Brick wall, Normal Modes and Emerging Thermality

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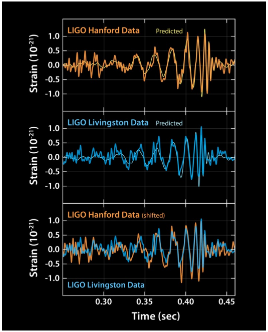
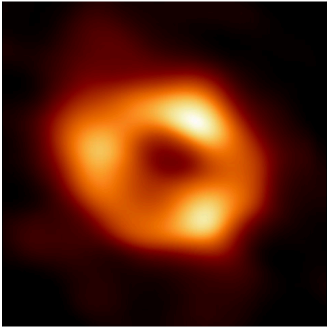
Motivation I

- Observations from LIGO and EHT





- Considering the observed precession, one might ask: Could this be the result of a black hole, or perhaps an extremely compact object?
- To what extent can an extremely compact object mimic the properties of a true black hole?

Introduction

Motivation II

- Black holes are chaotic objects, but demonstrating this property is very challenging for black holes in dimensions $d > 2$.
- Lower-dimensional gravity (dual to the SYK model) provides hints by analyzing the spectrum. The spectrum reveals a linear ramp in the Spectral Form Factor (SFF), and the level spacing distribution (LSD) follows Wigner-Dyson statistics.
- On the boundary side, to demonstrate this in higher dimensions (e.g., $4 + 1$ dimensions), one must solve 4-dim $\mathcal{N} = 4$ super Yang-Mills theory, which is a difficult task.
- On the bulk side, it is necessary to solve quantum gravity to determine the spectrum and perform spectral analysis—an extremely challenging!!

Introduction

Simpler question: Is there any evidence of chaos in the probe sector?

- OTOC computation shows that black holes are chaotic but it is a measure of early time chaos.
- How can chaos be observed at late times? One probe of late-time chaos is the Spectral Form Factor (SFF), where a linear ramp in the SFF serves as a measure of late-time chaos.
- The quantization of a probe scalar in black hole geometry leads to complex-valued quasi-normal modes, resulting in a decaying SFF at late times—no linear ramp is observed.
- Is there an alternative approach?

We quantized a probe scalar field by introducing a **brick wall** in the geometry, resulting in real-valued normal modes. The Spectral Form Factor (SFF) constructed from these modes exhibits a linear ramp.
While this does not solve the main problem, it is an interesting result. Let us focus on this aspect.

Introduction

- Our measuring tool is the correlation functions. Basically we focus on Spectral form factor (SFF), Level Spacing Distribution (LSD) and Green's function.
- Let's define SFF as:

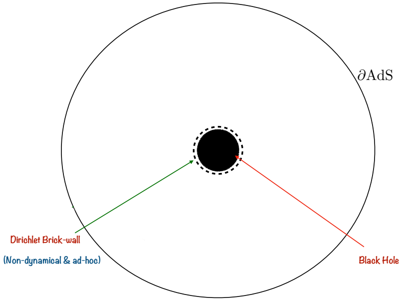
$$g(\beta, t) = \frac{|Z(\beta, t)|^2}{|Z(\beta, 0)|^2}. \tag{1}$$

  For a given quantum mechanical system, $Z(\beta, t) = \mathrm{Tr}\left[e^{-(\beta - it)H}\right]$, where β, t, and H represent the inverse temperature, time, and Hamiltonian of the system, respectively.
- SFF measures correlations among all the energy levels.
- **Properties of SFF**
  Smooth density of states $\implies$ **vanishing** SFF at late time.
  discreate spectra $\implies$ **non vanishing** SFF at late time

- Poisson distribution of energy levels $\implies$ **NO** ramp
  Wigner-Dyson distribution $\implies$ ramp of **slope 1**

The brick wall model

- Consider a BTZ black hole geometry
- Instesd of the full geometry, consider a brickwall at some $r_0 > r_H$
- Our region of interest is $r_0 < r < \infty$
- This model do not have any horizon.
- We will quantize a probe scalar field in this region.



Quantization of a probe scalar

Let us consider the BTZ metric,

$$ds^2 = -(r^2 - r_H^2)dt^2 + \frac{dr^2}{(r^2 - r_H^2)} + r^2 d\psi^2$$

where $r = r_H$ is the position of the horizon an ψ is a compact direction. We will quantize a scalar field $\Phi = \sum_{\omega,m} e^{-i\omega t} e^{im\psi} \phi_{\omega,m}(r)$ in this geometry.

$$\Box\Phi = \mu^2\Phi$$

with the following boundary conditions:

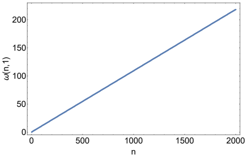
$$\phi(r) = \begin{cases} r^{-\frac{1}{2}-\sqrt{1+\mu^2}} & \text{at } r \longrightarrow \infty \\ 0 & \text{at } r = r_0 \text{(instead of usual ingoing bdry condition)} \end{cases}$$

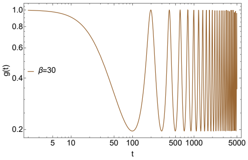
This gives rise to normal modes as opposed to quasi-normal modes. These normal modes are **real** numbers and charaterized by **two quantum numbers** m (due to periodicity in ψ) and n (pricipal quantum number).

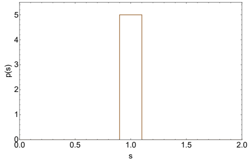
Spectrum, SFF and LSD along *n* and *m*

Here we have separately plotted SFF and LSD along the both directions.

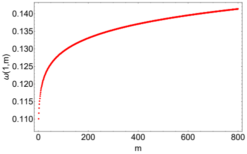
- along *n* direction

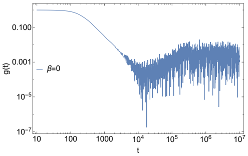


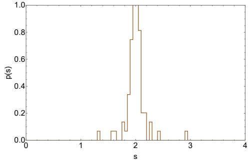




- along *m* direction

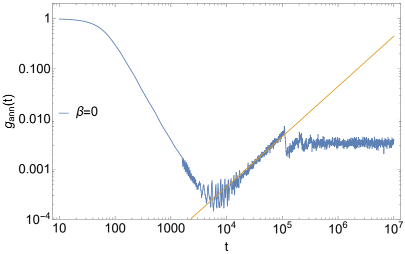






- Non trivial Dip-Ramp-Plateau structure is coming from the modes along the compact direction so compact direction is necessary.

Averaging decreases fluctuation



Annealed SFF of BTZ normal modes with $J_{cut} = 300$, $\beta = 0$ and $n = 1$. Averaging is done over hundred randomly chosen z_0 from a normal distribution with mean $\mu = 20$ and variance $\sigma = 0.1$. The equation of yellow straight line is $\log g_{\mathrm{ann}}(t) = \log t + \text{constant}$.

Analytic Approximation

- In the limit of small ω and $r_H/L \sim O(1)$, the modes are given by,

$$\omega_{n,m} = -\frac{2n\pi r_H}{L^2 \left(\log(\frac{2\epsilon}{r_H}) + \log(\frac{mL}{r_H})^2\right)}$$

  This clearly demonstrates the aforementioned behavior of the modes along m and n direction.
- In the limit $\epsilon \to 0$, this can be approximated as,

$$\omega_{n,m} = -\frac{2n\pi r_H}{L^2 \log(\frac{2\epsilon}{r_H})}$$

  which implies that the spectrum is approximately degenerate along the m direction.
- This expression can be utilized to compute the partition function and subsequently the entropy, which matches exactly with the BTZ black hole entropy.

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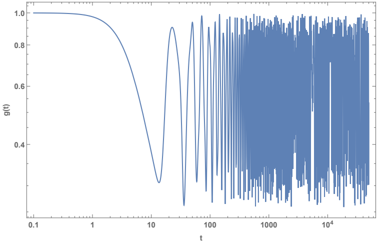
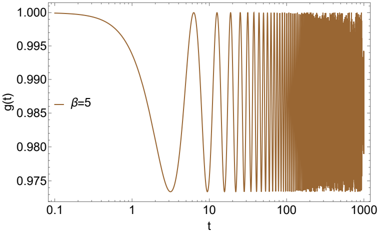
Position of the brick wall is important

- The Dip-Ramp-Plateau structure in the SFF is very much dependent of the position of the brick wall.
- These two figures show how the structure of SFF becomes empty AdS like when the wall is far from the horizon.

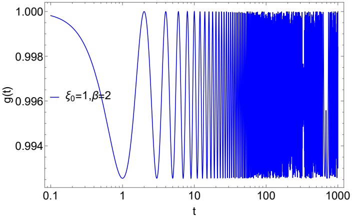


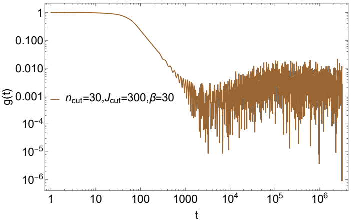


Left : SFF for BTZ black hole with with large r_0. Right: SFF for empty AdS with dispersion relation $\omega_{nl} = (\Delta + l + 2n)$.

Is this generic? (2d Rindler $\times S^1$)

- Near horizon geometry of all the non-extremal black holes are described by the $2d$ Rindler geometries
- quantization of probe scalar in $2d$ Rindler gives rise to the SFF of the left figure.
- On the other hand, quantization of probe scalar in 2D Rindler $\times S^1$ yields identical features to those in BTZ (see right figure).





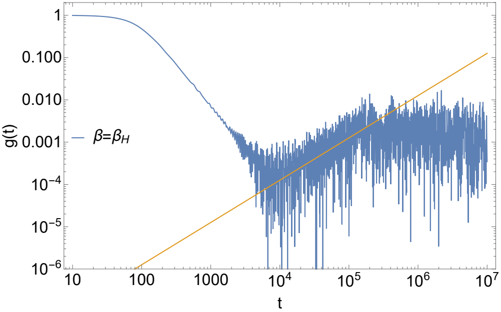
- This implies that the previous results are not only particularly specific to the BTZ geometry but also a generic feature of stretched horizons.

Brickwall in rotating BTZ

We have quantized a probe scalar field in the rotating BTZ background and made the following observations:

- We observe a non-trivial Dip-Ramp-Plateau structure with a ramp of slope one in the SFF obtained from the grand canonical partition function.
- This behavior appears to remain stable close to extremality. However, at extremality, we find a loss of the DRP structure, and the corresponding SFF now resembles that of an integrable system, with normal modes appearing to be linear and thus harmonic oscillator-like.
- This is in agreement with the existing literature on actual microstate geometries.

Brickwall in rotating BTZ



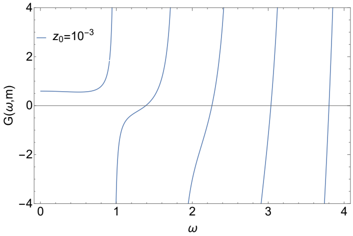
SFF of the normal modes along m- direction of the rotating BTZ black hole with a brick wall.

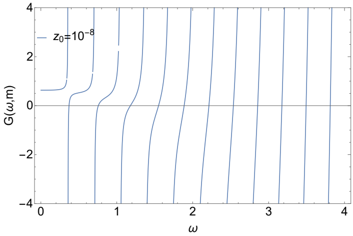
Emerging Thermality

- Chaos implies thermality. How can we observe this?
- A black hole exhibits thermal features due to the presence of the horizon.
- Is it possible to observe any hint of thermality in the absence of the horizon (but with a large redshift)?
- To explore this, let's focus on the Green's function.

Emerging thermality

Following Son-Starients prescription, we have calculated the boundary two point function of an operator that is dual to the scalar field in the bulk.

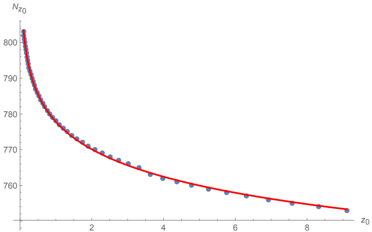


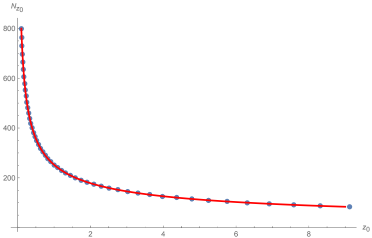


- Figures are showing the pole Structure of $G_\omega(n, m)$ for fixed $m = 1$. Poles are coming closer and closer as we move the position of the stretched horizon towards the event horizon.
- In terms of the bulk picture, this poles corresponds to the normal modes.

Emerging thermality

Pole accumulation along n and m as we move the brickwall:





- Figures showing pole accumulation along the n (left) and m (right) directions.
- The pole accumulation rate along n can be fitted with $a + b\log(1/z_0)$, whereas along m, it is $a + bz_0^{-1/2}$, which is exponentially higher.
- This may be the cause of exhibiting RMT-like behavior along the m direction.

Emerging Thermality

- When the brick wall is very close to the event horizon, the poles are so dense that we can approximate the discrete poles as a branch cut, and the discontinuity around the branch cut is given by the following.

$$G(\omega + i\epsilon, m) - G(\omega - i\epsilon, m) \simeq -\frac{1}{\pi}\left(\frac{\omega}{\omega_n}\right)^{\Delta} \mathrm{Im} G_{\mathrm{bh}}(\omega, m)\Big|_{\omega=\omega_n} \tag{2}$$

- We can read off the black hole's quasi-normal modes by computing the discontinuity around the branch cut.

Emerging thermality

- In the aforementioned limit, the following can be show,

$$\operatorname{Im} G(\omega, m) = -\operatorname{Im} G_{\text{bh}}(\omega, m), \tag{3}$$

- which states that in the limit when the wall is sufficiently close to the horizon of the black hole, the imaginary part of the scalar Green's function is indistinguishable from that of the thermal Green's function of the black hole for a sufficiently low energy asymptotic observer.

- The equation (3) automatically ensures that the position space Green's function is equal to the thermal Green's function with temperature equal to the Hawking temperature of the black hole.

Brickwall in AdS-Schwarzschild black hole

- In 3 dimensions, the degrees of freedom of the graviton are zero, so there is no dynamical graviton.
- **Question**: Do these features remain for dimensions≥ 4?
- Rindler calculations suggest that this may be a generic fact, but it is beneficial to see this explicitly.
- Here we will provide an explicit example for 5-dimensional AdS-Schwarzschild black holes.

Probe scalar in 5d AdS-Schwarzschild black hole

- **Metric** : $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_3^2\,, \qquad f(r) = \left(1 - \frac{r_H^2}{r^2}\right)(r^2 + r_H^2 + 1).$
- **Probe scalar**:$\Box\Phi \equiv \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}\partial^\nu\Phi\right) = \mu^2\Phi.$
- As the metric is time independent and spherically symmetric we can take the ansatz: $\Phi(t, r, \Omega) \sim e^{-i\omega t}Y_{l,\vec{m}}(\Omega)\phi_{\omega l}(r)$.
- radial equation: $\frac{1}{r^3}\frac{d}{dr}\left(r^3 f(r)\frac{d\phi(r)}{dr}\right) + \left(\frac{\omega^2}{f(r)} - \frac{l(l+2)}{r^2} - \mu^2\right)\phi(r) = 0.$
- This can be written as a Heun equation in a new radial coordinate

$$\left(\partial_z^2 + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} - \frac{\frac{1}{2} - a_0^2 - a_1^2 - a_t^2 + a_\infty^2 + u}{z(z-1)} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} + \frac{u}{z(z-t)} + \frac{\frac{1}{4} - a_0^2}{z^2}\right)\chi(z) = 0$$

where, $z = \frac{r^2}{r_H^2 + r^2 + 1}$ and $\phi(r) = \frac{\chi(z)}{\sqrt{r^3 f(r)\frac{dz}{dr}}}$

- **Near horizon** : $\chi_{\text{hor}}(z) = c_1\,(t-z)^{\frac{1}{2}+a_t} + c_2\,(t-z)^{\frac{1}{2}-a_t} + \ldots$
  The first term corresponds to the outgoing mode, while the second term corresponds to the ingoing mode near the horizon.
- near boundary : $\chi_{\text{bdry}}(z) = c_3\left(\frac{1-z}{1+r_H^2}\right)^{\frac{1}{2}+a_1} + c_4\left(\frac{1-z}{1+r_H^2}\right)^{\frac{1}{2}-a_1}$
  Where the first term corresponds to the normalizable mode and second term is non-normalizable.
- **Boundary conditions**:

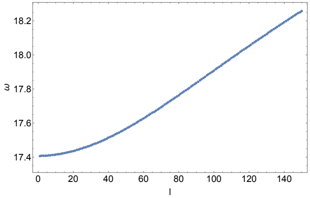
$$\chi_{\text{hor}}(z_0) = 0 \qquad \text{(Dirichlet boundary condition instead of ingoing)}$$

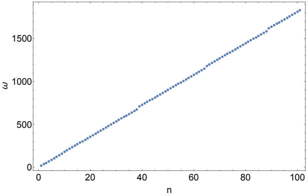
$$\chi_{\text{bdry}}(z) \sim (1-z)^{\frac{1}{2}+a_1} \qquad \text{(Normalizable at the boundary)}$$

- To find the normal modes, we need connection formulas that link the solutions around one singular point to another. These can be determined using techniques from Liouville CFT. [Zhiboedov, Dodelson, …]

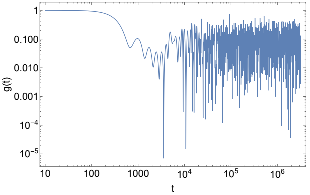
Normal modes and SFF

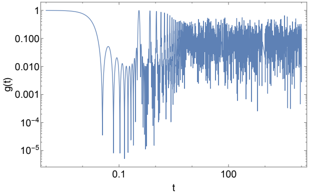
- Normal modes are labeled by two quantum number l and n.
- Spectrum along l and n ($r_H/L = 100$):





- SFF

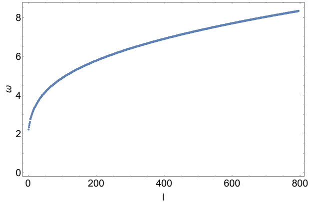


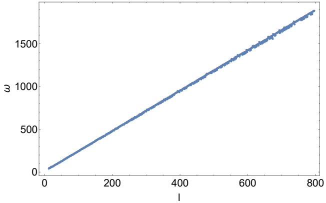


- We were not able to get sufficient number of modes for small r_H/L with this method but we can do the WKB analysis.

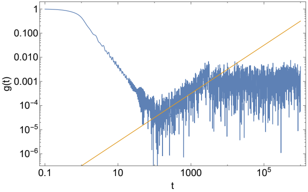
WKB analysis

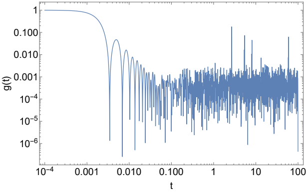
- spectrum along l and n





- SFF along l and n





- The SFF shows a clear dip-ramp-plateau with a ramp of slope$\sim$ 1 for the modes along the l direction. On the other hand, the SFF resembles that of a simple harmonic oscillator (SHO) for the spectrum along the n direction, as

Analytic Structure of the Green's function

- The Green's function has poles on the real axis, and as the brickwall moves closer to the horizon, these poles come increasingly closer.

- When the stretched horizon is very close to the event horizon, the poles are so dense that we can approximate the discrete poles as a branch cut, and the discontinuity around the branch cut is given by the following.

$$G(\omega + i\epsilon, m) - G(\omega - i\epsilon, m) \sim -\mathrm{Im} G_{\mathrm{bh}}(\omega, m)\Big|_{\omega=\omega_n}$$

- In this limit $\mathrm{Im}\, G(\omega, m) = -\mathrm{Im}\, G_{\mathrm{bh}}(\omega, m)$,
- Which implies that the two point function looks like a thermal two point function with the same Hawking temperature as the black hole.

Probe scalar in collapsing black hole

- We have considered a probe scalar field in a $2 + 1$ dimensional collapsing black hole. At any instant of time, the geometry outside the shell is BTZ, while inside it is global AdS.
- Due the smooth and consistent gluing, we can no longer impose a Dirichlet boundary condition for the probe scalar at the shell. Instead, we have to use the right junction condition.
- Nevertheless, we observe that the key features of pole accumulation in the correlator persist in this model as well.

Probe scalar in collapsing black hole

- metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\psi^2 \ , \tag{4}$$

where

$$f(r) = \begin{cases} f_1 = 1 + r^2, & \text{for } r < r_s \ . \\ f_2 = r^2 - r_H^2, & \text{for } r > r_s \ . \end{cases} \tag{5}$$

- Let ϕ_1 and ϕ_2 represent solutions inside and outside the shell, respectively. The matching conditions are:

$$\begin{aligned} \phi_1|_{r=r_s} &= \phi_2|_{r=rs} \ , \\ f_1(r)\partial_r\phi_1|_{r=r_s} &= f_2(r)\partial_r\phi_2|_{r=r_s} \ . \end{aligned} \tag{6}$$

Frame Title

We are interested in the Green's function of the system. Following Son-Starinets prescription and using the conditions (6), we can write the Green's function, which is the ratio of the normalizable mode to the non-normalizable mode, as:

$$G(\omega, m) = -\frac{\phi_1 f_2 \partial_r \phi_2^{(-)} - \phi_2^{(-)} f_1 \partial_r \phi_1}{\phi_1 f_2 \partial_r \phi_2^{(+)} - \phi_2^{(+)} f_1 \partial_r \phi_1} \,, \tag{7}$$

where everything is evaluated at $r = r_s$. Here:

- ϕ_1: solution of the Klein-Gordon (K-G) equation that is regular at the origin $r = 0$.
- $\phi_2^{(+)}$: normalizable part of the solution of the Klein-Gordon equation outside the shell which is regular at the conformal boundary, $r \to \infty$.
- $\phi_2^{(-)}$: non-normalizable part of the solution of the Klein-Gordon equation outside the shell which blows up at the boundary, $r \to \infty$.

Green's function

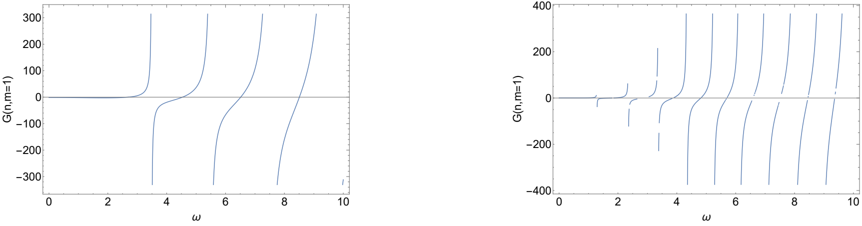


图 1: Plot of the Green's function for fixed $m = 1$. $r_s = 10$ for the left, whereas $r_s = 1.01$ for the right. This shows the poles are coming closer and closer as the shell starts moving from the boundary towards the horizon. Other parameters are $\mu = 1.1$ and $r_H = 1$.

Modes

- Case I ($r_s \to \infty$):
  Modes are given by:

$$\omega_n = \pm(1 + m + 2n + \sqrt{\mu^2 + 1}) \ .$$

- Case II ($r_s \to r_H$):
  Modes are those ω where the Denominator of the Green's function:

$$\phi_1 f_2 \partial_r \phi_2^{(+)} - \phi_2^{(+)} f_1 \partial_r \phi_1 \Big|_{r_s} = 0 \ .$$

  In the limit $r_s \to r_{\mathrm{H}}, f_2 \to 0$ which yields:

$$\phi_2^{(+)} \partial_r \phi_1 \Big|_{r_s} = 0 \ .$$

  Which implies two in-equivalent boundary conditions:

  1. Dirichlet: $\phi_2^{(+)} = 0$ (same as Brickwall boundary condition.)
  2. Neumann: $\partial_r \phi_1 = 0$. (Modes are linear along the both quantum numbers)

Some future directions

- We have observed that for large r_H/L, the spectrum loses its logarithmic dependence on the angular momentum quantum number (though quasi-degeneracy remains). As a result, the SFF does not exhibit a linear ramp structure. However, for the high-lying part of the spectrum, it remains logarithmic. It would be useful to have an analytic expression for the modes when $r_H/L \gg 1$. It seems that the part of the spectrum responsible for the linear ramp changes with r_H/L. Gaining a better understanding of this fact would be valuable.
- For the BTZ example, it can be shown that the scrambling time can be extracted from the slope of the ω vs. n plot. Is it possible to extract this time for higher-dimensional black holes as well?
- One of the most important questions is what causes the quasi-degeneracy along the angular quantum number direction. Perhaps the probability computed from the WKB wavefunction can provide some insight into this (currently under investigation).

Thank You.