dot 2}_{\dot 1}\bar L^{\dot 1}{}_{\dot 2}} e^{r_i^j R_j^i}|\phi_r\rangle$
- We have several pairs of exponentials that satisfy
- $e^A e^B = \prod_{j=1}^\infty e^{\frac{1}{2^{j-1}}B_j} \prod_{j=1}^\infty e^{\frac{1}{2^{j-1}j}C_j} \prod_{j=1}^\infty e^{\frac{1}{2^{j-1}}A_j}$

$\mathcal{N} = 0$

- $$\begin{aligned} &\langle \Delta; h, h; \bar{h}, \bar{h} | e^{\bar{l}^{\dot{1}}_{\dot{2}} \bar{L}^{\dot{2}}{}_{\dot{1}}} e^{l^{2}_{1} L_{2}{}^{1}} e^{k_{\alpha\dot{\alpha}} K^{\dot{\alpha}\alpha}} e^{p^{\dot{\alpha}\alpha} P_{\alpha\dot{\alpha}}} e^{l^{1}_{2} L_{1}{}^{2}} e^{\bar{l}^{\dot{2}}_{\dot{1}} \bar{L}^{\dot{1}}{}_{\dot{2}}} | \Delta; h, h; \bar{h}, \bar{h} \rangle \\ = \quad & \left((1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1})(1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) - 16 k_{1\dot{\beta}} p^{\dot{\beta}2} k_{2\dot{\gamma}} p^{\dot{\gamma}1} \right)^{-(\Delta + h + \bar{h})} \times \\ & \left((1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1}) + 4 l^{2}_{1} k_{2\dot{\beta}} p^{\dot{\beta}1} + 4 l^{1}_{2} k_{1\dot{\beta}} p^{\dot{\beta}2} + l^{2}_{1} l^{1}_{2} (1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) \right)^{2h} \times \\ & \left((1 - 4p^{\dot{2}\beta} k_{\beta\dot{2}}) + 4 \bar{l}^{\dot{1}}_{\dot{2}} p^{\dot{2}\beta} k_{\beta\dot{1}} + 4 \bar{l}^{\dot{2}}_{\dot{1}} p^{\dot{1}\beta} k_{\beta\dot{2}} + \bar{l}^{\dot{1}}_{\dot{2}} \bar{l}^{\dot{2}}_{\dot{1}} (1 - 4p^{\dot{1}\beta} k_{\beta\dot{1}}) \right)^{2\bar{h}} \end{aligned}$$
- Note the product structure of terms - this leads to a sum of terms when taking the logarithm
- $K = \log\left(\langle \phi_t | \phi_t \rangle\right)$

$\mathcal{N} = 1$

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- We choose $R^1_1|\phi_r\rangle = R|\phi^r_r\rangle$

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$$
\begin{aligned}
&\langle\psi_0|e^{\bar{l}^{\dot{1}}_{\dot{2}}\bar{L}^{\dot{2}}{}_{\dot{1}}}\, e^{l^2_1 L_2{}^1}\, e^{\bar{s}_{i\dot{\alpha}}\bar{S}^{i\dot{\alpha}}}\, e^{s^i_\alpha S^\alpha_i}\, e^{k_{\alpha\dot{\alpha}}K^{\dot{\alpha}\alpha}}\, e^{p^{\dot{\alpha}\alpha}P_{\alpha\dot{\alpha}}}\, e^{q^\alpha_i Q^i_\alpha}\, e^{\bar{q}^{i\dot{\alpha}}\bar{Q}_{i\dot{\alpha}}}\, e^{l^1_2 L_1{}^2}\, e^{\bar{l}^{\dot{2}}_{\dot{1}}\bar{L}^{\dot{1}}{}_{\dot{2}}}|\psi_0\rangle \\
= &\left((1-4k_{1\dot{\beta}}p^{\dot{\beta}1}-s_1q^1)(1-4k_{2\dot{\beta}}p^{\dot{\beta}2}-s_2q^2)-(4k_{1\dot{\beta}}p^{\dot{\beta}2}+s_1q^2)(4k_{2\dot{\beta}}p^{\dot{\beta}1}+s_2q^1)\right)^{-\Delta}\times \\
&\left(\frac{(1-4k_{1\dot{\beta}}p^{\dot{\beta}1}-s_1q^1)+l^2_1(4k_{2\dot{\beta}}p^{\dot{\beta}1}+s_2q^1)+l^1_2(4k_{1\dot{\beta}}p^{\dot{\beta}2}+s_1q^2)+l^2_1l^1_2(1-4k_{2\dot{\beta}}p^{\dot{\beta}2}-s_2q^2)}{\sqrt{(1-4k_{1\dot{\beta}}p^{\dot{\beta}1}-s_1q^1)(1-4k_{2\dot{\beta}}p^{\dot{\beta}2}-s_2q^2)-(4k_{1\dot{\beta}}p^{\dot{\beta}2}+s_1q^2)(4k_{2\dot{\beta}}p^{\dot{\beta}1}+s_2q^1)}}\right)^{2h}\times \\
&\left(\frac{(1-4\tilde{p}^{\dot{2}\beta}\tilde{k}_{\beta\dot{2}}-\bar{s}_{\dot{2}}\bar{q}^{\dot{2}})+\bar{l}^{\dot{1}}_{\dot{2}}(4\tilde{p}^{\dot{2}\beta}\tilde{k}_{\beta\dot{1}}+\bar{s}_{\dot{1}}\bar{q}^{\dot{2}})+\bar{l}^{\dot{2}}_{\dot{1}}(4\tilde{p}^{\dot{1}\beta}\tilde{k}_{\beta\dot{2}}+\bar{s}_{\dot{2}}\bar{q}^{\dot{1}})+\bar{l}^{\dot{1}}_{\dot{2}}\bar{l}^{\dot{2}}_{\dot{1}}(1-4\tilde{p}^{\dot{1}\beta}\tilde{k}_{\beta\dot{1}}-\bar{s}_{\dot{1}}\bar{q}^{\dot{1}})}{\sqrt{(1-4\tilde{p}^{\dot{1}\beta}\tilde{k}_{\beta\dot{1}}-\bar{s}_{\dot{1}}\bar{q}^{\dot{1}})(1-4\tilde{p}^{\dot{2}\beta}\tilde{k}_{\beta\dot{2}}-\bar{s}_{\dot{2}}\bar{q}^{\dot{2}})-(4\tilde{p}^{\dot{2}\beta}\tilde{k}_{\beta\dot{1}}+\bar{s}_{\dot{1}}\bar{q}^{\dot{2}})(4\tilde{p}^{\dot{1}\beta}\tilde{k}_{\beta\dot{2}}+\bar{s}_{\dot{2}}\bar{q}^{\dot{1}})}}\right)^{2\bar{h}}\times \\
&\left(\frac{\left((1-4k_{1\dot{\beta}}p^{\dot{\beta}1}-s_1q^1)(1-4k_{2\dot{\beta}}p^{\dot{\beta}2}-s_2q^2)-(4k_{1\dot{\beta}}p^{\dot{\beta}2}+s_1q^2)(4k_{2\dot{\beta}}p^{\dot{\beta}1}+s_2q^1)\right)}{\left((1-4k_{1\dot{\beta}}p^{\dot{\beta}1})(1-4k_{2\dot{\beta}}p^{\dot{\beta}2})-(4k_{1\dot{\beta}}p^{\dot{\beta}2})(4k_{2\dot{\beta}}p^{\dot{\beta}1})\right)(1+2q^\alpha k_{\alpha\dot{\alpha}}\bar{q}^{\dot{\alpha}})^{-1}(1+2\bar{s}_{\dot{\alpha}}p^{\dot{\alpha}\alpha}s_\alpha)^{-1}}\right. \\
&\left.+\frac{(1-4p^{\dot{1}\beta}k_{\beta\dot{1}})(1-4p^{\dot{2}\beta}k_{\beta\dot{2}})-(4p^{\dot{2}\beta}k_{\beta\dot{1}})(4p^{\dot{1}\beta}k_{\beta\dot{2}})}{(1-4p^{\dot{1}\beta}k_{\beta\dot{1}}-\bar{s}_{\dot{1}}\bar{q}^{\dot{1}})(1-4p^{\dot{2}\beta}k_{\beta\dot{2}}-\bar{s}_{\dot{2}}\bar{q}^{\dot{2}})-(4p^{\dot{2}\beta}k_{\beta\dot{1}}+\bar{s}_{\dot{1}}\bar{q}^{\dot{2}})(4p^{\dot{1}\beta}k_{\beta\dot{2}}+\bar{s}_{\dot{2}}\bar{q}^{\dot{1}})}-1\right)^{\frac{\Delta}{2}+R}
\end{aligned}
$$

$\mathcal{N} = 1$

- An instructive limit is $\bar{q}^{i\dot{\alpha}} \to 0, \bar{s}_{i\dot{\alpha}} \to 0$

$$
\begin{aligned}
&\langle\psi_0| e^{\bar{l}^{\dot{1}}_{\dot{2}} \bar{L}^{\dot{2}}{}_{\dot{1}}} e^{l^2_1 L_2{}^1} e^{s^i_\alpha S^\alpha_i} e^{k_{\alpha\dot{\alpha}} K^{\dot{\alpha}\alpha}} e^{p^{\dot{\alpha}\alpha} P_{\alpha\dot{\alpha}}} e^{q^\alpha_i Q^i_\alpha} e^{l^1_2 L_1{}^2} e^{\bar{l}^{\dot{2}}_{\dot{1}} \bar{L}^{\dot{1}}{}_{\dot{2}}} |\psi_0\rangle \\
= &\Big((1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1} - s_1 q^1) + l^2_1 (4k_{2\dot{\beta}} p^{\dot{\beta}1} + s_2 q^1) + l^1_2 (4k_{1\dot{\beta}} p^{\dot{\beta}2} + s_1 q^2) + l^2_1 l^1_2 (1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2} - s_2 q^2) \Big)^h \times \\
&\Big((1 - 4\tilde{p}^{\dot{2}\beta} \tilde{k}_{\beta\dot{2}}) + \bar{l}^{\dot{1}}_{\dot{2}} (4\tilde{p}^{\dot{2}\beta} \tilde{k}_{\beta\dot{1}}) + \bar{l}^{\dot{2}}_{\dot{1}} (4\tilde{p}^{\dot{1}\beta} \tilde{k}_{\beta\dot{2}}) + \bar{l}^{\dot{1}}_{\dot{2}} \bar{l}^{\dot{2}}_{\dot{1}} (1 - 4\tilde{p}^{\dot{1}\beta} \tilde{k}_{\beta\dot{1}}) \Big)^{\bar{h}} \times \\
&\Bigg(\frac{(1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1})(1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) - (4k_{1\dot{\beta}} p^{\dot{\beta}2})(4k_{2\dot{\beta}} p^{\dot{\beta}1}) + 2(\frac{\Delta}{2} - R + h - 1)(\frac{\Delta}{2} - R + h) s_1 s_2 q^2 q^1}{\Big((1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1})(1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) - (4k_{1\dot{\beta}} p^{\dot{\beta}2})(4k_{2\dot{\beta}} p^{\dot{\beta}1}) \Big)^{\Delta + h + \bar{h} + 1}} \\
&\quad + (\frac{\Delta}{2} + h - R) \frac{(1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) s_1 q^1 + 4k_{2\dot{\beta}} p^{\dot{\beta}1} s_1 q^2 + 4k_{1\dot{\beta}} p^{\dot{\beta}2} s_2 q^1 + (1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1}) s_2 q^2}{\Big((1 - 4k_{1\dot{\beta}} p^{\dot{\beta}1})(1 - 4k_{2\dot{\beta}} p^{\dot{\beta}2}) - (4k_{1\dot{\beta}} p^{\dot{\beta}2})(4k_{2\dot{\beta}} p^{\dot{\beta}1}) \Big)^{\Delta + h + \bar{h} + 1}} \Bigg)
\end{aligned}
$$

$\mathcal{N} = 2$

- The R-charge generators form a $u(2)$
- The reference state can be chosen such that
  $R^1_1|\phi_r\rangle = r - R \quad , \quad R^2_2|\phi_r\rangle = r + R \quad , \quad R^2_1|\phi_r\rangle = 0$
- An explicity (though bulky) expression can be found for the relevant overlap
- $\langle\psi_0| e^{r_2^{\ 1} R_1^{\ 2}}\, e^{\bar{l}^{\dot{1}}_{\ \dot{2}} \bar{L}^{\dot{2}}_{\ \dot{1}}}\, e^{l_1^{\ 2} L_2^{\ 1}}\, e^{\bar{s}_{i\dot{\alpha}} \bar{S}^{i\dot{\alpha}}}\, e^{s^i_{\ \alpha} S_i^{\ \alpha}}\, e^{k_{\dot{\alpha}\alpha} K^{\alpha\dot{\alpha}}}\, e^{p^{\alpha\dot{\alpha}} P_{\dot{\alpha}\alpha}}\, e^{q_i^{\ \alpha} Q^i_{\ \alpha}}\, e^{\bar{q}^{i\dot{\alpha}} \bar{Q}_{i\dot{\alpha}}}\, e^{l_2^{\ 1} L_1^{\ 2}}\, e^{\bar{l}^{\dot{2}}_{\ \dot{1}} \bar{L}^{\dot{1}}_{\ \dot{2}}}\, e^{r_1^{\ 2} R_2^{\ 1}} |\psi_0\rangle$

Outlook

- BCH techniques provide powerful tools. These may be applied in other circuits (or Krylov complexity computations)
- $\mathcal{N} = 4$ is an important symmetry group for the AdS/CFT correspondence
- Do the manifolds give rise to conjugate points? What is the role played by spin and supersymmetry? How do the small and large scaling dimensions compare?

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- Do the manifolds give rise to conjugate points? What is the role played by spin and supersymmetry? How do the small and large scaling dimensions compare?
- ...note that the relevant manifolds have a constant scalar curvature...

Thank you for your attention!

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