MULTI-INVARIANTS AND BULK REPLICA SYMMETRY

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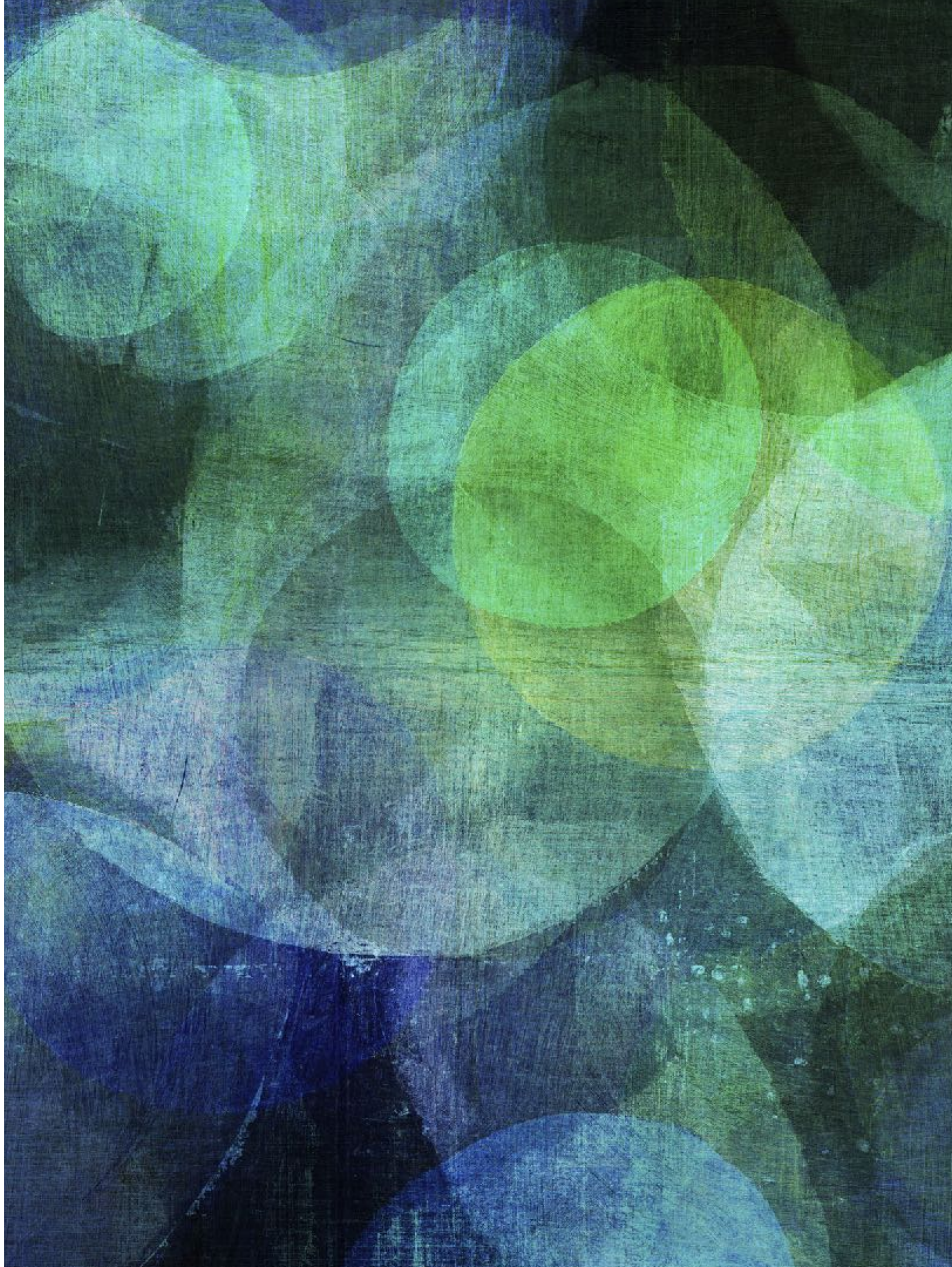
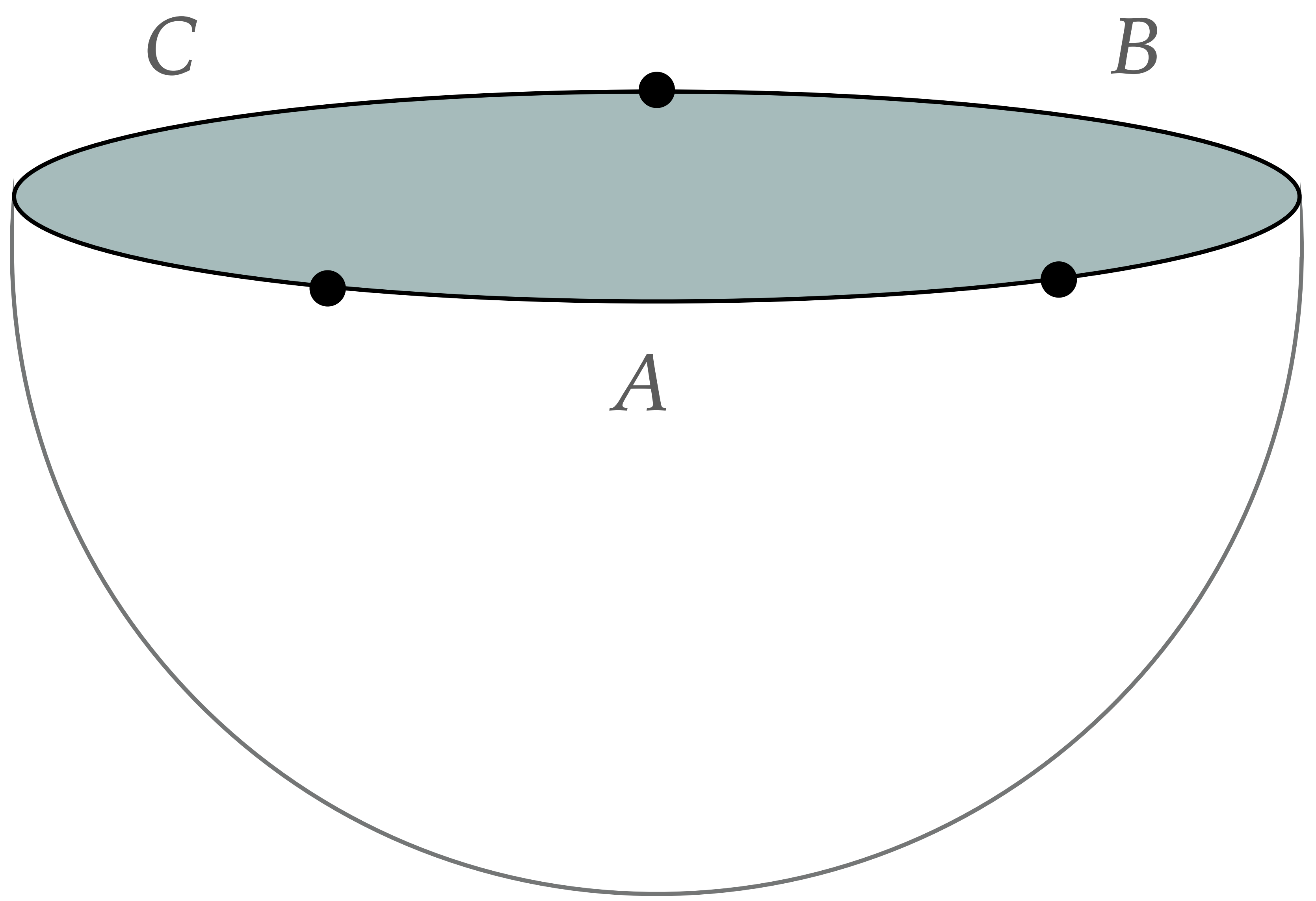


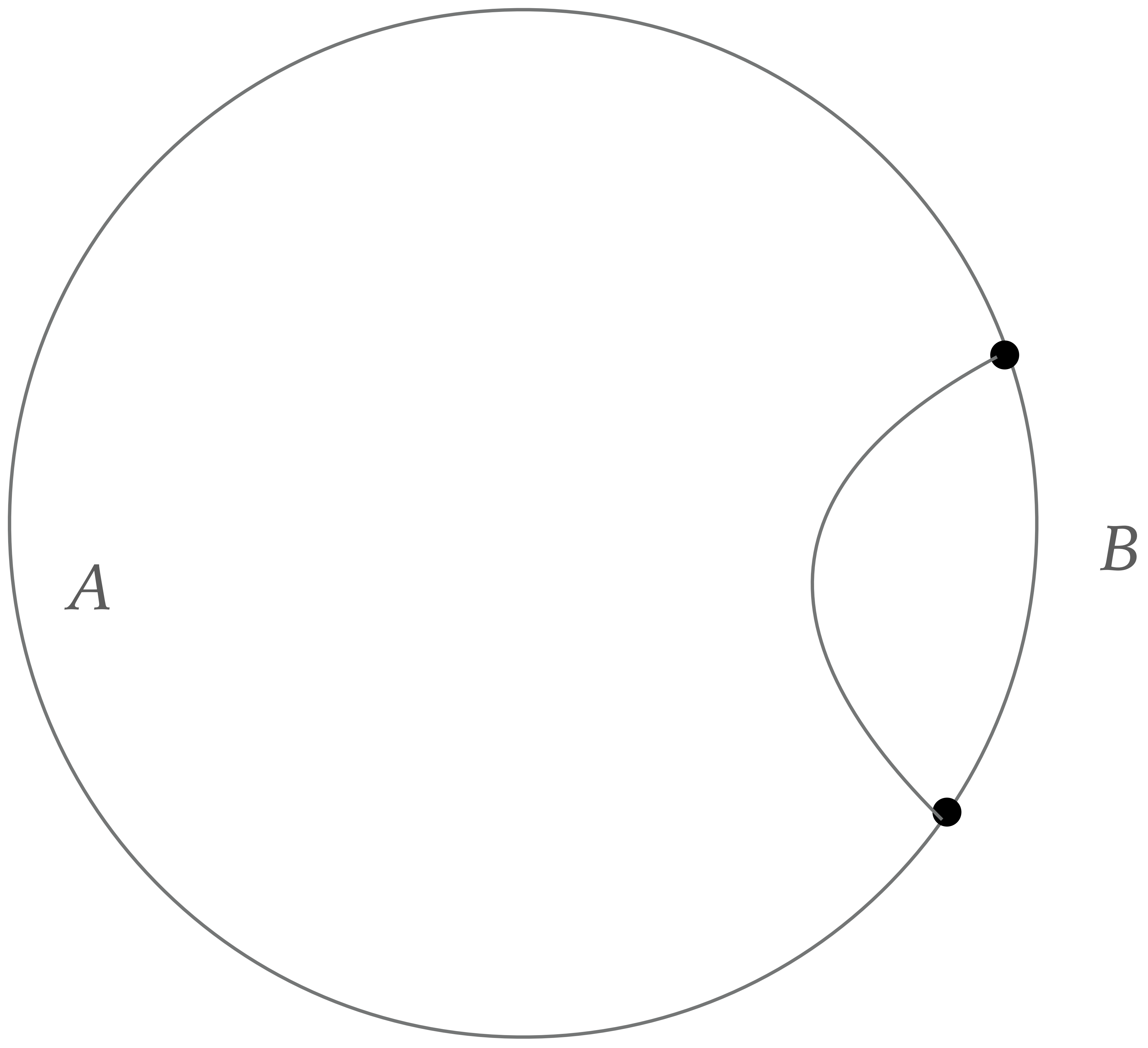
A HOLOGRAPHIC STATE

- Consider a CFT with a large central charge and large gap in the spectrum
- Conjecturally, it is dual to a holographic gravitational dual
- By holographic state, we mean a state in such a CFT that is described by a bulk geometry.
- Vacuum state in 2D CFT (canonical example, can be generalised to higher dimensions and other states such as thermal state etc.)
- We divide boundary into multiple regions A, B, C
- The Hilbert space factorizes as $H_A \otimes H_B \otimes H_C$
- This is the holographic multi-partite state considered up to local unitary transformations



A HOLOGRAPHIC STATE

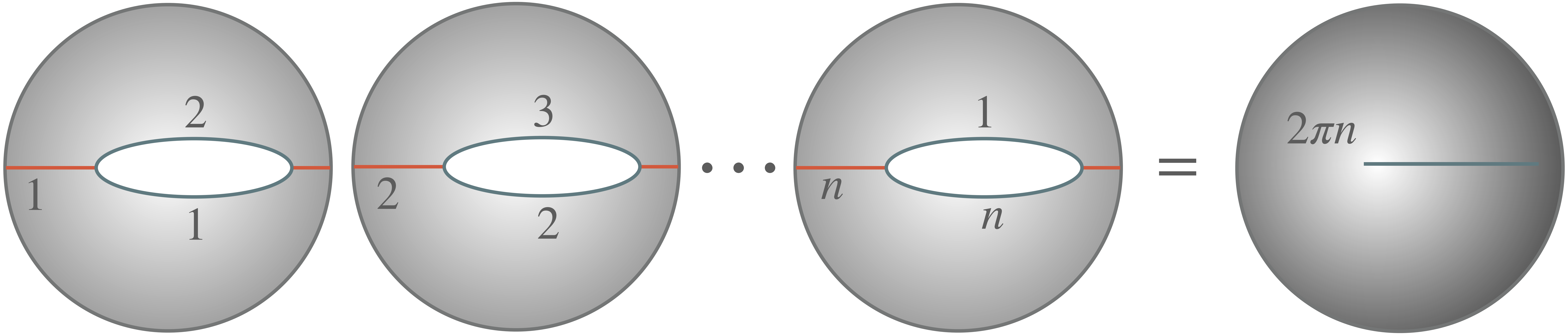
- Question: Does this multi-partite state have any special properties compared to a generic multi-partite quantum state? i.e. what makes the holographic state, holographic?
- To explore this question, we need to first figure out the appropriate multi-partite invariants that admit a convenient i.e. geometric description for the holographic state.
- Example: $\mathrm{Tr}\rho^n$ for bi-partite state. It is evaluated by the action of an orbifold geometry with the orbifold singularity beginning and ending on the end-points of the region of interest



A QUICK REVIEW OF RENYI ENTROPY

- Let us review the construction $\mathrm{Tr}\rho^n$ as the orbifold singularity

$$\mathrm{Tr}\rho^n = \sum \psi_{i_1 j_1}\ldots\psi_{i_{n-1}j_{n-1}}\psi_{i_n j_n}\bar{\psi}^{i_1 j_2}\ldots\bar{\psi}^{i_{n-1}j_n}\bar{\psi}^{i_n j_1}$$



- Sphere with two points of cone angle $2\pi n$. This is Weyl equivalent to a smooth sphere.

$\mathbb{Z}_n$ REPLICA SYMMETRY

- The boundary manifold $\mathscr{M}$ is a sphere with $\mathbb{Z}_n$ replica symmetry action. According to AdS/CFT, the partition function on this manifold is computed by a gravitational solution $\mathscr{B}$ that "fills in" this sphere.

$$Z_{\mathscr{M}} = e^{-S_{\text{grav}}(\mathscr{B})}$$

- The $\mathbb{Z}_n$ replica symmetry of the boundary extends into the bulk i.e. acts on $\mathscr{B}$. We quotient $\mathscr{B}$ by this action. This action has two fixed points on the boundary. After quotienting, $\mathscr{B}$ turns into the the orbifold $\tilde{\mathscr{B}}$ which has a conical singularity emanating at a boundary point and ending at another. As a result,

$$\mathscr{E} \equiv (\text{Tr}\rho^{\text{n}})^{1/n} = e^{-S_{\text{grav}}(\tilde{\mathscr{B}})}$$

KEY PROPERTIES

- We are interested in characterizing multi-partite entanglement invariants that admit a similar sort of description viz. it is computed by an orbifold geometry whose boundary is the "original" sphere.
- There are two key conditions that must hold for this:
  1. The replica symmetry R must act freely and transitively on all the replicas. This will ensure that after quotienting by R, the boundary $\mathscr{M}$ will become S^2.
  2. The action of replica symmetry on $\mathscr{M}$ must extend to the bulk solution $\mathscr{B}$ that fills it in. This ensure that the bulk solution $\mathscr{B}$ can be quotiented by R to produce the orbifold $\tilde{\mathscr{B}}$
- It turns out that the first condition can be satisfied for any replica symmetry R. It is the second condition that is non-trivial to impose. When it is satisfied, we say that the replica symmetry is preserved in the bulk. When it is not, we say that the bulk replica symmetry is broken.

KEY PROPERTIES

- The question of whether the bulk replica symmetry is preserved or broken not only depends on the multi-partite invariant but also on the configuration of the boundary regions. Two types of questions can be asked.

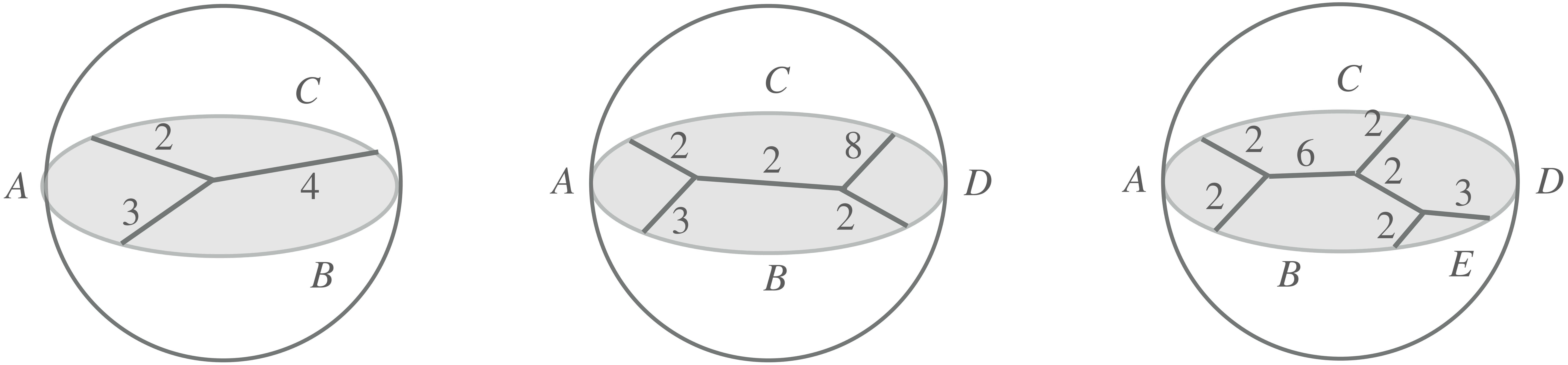
  A. Given a configuration of boundary regions, which multi-partite invariants preserve bulk replica symmetry?

  B. Which multi-partite invariants preserve bulk replica symmetry for any configuration of boundary regions?

- We answer both these questions.

HOW DOES THE ORBIFOLD LOOK LIKE?

- In general the replicated boundary $\mathscr{M}$ is a genus g surface and the bulk solution $\mathscr{B}$ is a handlebody that fills it in. When the $\mathscr{B}$ preserves replica symmetry, the orbifold $\tilde{\mathscr{B}}$ is a topologically a ball with a tri-valent network of singularities. Each edge of the network is labeled by the order of the element that it is fixed under. Some examples of the orbifolds is given below.



BI-PARTITE VS MULTI-PARTITE

- The density matrix is defined with respect to a bi-partition of parties. Therefore it is a bi-partite object even if it is defined for multi-partite states.
- Local unitary invariants computed from a density matrix such as entanglement entropy and Renyi entropy are therefore bi-partite
- The key to construct genuine multi-partite invariants is to use the full *tensorial* structure of the state.

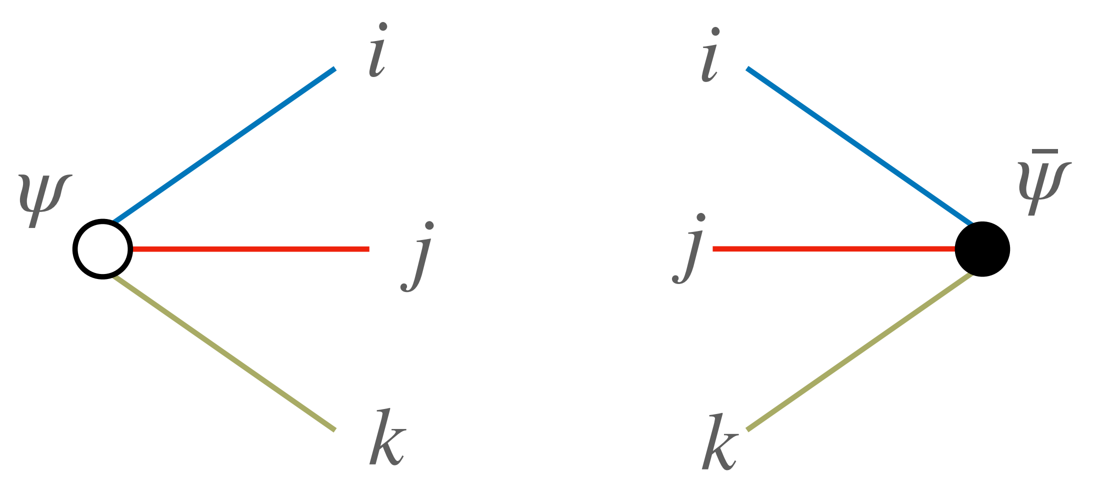
$$I_1 \equiv \psi_{i_1 j_1 k_1} \bar{\psi}^{i_2 j_1 k_1} \psi_{i_2 j_2 k_2} \bar{\psi}^{i_3 j_2 k_2} \psi_{i_3 j_3 k_3} \bar{\psi}^{i_1 j_3 k_3} = \mathrm{Tr} \rho_A^3$$

$$I_2 \equiv \psi_{i_1 j_1 k_1} \bar{\psi}^{i_1 j_2 k_3} \psi_{i_2 j_2 k_2} \bar{\psi}^{i_2 j_3 k_1} \psi_{i_3 j_3 k_3} \bar{\psi}^{i_3 j_1 k_2}$$

- Both I_1 and I_2 are constructed out of 3 copies of $\psi's$ and $\bar{\psi}'s$. But I_1 can be expressed in terms of a matrix while I_2 can not. We get different values of I_2 for the previous example of isopectral states.

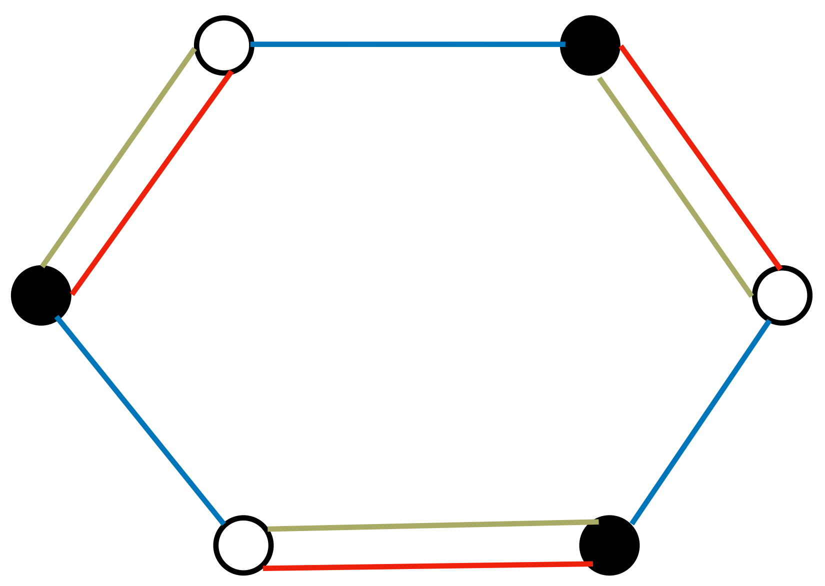
MULTI-PARTITE MEASURES

- Index contractions can be cumbersome. It is convenient to have a graphical language for the same.
- The wave function ψ is denoted with a white dot and $\bar{\psi}$ with a black do. Assign each party a colour and indices of of a given party are represented as edges of appropriate colour.

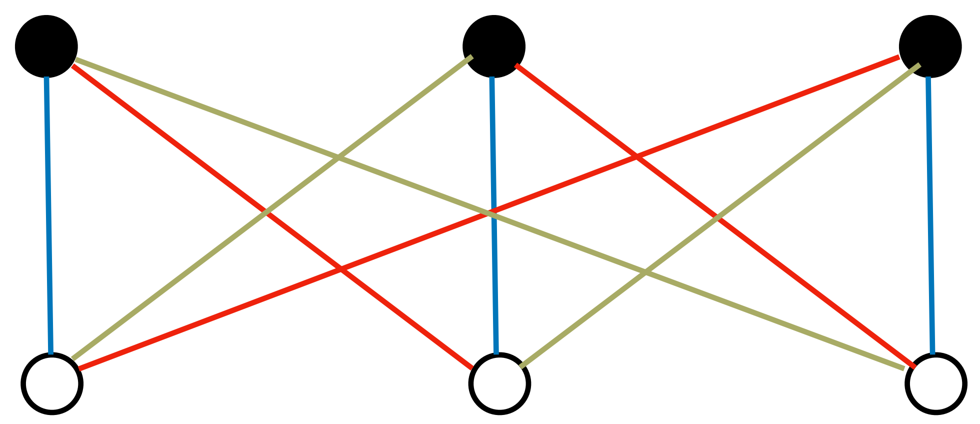


$$I_1 \equiv \psi_{i_1 j_1 k_1} \bar{\psi}^{i_2 j_1 k_1} \psi_{i_2 j_2 k_2} \bar{\psi}^{i_3 j_2 k_2} \psi_{i_3 j_3 k_3} \bar{\psi}^{i_1 j_3 k_3} = \mathrm{Tr} \rho_A^3$$

$$I_2 \equiv \psi_{i_1 j_1 k_1} \bar{\psi}^{i_1 j_2 k_3} \psi_{i_2 j_2 k_2} \bar{\psi}^{i_2 j_3 k_1} \psi_{i_3 j_3 k_3} \bar{\psi}^{i_3 j_1 k_2}$$

$I_1 =$ 

- With this notation, I_1 is a necklace. That is why it can be interpreted as a trace of a power of a matrix. However, I_2 is not a necklace and can't be constructed using matrices.

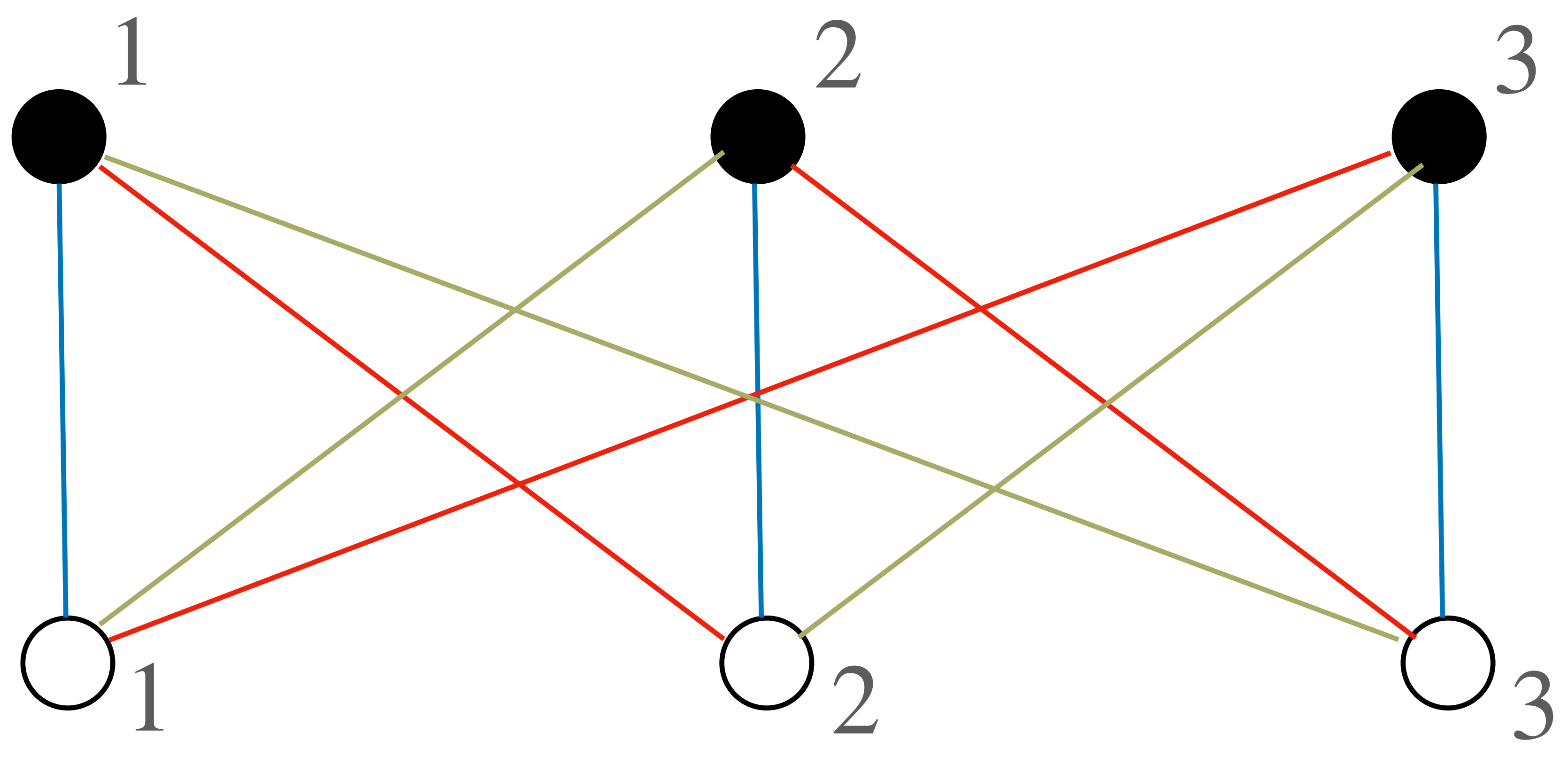
$I_2 =$ 

- Any graph that is not a necklace is a multi-partite invariant.

DESCRIPTION IN TERMS OF PERMUTATIONS

- Any multi-partite measure can equivalently be described in terms of permutation tuple.
- Index each replica with a number $1, \ldots, n$. The index contraction is described by assigning a permutation element $\sigma_a \in S_n$ to a party a. The party a index of i^{th} bra is contracted with that of the $(\sigma_a \cdot i)^{\text{th}}$ ket.

$\mathcal{E}_2 =$



$$\sigma_{\text{blue}} = (1)(2)(3)$$

$$\sigma_{\text{green}} = (123)$$

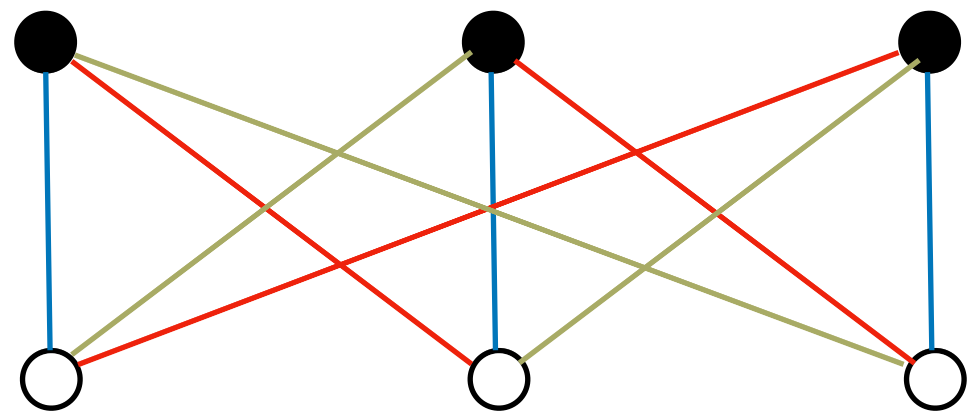
$$\sigma_{\text{blue}} = (321)$$

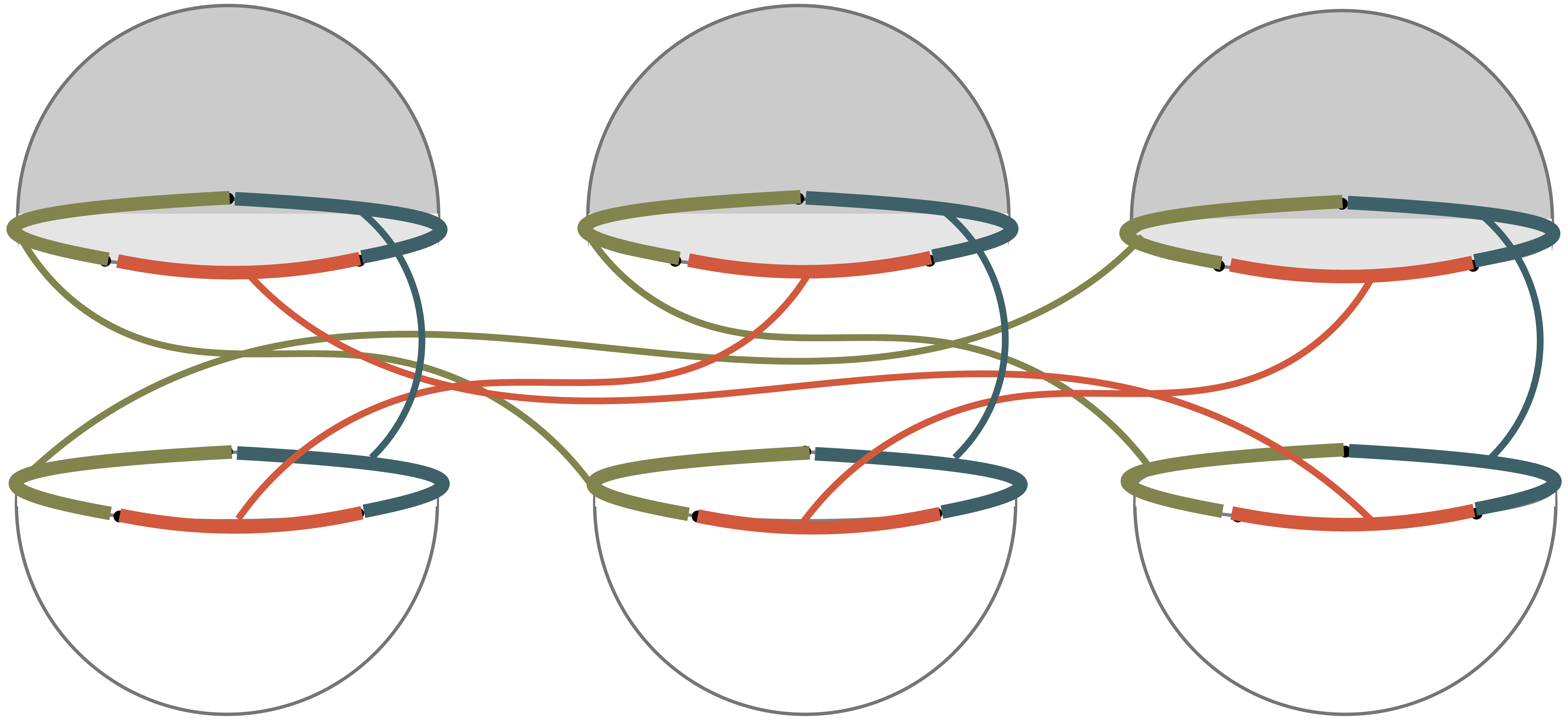
- This labelling by permutation tuple had redundancy $(\sigma_1, \sigma_2, \sigma_3) \sim (\sigma_1 \cdot h, \sigma_2 \cdot h, \sigma_3 \cdot h)$. This corresponds to relabelling of kets once the bra labelling is fixed.

MULTIPARTITE MEASURES IN QFT

- Let us see how to formulate any multi-partite measure in quantum field theory as a path integral. Let us take the example of $\mathcal{E}_2$.

$\mathcal{E}_2 =$

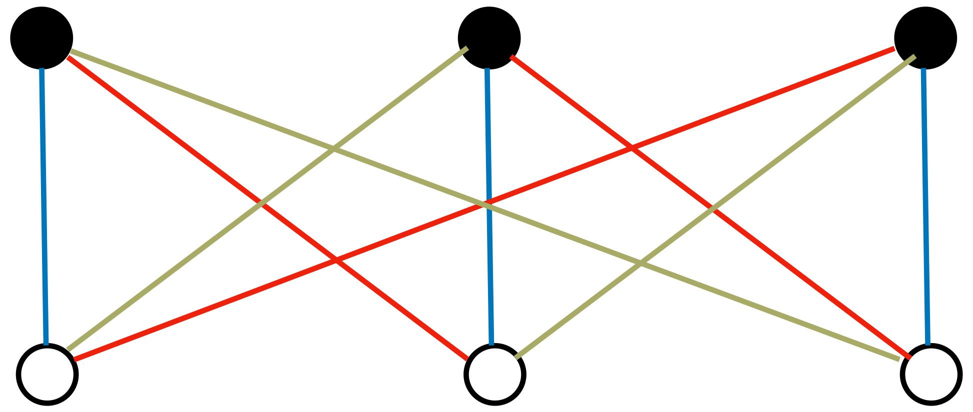


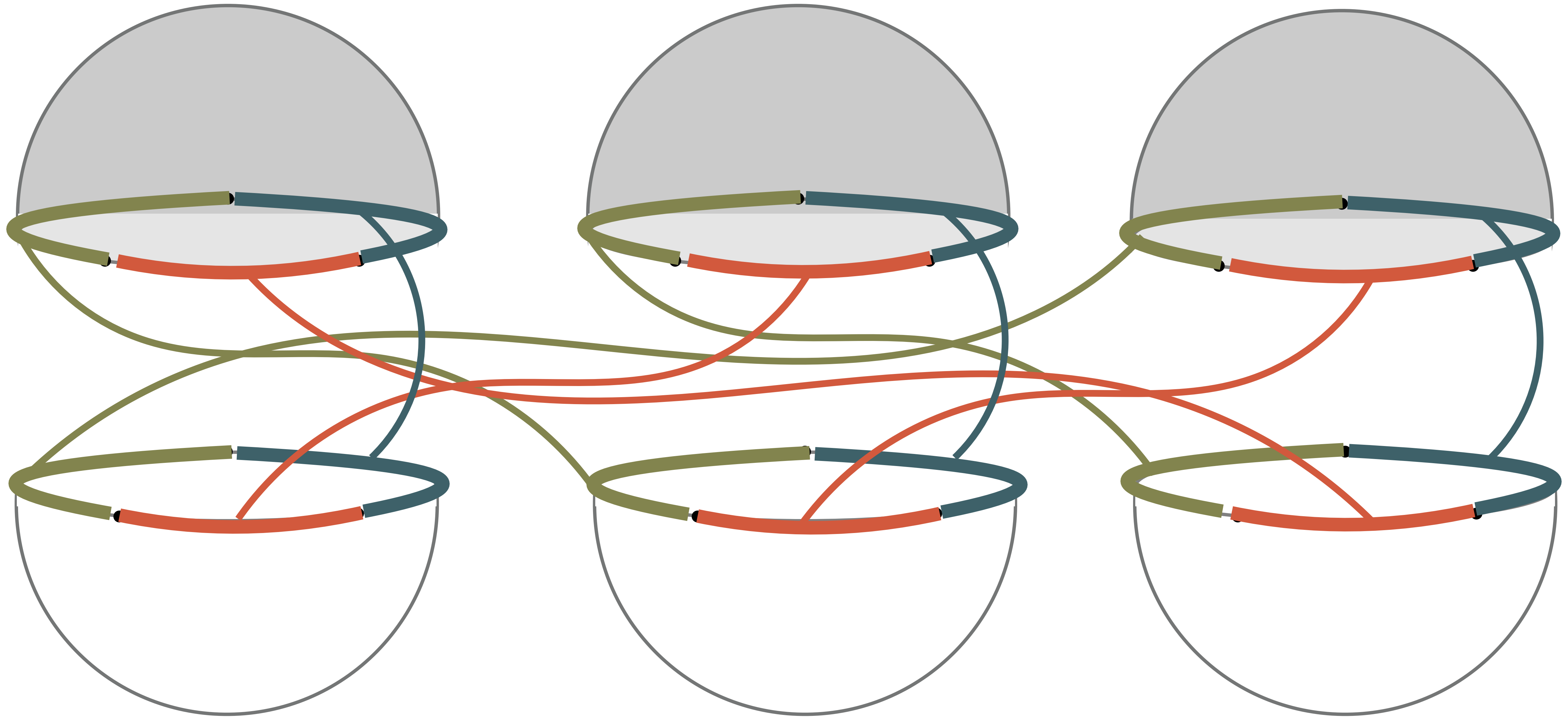


- $\mathcal{E}_2$ is computed by path integral over the glued surface, appropriately normalized.
- Genus of the resulting surface can be computed using Riemann-Hurwitz formula. In this case, the genus of the glued surface is 1. It is tessellated by “bra hemispheres” and “ket hemispheres”. The hemispheres appear as n-gon, each side corresponding to a region.

MULTIPARTITE MEASURES IN HOLOGRAPHY

- Computing path integral over glued surface is a difficult task and depends on the details of the theory but if the theory is holographic a universality emerges due to the geometric dual description

$\mathscr{E}_2 =$ 

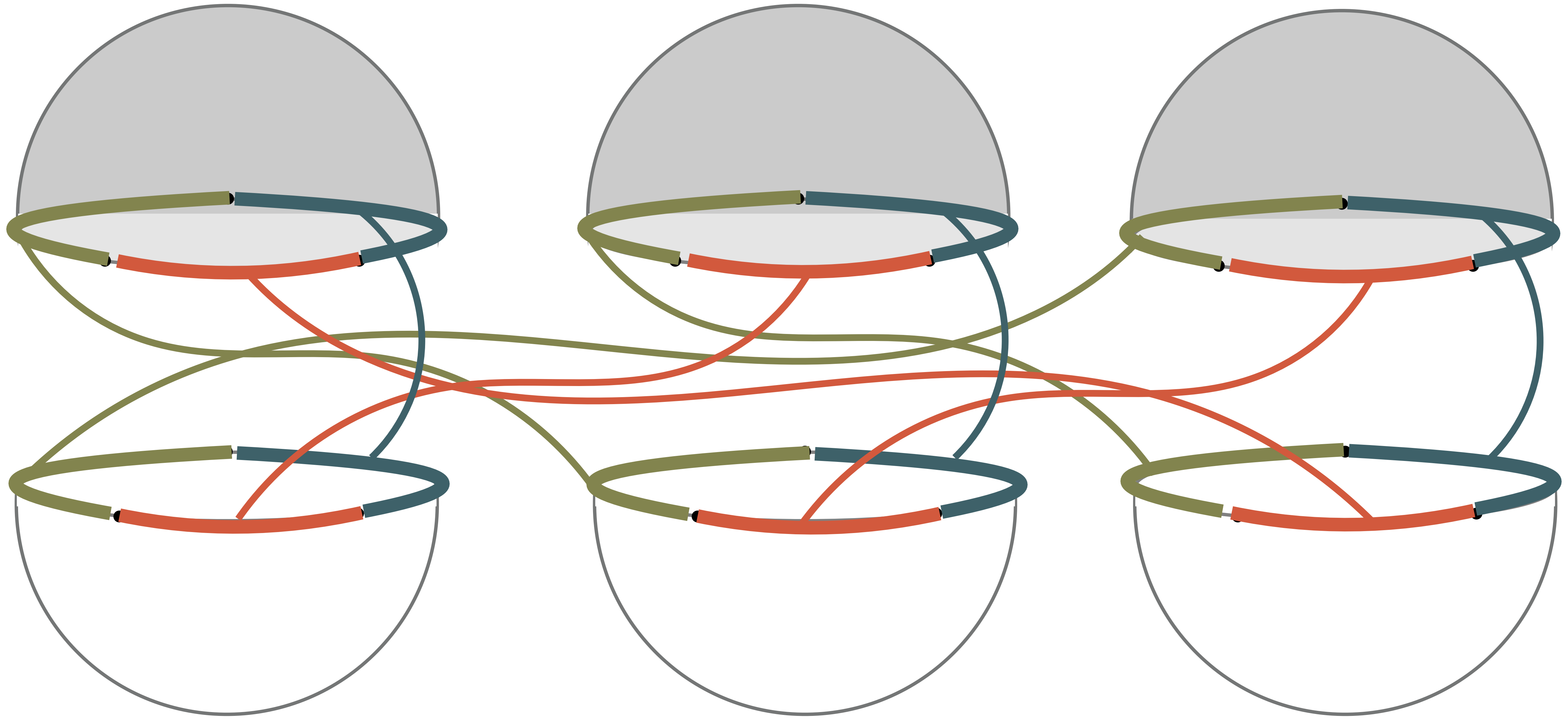


- The partition function is $e^{-S_{\mathscr{B}}}$ where $S_{\mathscr{B}}$ is the action of the dominant bulk solution $\mathscr{B}$ such that $\partial\mathscr{B} = \mathscr{M}$

- If the measure $\mathscr{E}$ has a symmetry $\mathscr{G}$ that acts freely and transitively on the replicas then $\mathscr{M}/\mathscr{G}$ is the original manifold which is S^2.

REPLICA SYMMETRY

- If the replica symmetry is also enjoyed by the bulk solution $\mathscr{B}$ then we can quotient it by $\mathscr{G}$ then because $\mathscr{M}/\mathscr{G} = S^2$, we get a hyperbolic geometry with S^2 boundary and possible conical singularities in the interior



- The conical singularities appear because, the action of $\mathscr{G}$ on $\mathscr{B}$ may have fixed points.

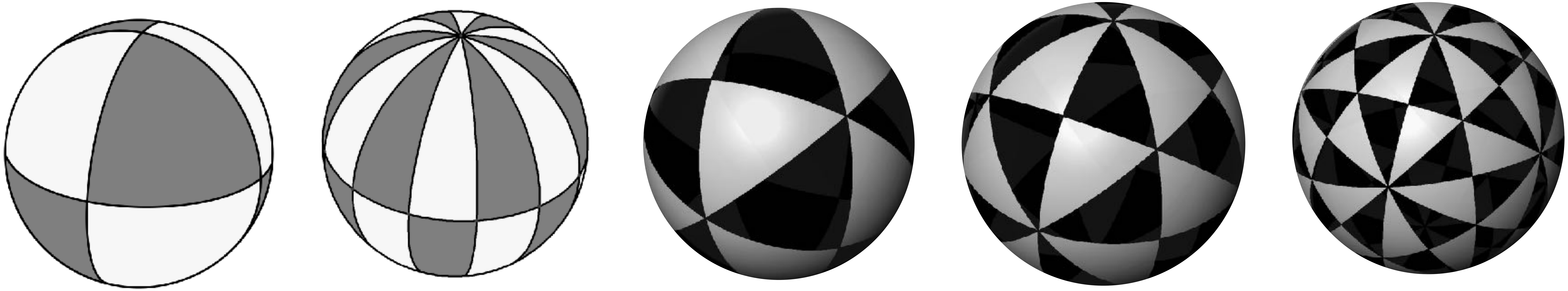
- This motivates the construction of multipartite measures with a replica symmetry that acts freely and transitively on replicas. Of course, this does not guarantee that $\mathscr{B}$ is replica symmetric. e.g. the bulk solution for measure $\mathscr{E}_2$ does does not preserve replica symmetry.

REPLICA SYMMETRY

- The problem of constructing measures with desired replica symmetry can be readily solved. The non-trivial problem is to analyse whether the replica symmetry is preserved by the bulk $\mathscr{B}$.
- Let us first look at the problem of having a replica symmetry action that is free and transitive. Take a finite group $\mathcal{G}$ and take $\mathcal{G}$ worth of replicas. Index each replica by a group element. Assign each party a a generator g_a of the group.
- We specify that the party a index of h^{th} bra is contracted with that of the $(g_a \cdot h)^{\text{th}}$ ket. The difference with the general case is that $h, g_a \in \mathcal{G}$ and $\cdot$ stands for right-multiplication in $\mathcal{G}$
- The advantage of this is that the action of the group on replicas by left-multiplication commutes with the right-action and gives rise the the replica symmetry. In this way, a measure with any given replica symmetry can be constructed. We further need to specify a generator of $\mathcal{G}$ for each party.

THE CASE OF GENUS 0 $\mathscr{M}$

- Finite groups acting on the spheres are finite subgroups of $SO(3)$
- These are symmetry groups of regular polytopes.

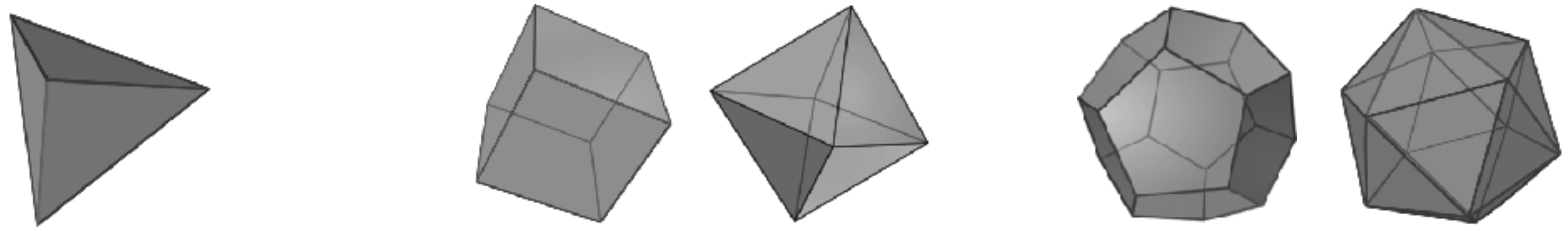


(2,2,2) (2,2,n) (2,3,3) (2,3,4) (2,3,5)

Wikipedia:

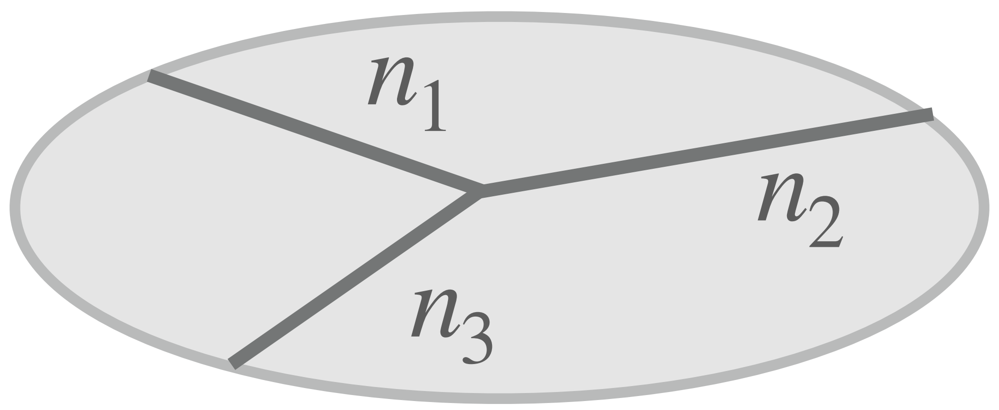
Triangle groups,

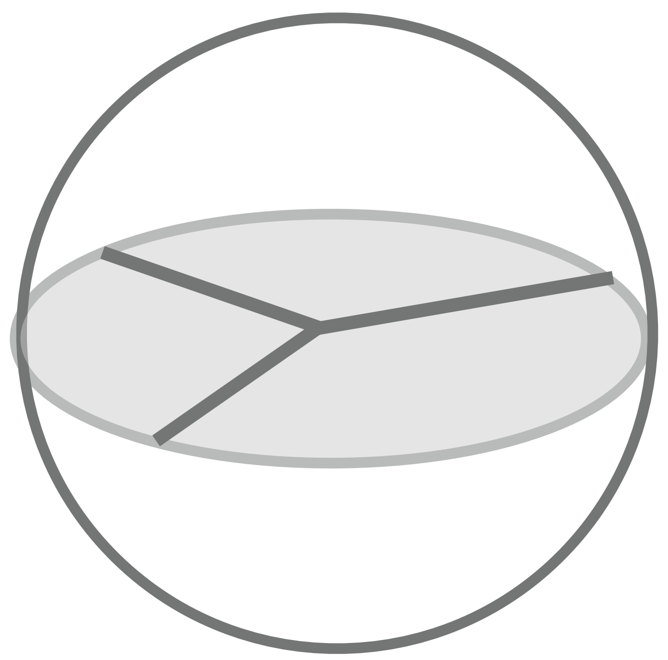
Regular polytopes



QUOTIENTING THE BULK

- Action of each of these symmetry groups on S^2 extends into the hyperbolic ball.
- After quotienting, we get a geometry that is topologically a ball with a tri-valent vertex of singularities as below.



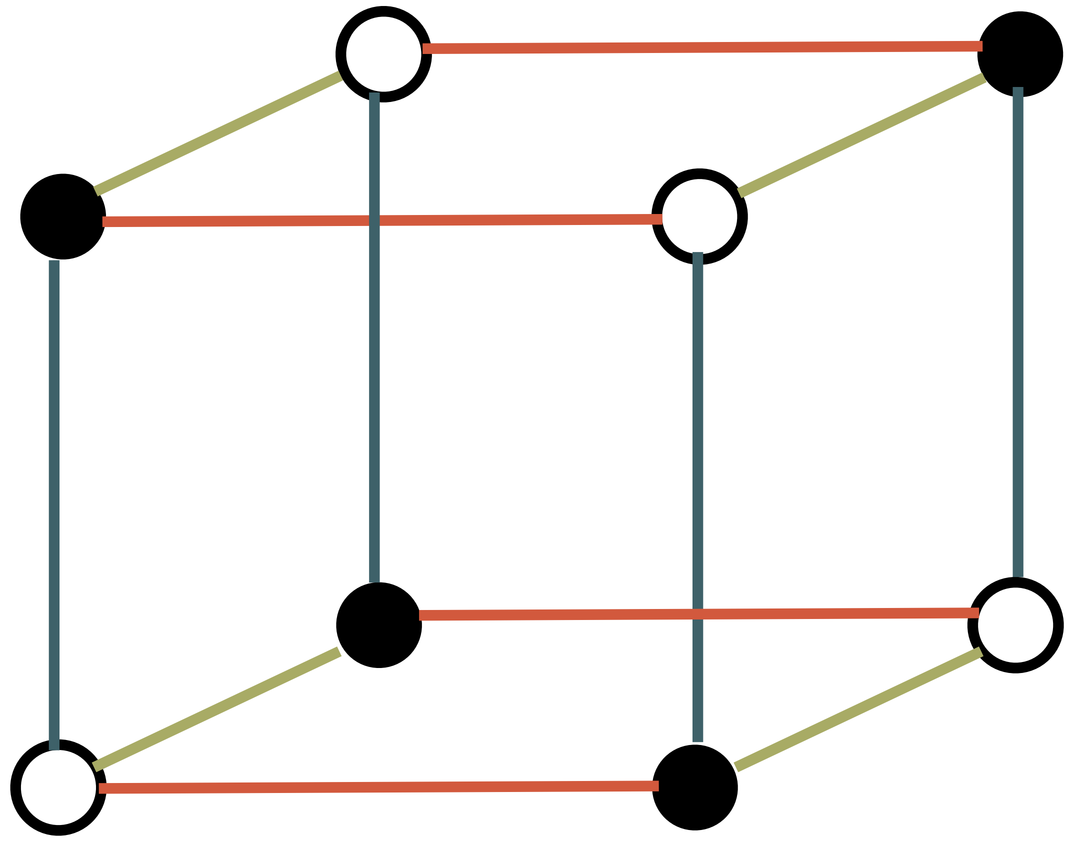


- Each edge with label n represents a singularity with cone angle $\dfrac{2\pi}{n}$
- The trivalent vertices (n_1, n_2, n_3) are

(2,2,n) (2,3,3) (2,3,4) (2,3,5)

REPLICA PRESERVING MEASURES: 3-PARTY EXAMPLE

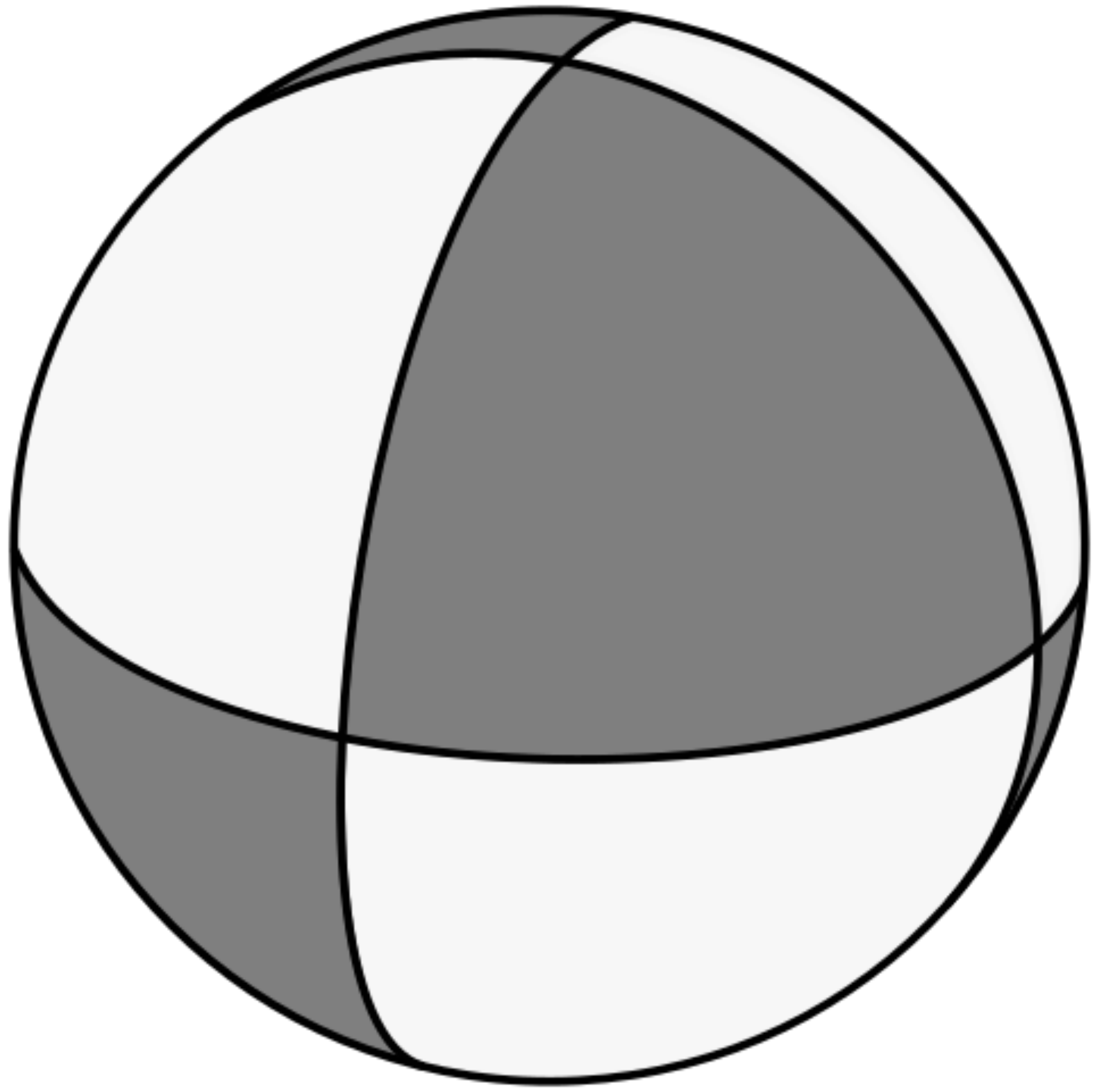
- Starting with the symmetry (2,2,2), we can reverse engineer the multi-invariant.

$\mathcal{E}_3 =$ 

3d - cube

Genus 0 surface

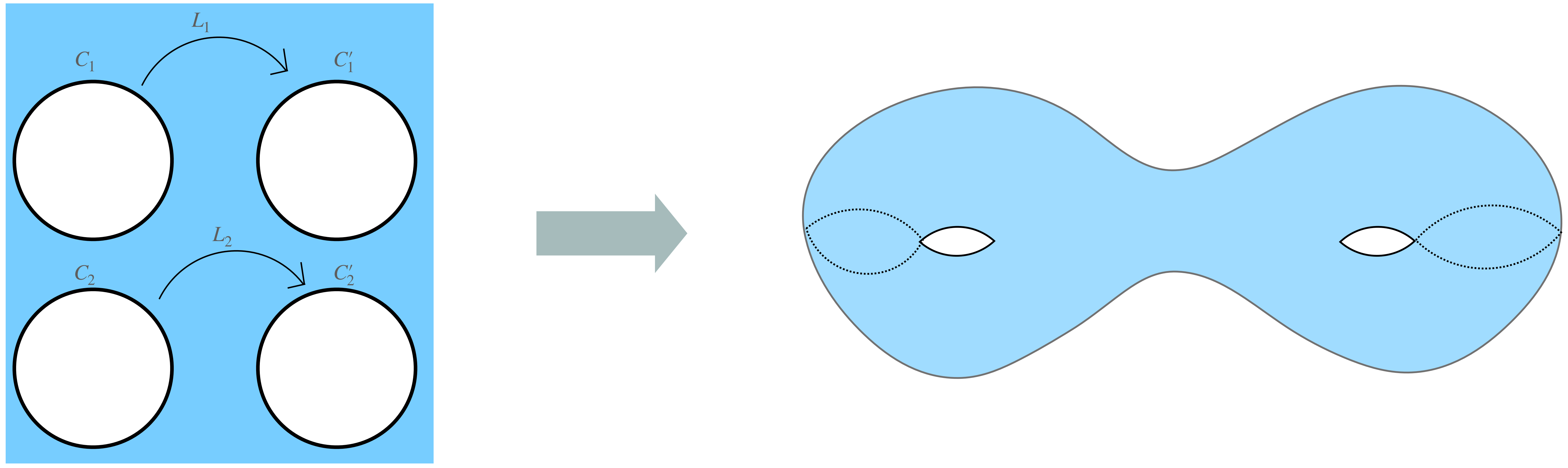
Replica symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$



WHAT ABOUT HIGHER GENUS $\mathscr{M}$?

- In general, a multi-invariant gives rise to the replicated manifold $\mathscr{M}$ of higher genus. It is filled in by a handlebody $\mathscr{B}$. We are interested in characterizing the handlebodies that have the replica symmetry R as their symmetry.

- To characterize symmetric handlebodies, it is useful to think of them as quotients of hypebolic space by a free subgroup of isometry group $SO(3,1)$. This is the so-called Schottkey construction of handlebodies.



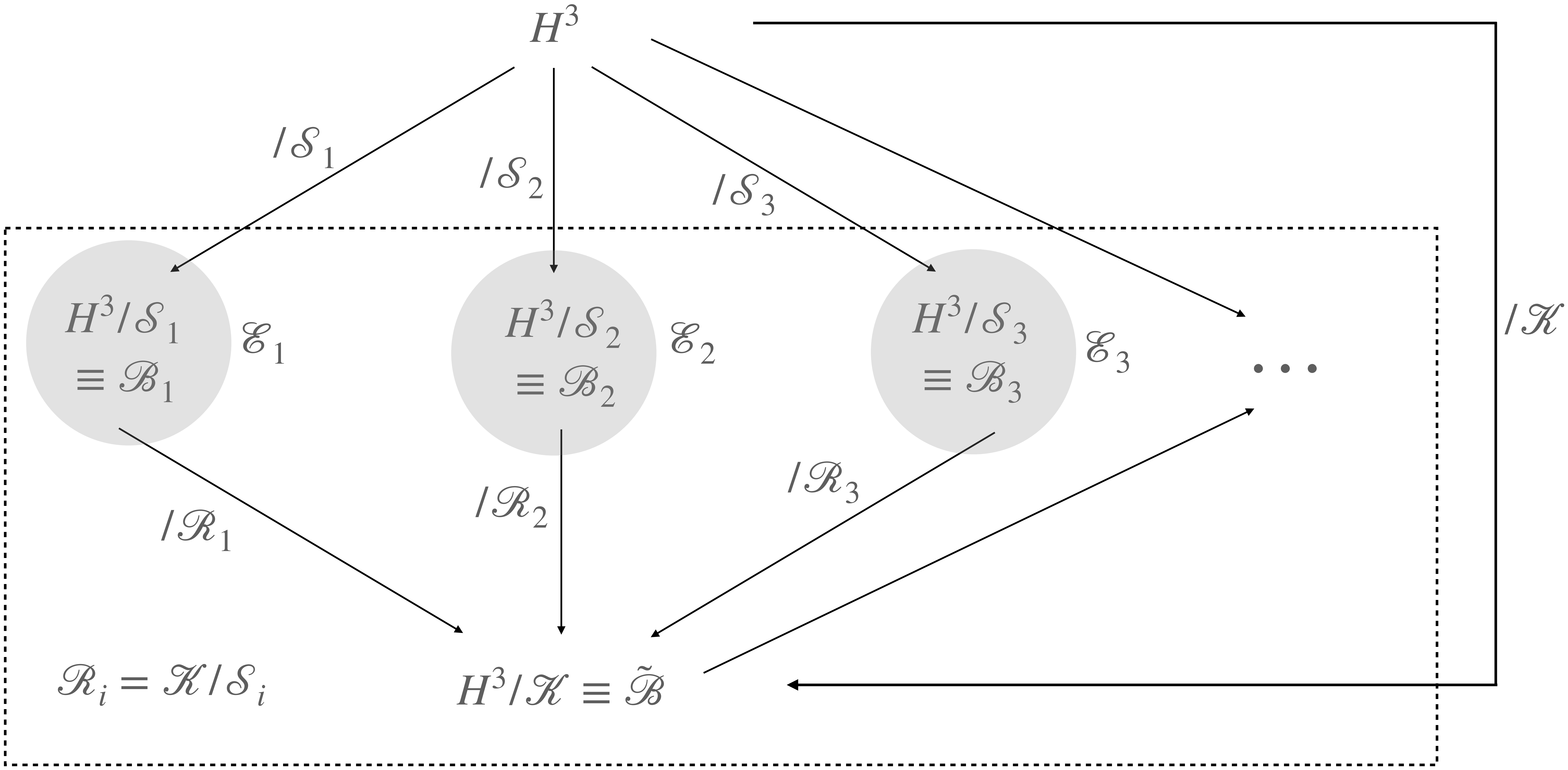
- A free group with g generators is called a Schottkey group S_g

- H^3/S_g is a genus g handlebody.

SYMMETRIC HANDLEBODIES

- Consider a discrete subgroup $\mathscr{K}$ of the bulk isometry group $SO(3,1)$. Let it have a free normal subgroup $\mathscr{S}_g$ of finite index i.e. such that the group $\mathscr{K}/\mathscr{S}_g \equiv \mathscr{R}$ is finite.
- Consider the quotient $H^3/\mathscr{K}$ in two steps. First quotient by $H^3/\mathscr{S}_g$. Because $\mathscr{S}_g$ is free, it is a Schottkey group and hence the quotient $H^3/\mathscr{S}_g$ is a genus g handlebody $\mathscr{B}$.
- By construction, $\mathscr{B}$ enjoys the action of "remaining" symmetry $\mathscr{R}$. This is exactly what we were looking for. The quotient $\tilde{\mathscr{B}} = \mathscr{B}/\mathscr{R} = H^3/\mathscr{K}$ is the orbifold dual that we are after!
- This construction has an interesting upshot. There are multiple multi-invariants that have the same dual description. This tells us that the holographic state is a very special type of multi-partite state. It satisfies *equalities* that a general multi-partite state doesn't!

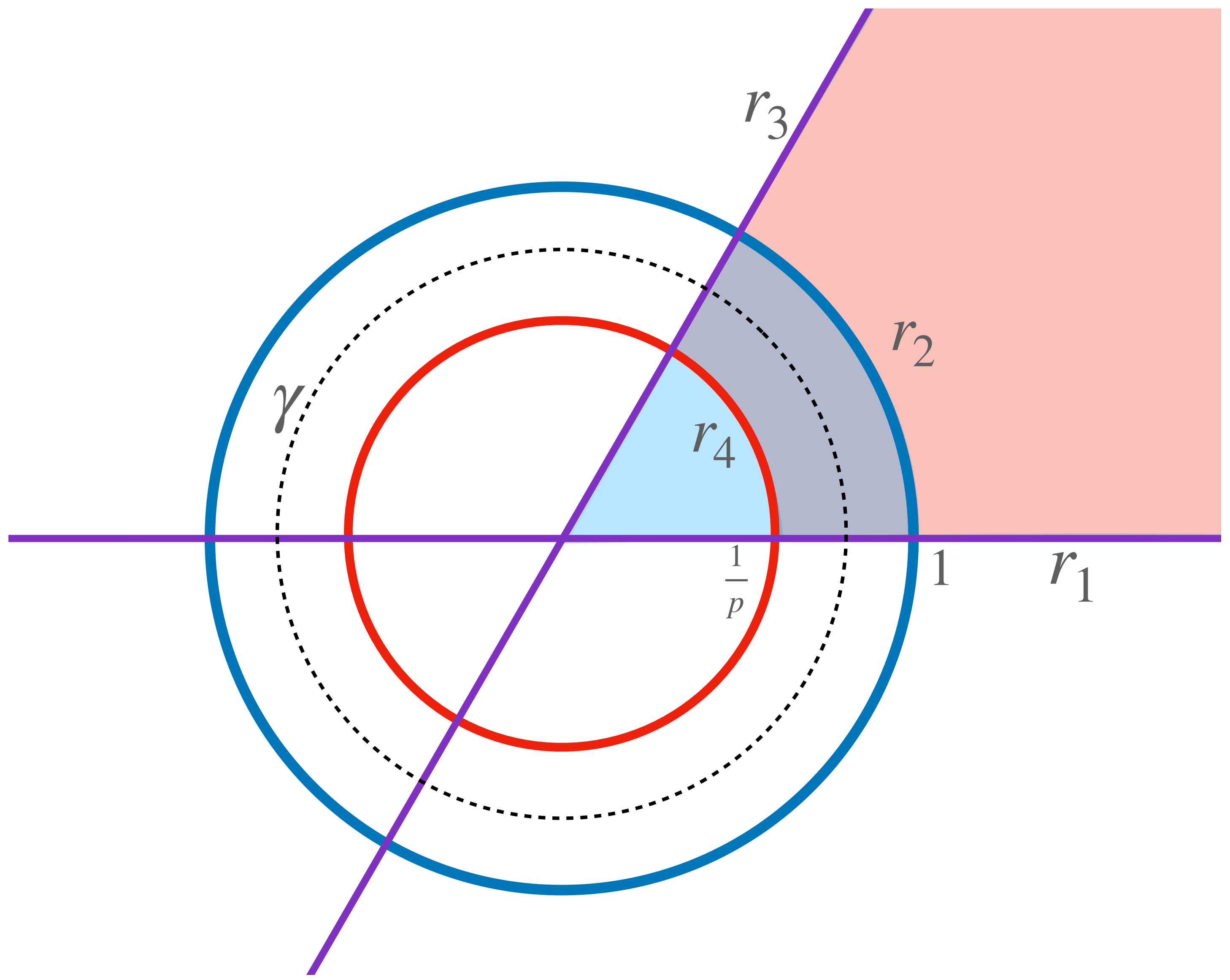
UPSHOT

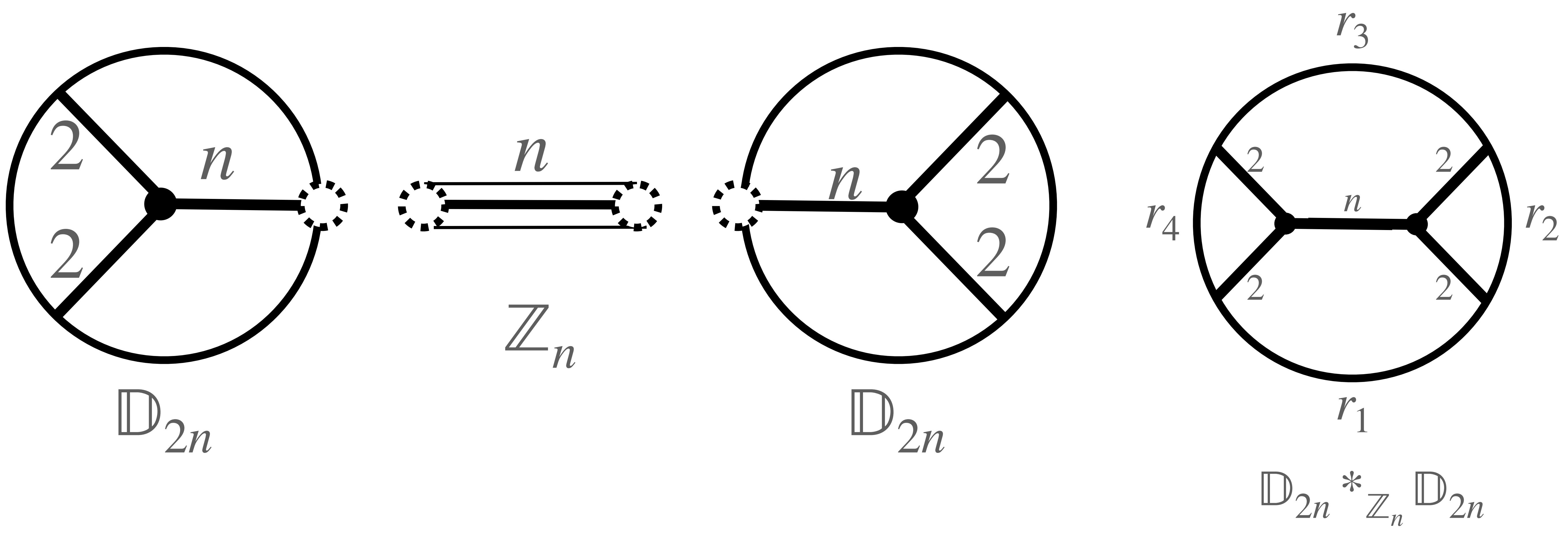


H^3
$/\mathscr{S}_1$
$/\mathscr{S}_2$
$/\mathscr{S}_3$
$/\mathscr{K}$
$H^3/\mathscr{S}_1 \equiv \mathscr{B}_1$
$\mathscr{E}_1$
$H^3/\mathscr{S}_2 \equiv \mathscr{B}_2$
$\mathscr{E}_2$
$H^3/\mathscr{S}_3 \equiv \mathscr{B}_3$
$\mathscr{E}_3$
$\cdots$
$/\mathscr{R}_1$
$/\mathscr{R}_2$
$/\mathscr{R}_3$
$\mathscr{R}_i = \mathscr{K}/\mathscr{S}_i$
$H^3/\mathscr{K} \equiv \tilde{\mathscr{B}}$

VIRTUALLY FREE KLEINIAN GROUPS

- The discussion establishes the importance of discrete subgroups of $SO(3,1)$ that have free normal subgroup of finite index. Such groups are called Virtually-free Kleinian groups
- They are constructed using a beautiful mathematical tool called Klein-Maskit recombination theorem.

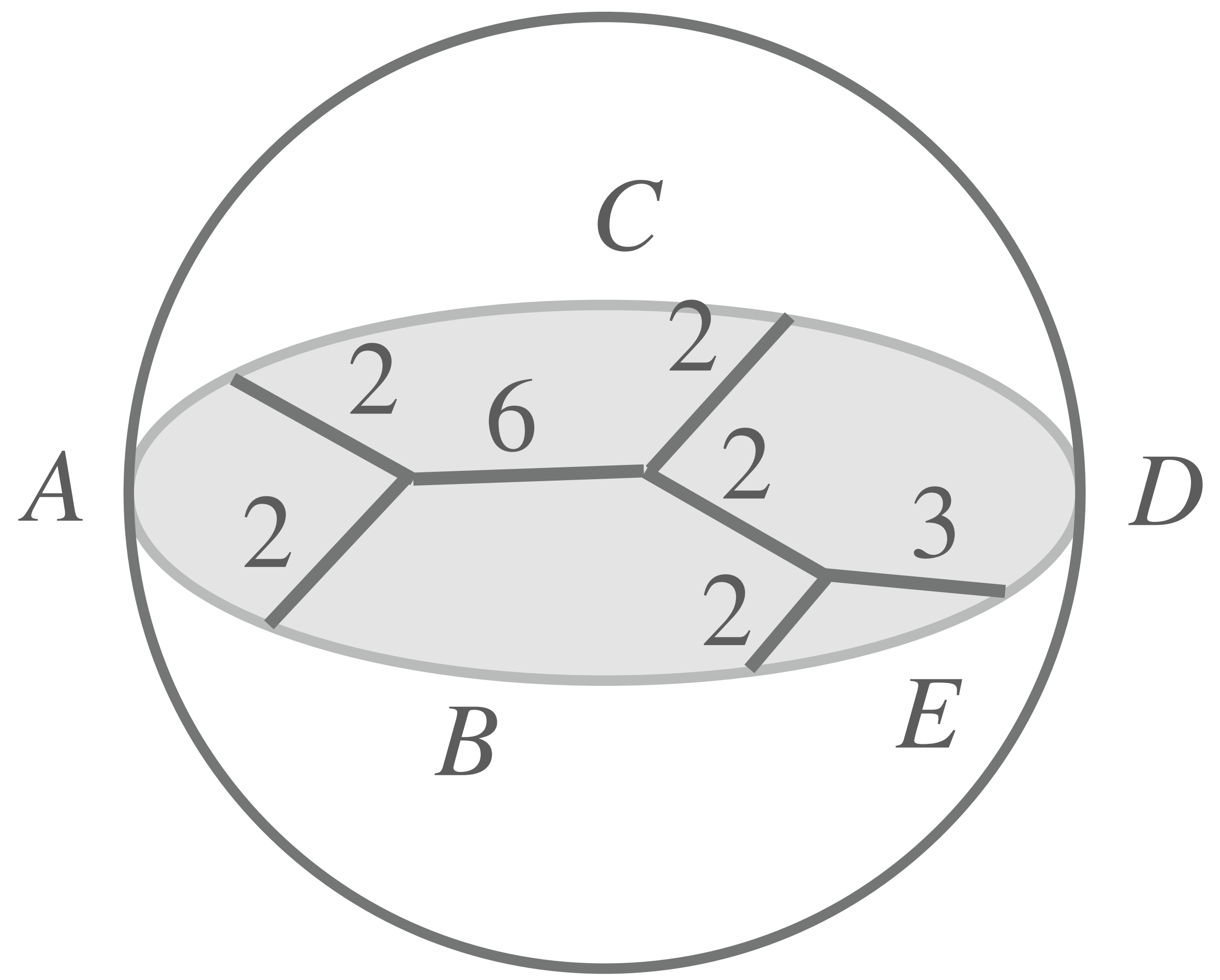




A GENERAL REPLICA SYMMETRY PRESERVING MULTI-INVARIANT

- The Klein-Maskit recombination can be carried out successively to create bulk replica symmetry preserving multi-invaraints for higher and higher number of parties.
- The orbifold geometries that are dual to them are topologically a ball with singularities forming a trivalent tree. Each vertex of the tree has to be one of the following types

$$(2,2,n), \quad (2,3,3), \quad (2,3,4), \quad (2,3,5)$$



- If we want to preserve the replica symmetry for all configurations of boundary regions then, as the sizes of regions are varied, the singularities have to undergo a "crossing" transition from one admissible graph to another
- This implies that the multi-invariants must be based on finite Coxeter groups!

SUMMARY

- We consider the problem of multi-partite invariants that preserve bulk replica symmetry.
- We solved this problem using the theory of virtually free Kleinian groups. This construction gave us infinitely many equalities of multi-invariants that the holographic state has to satisfy.
- We solved the problem of replica symmetry preserving invariants as Coxeter invariants.
- Although I didn't talk about it, we have checked the bulk predictions from conformal field theory in a number of cases.
- Interestingly Coxeter invariants also made an appearance in our earlier work that classified multi-invariants that are *entanglement monotones*. Why?
- Bulk replica symmetry in higher dimensions? Almost all our arguments generalize to higher dimensions. The one element that one has to worry about the is *uniformization*.
- Equalities of multi-invariants only depend on the fact that it is computed by a saddle point. In particular, they are robust against higher derivative corrections. It would be interesting to check them for large N vector models and/or large N symmetric product orbifolds.