Colored Jones Polynomials & the Volume Conjecture

Vishnu Jejjala

University of the Witwatersrand

1st Joint India/South Africa String Meeting

14th Joburg Workshop on String Theory

16 December 2024

Collaborators

Mark Hughes

P. Ramadevi

Pratik Roy

Vivek Singh

BRIGHAM YOUNG UNIVERSITY
FOUNDED
BYU
1875
PROVO, UTAH

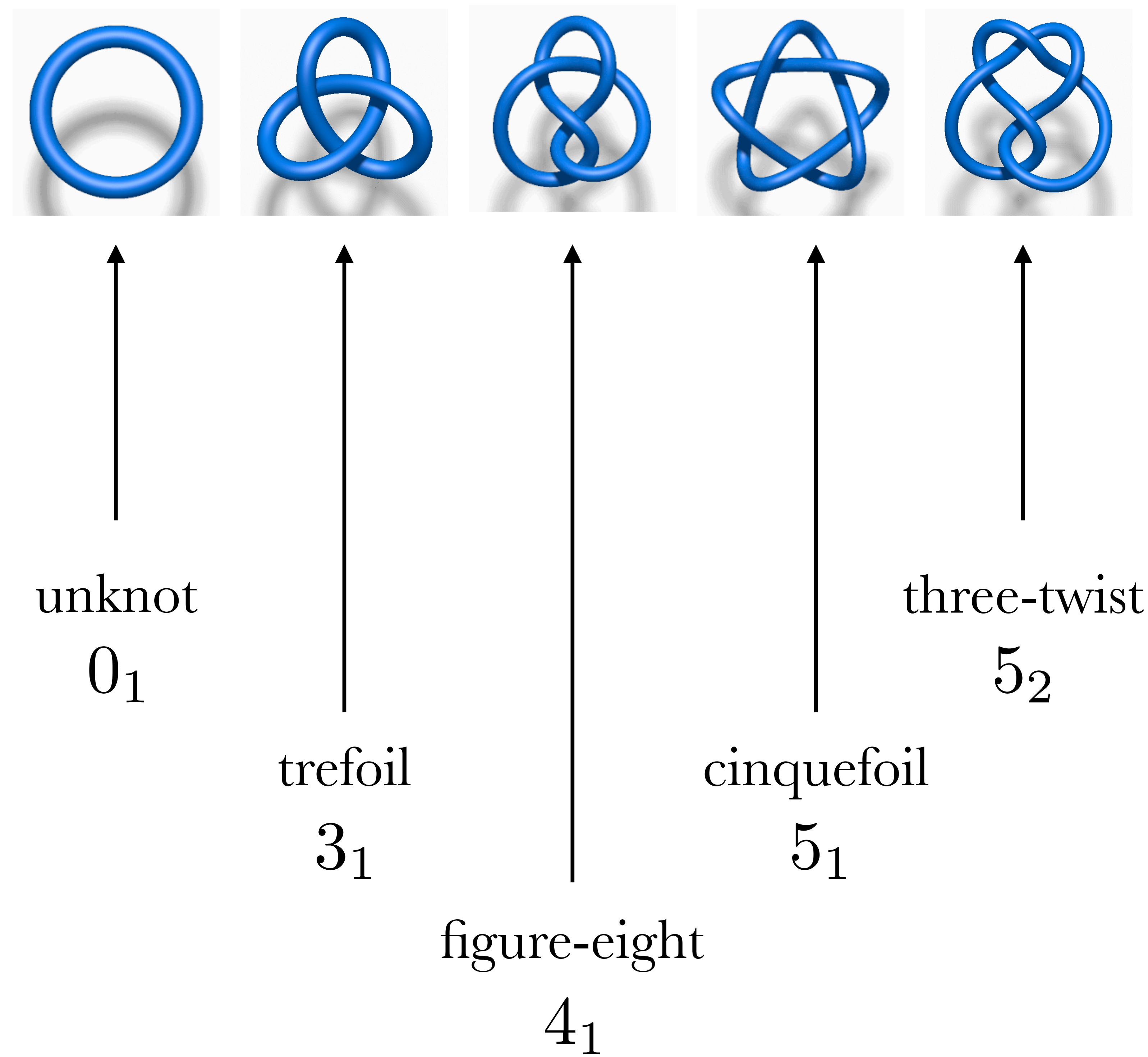
INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
भारतीय प्रौद्योगिकी संस्थान मुंबई
ज्ञानम् परमम् ध्येयम्

UNIVERSITY OF THE WITWATERSRAND
SCIENTIA ET LABORE
JOHANNESBURG

NEW YORK UNIVERSITY
ABU DHABI

Dramatis Personae

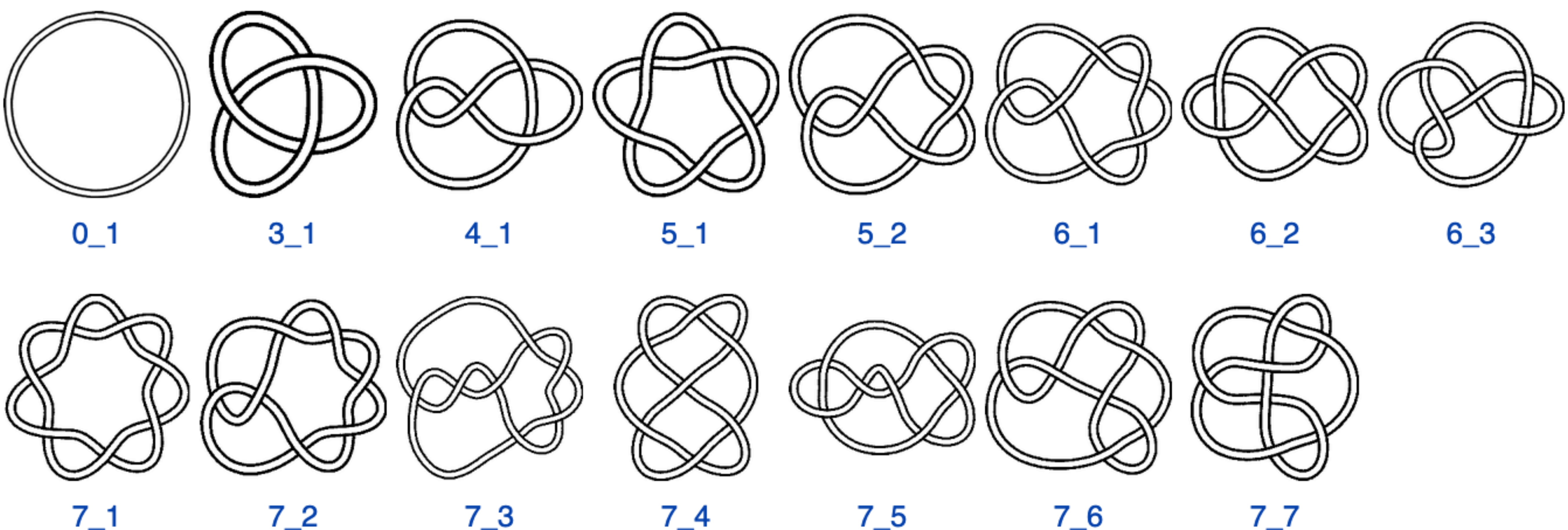
Knot: $S^1 \subset S^3$; *e.g.*,



Dramatis Personae

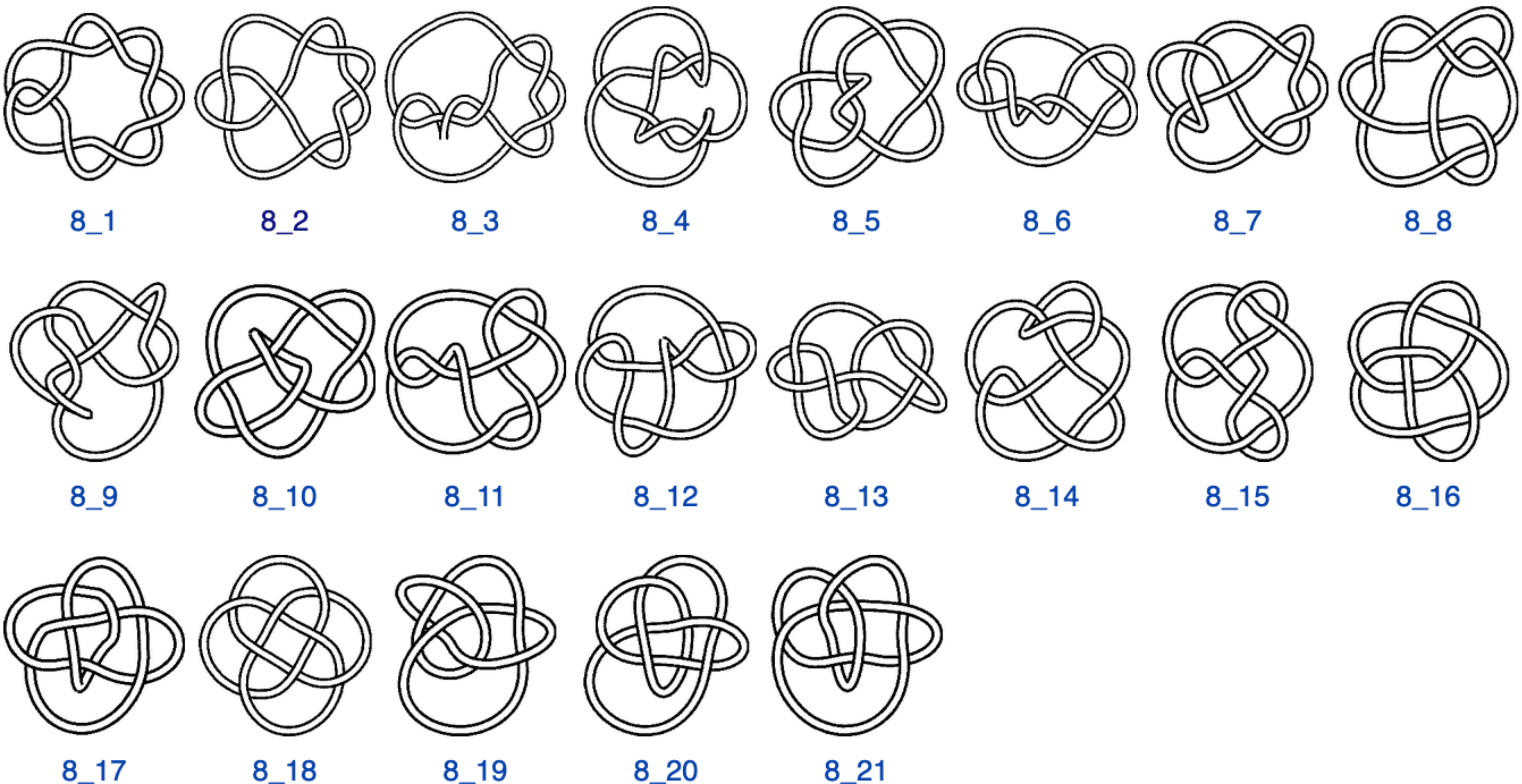
Knot: $S^1 \subset S^3$

- Organized by crossing number, a proxy for complexity



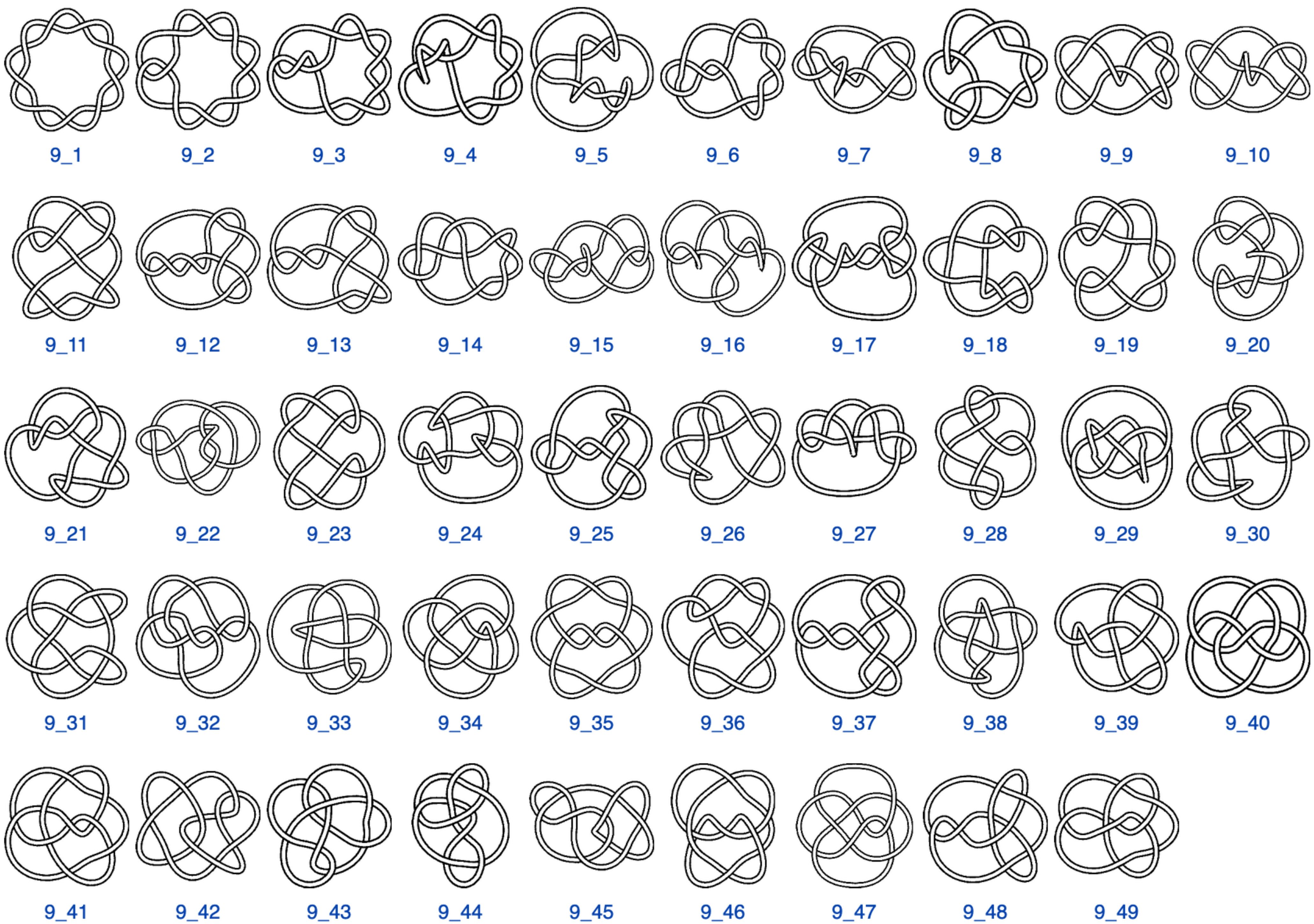
Dramatis Personae

Knot: $S^1 \subset S^3$



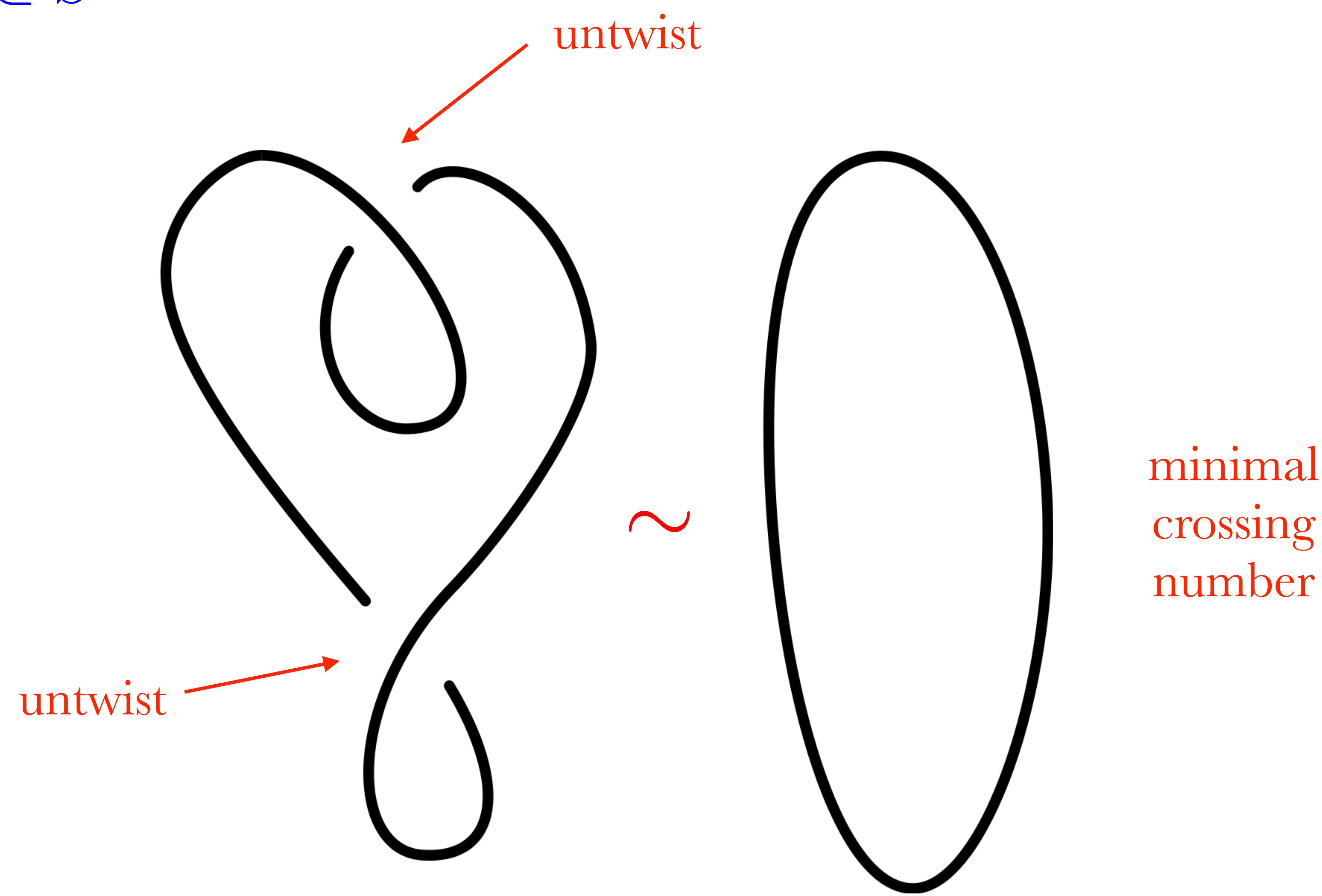
Dramatis Personae

Knot: $S^1 \subset S^3$



Dramatis Personae

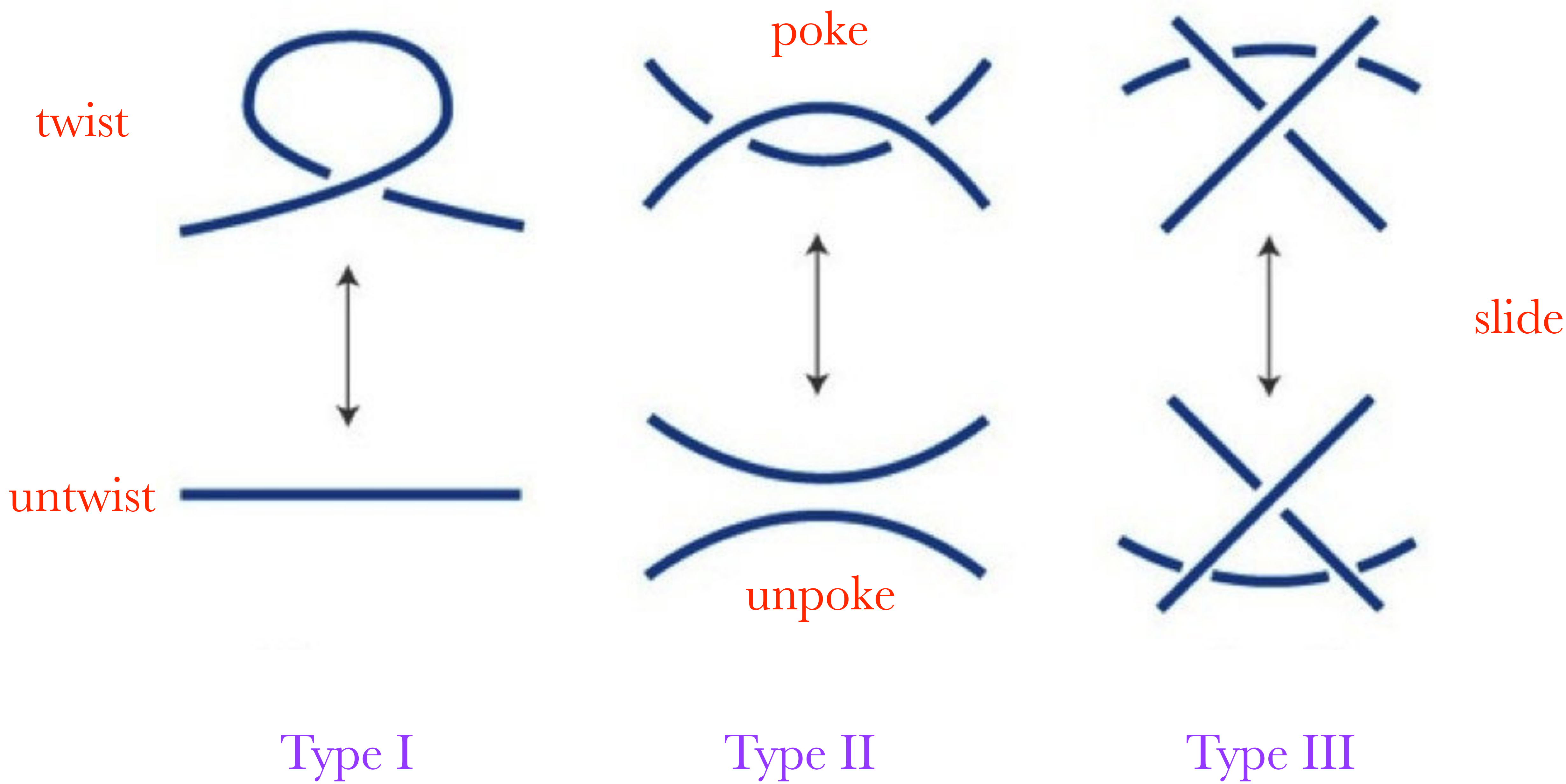
<u>Knot:</u> $S^1 \subset S^3$



Dramatis Personae

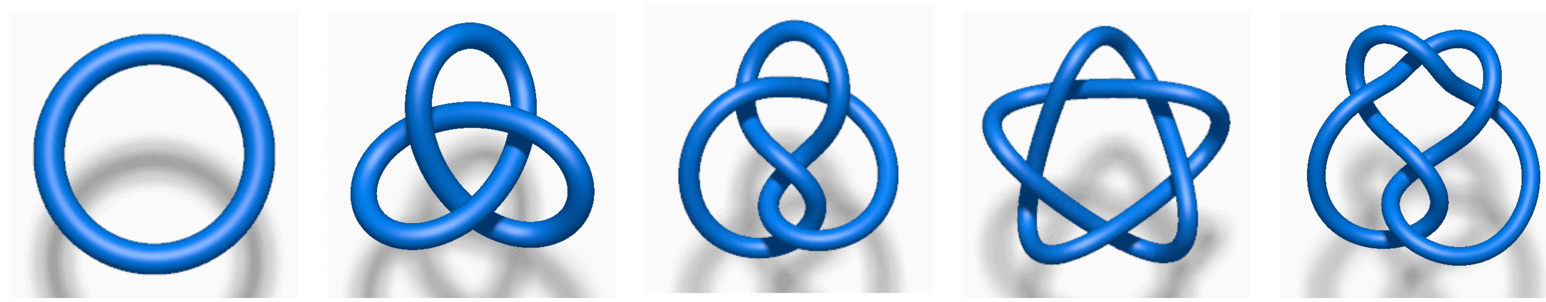
Knot: $S^1 \subset S^3$

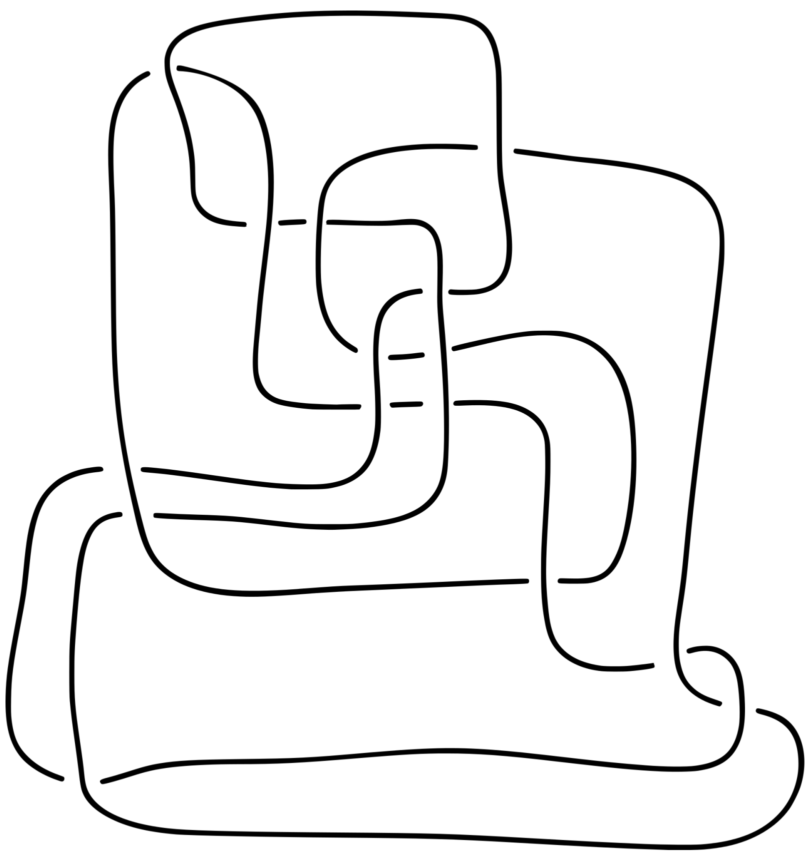
Reidemeister moves



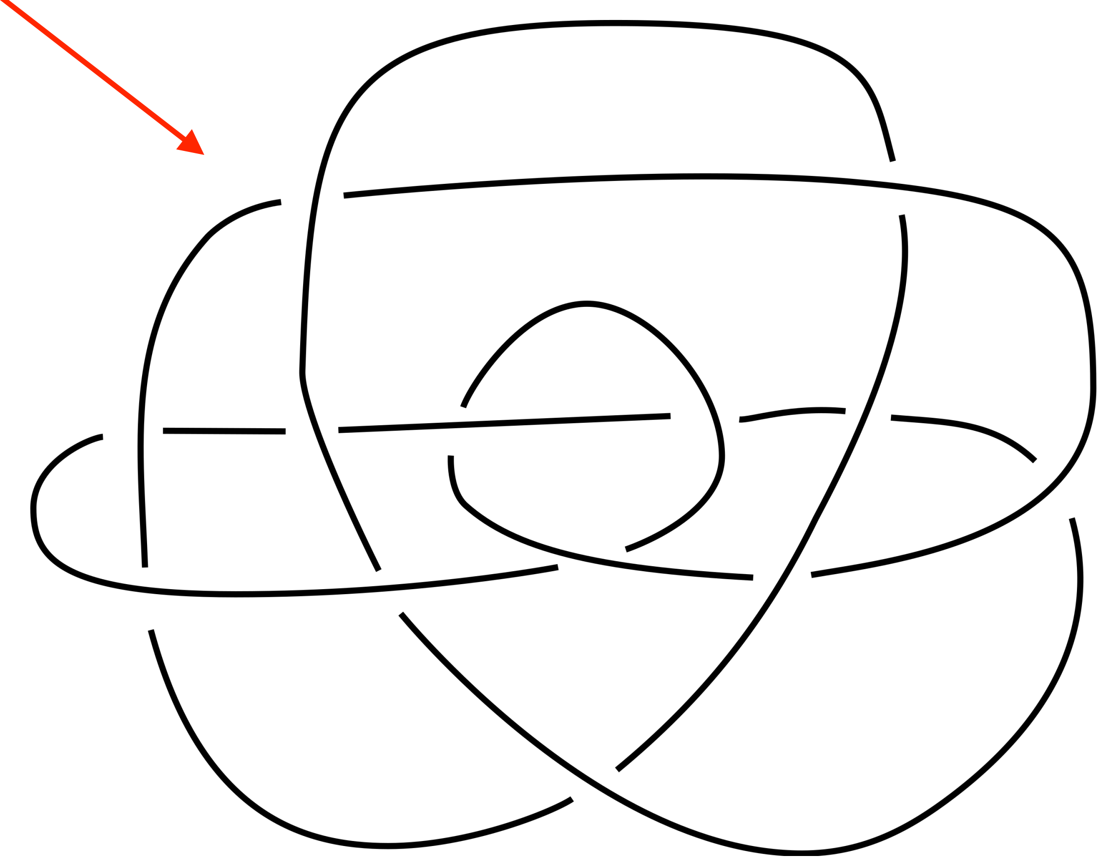
Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,





Thistlethwaite unknot

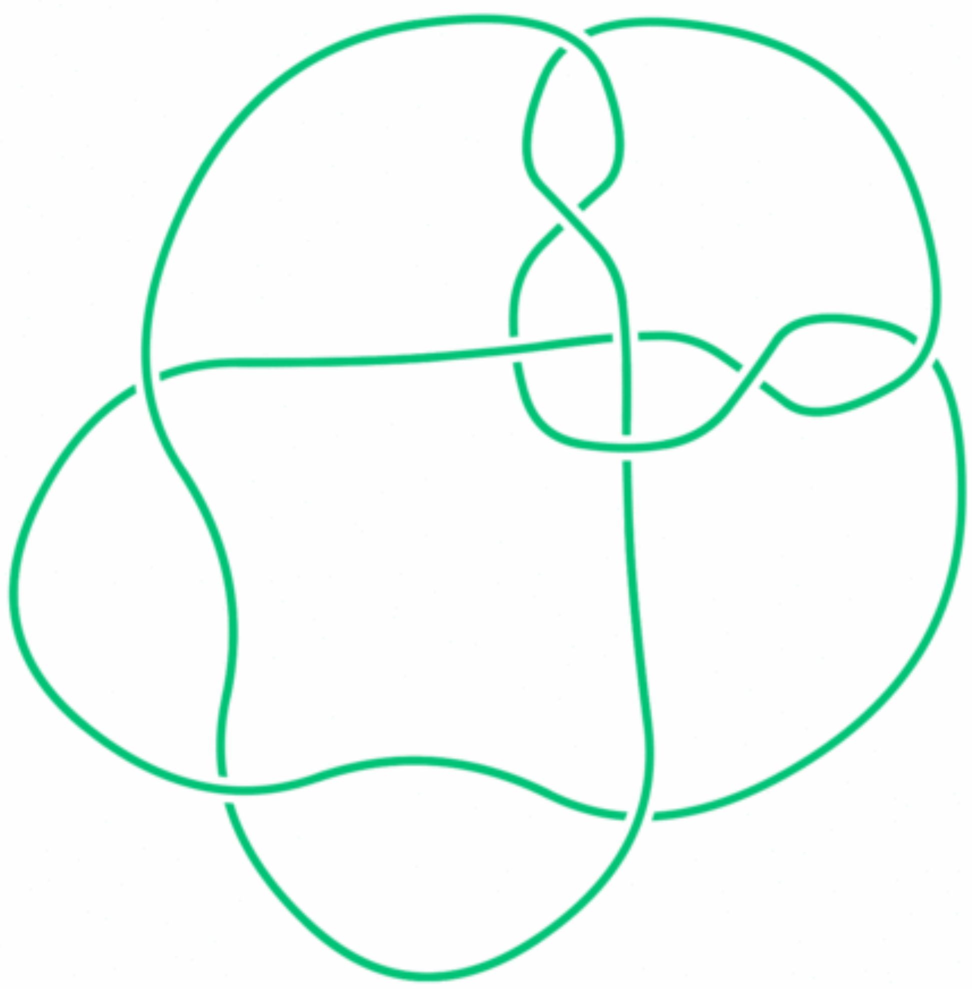


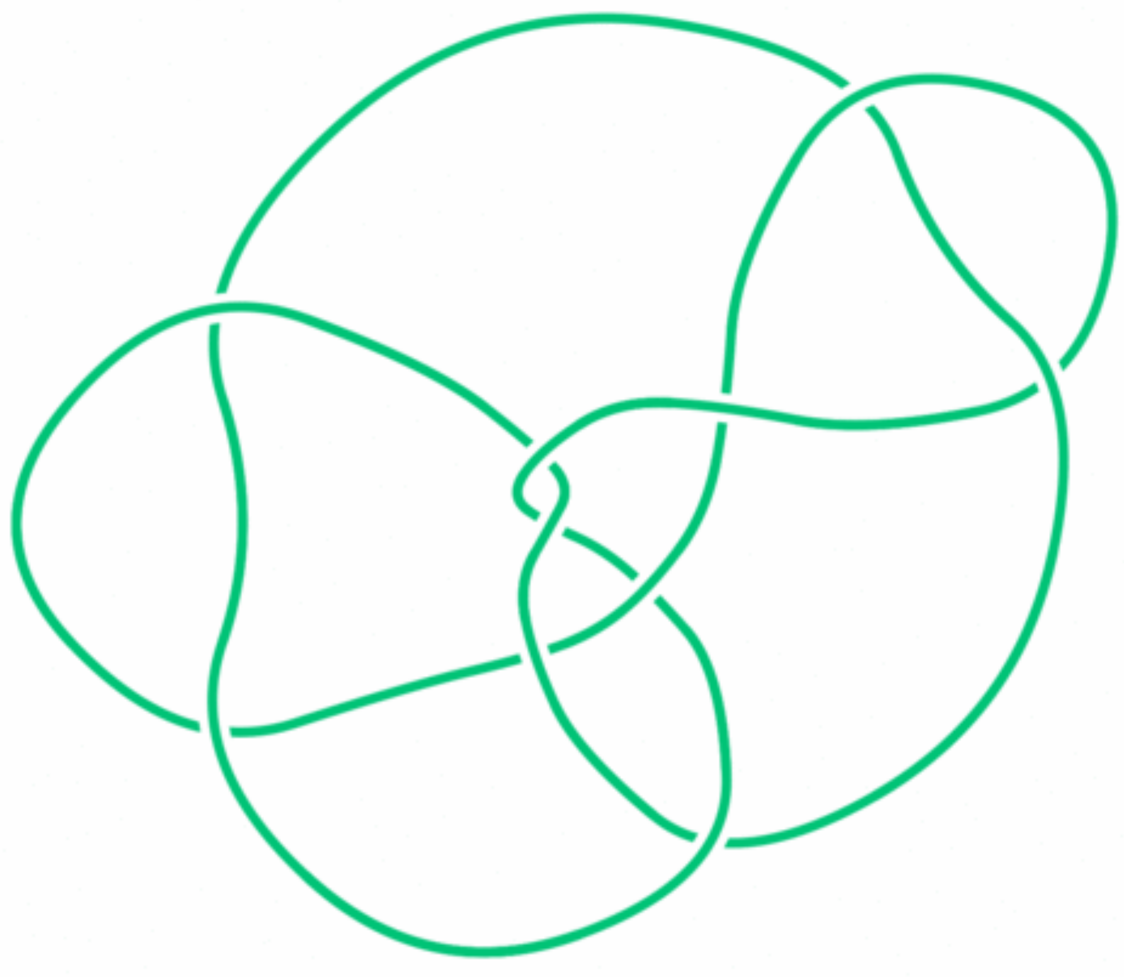
Ochiai unknot

Dramatis Personae

Knot: $S^1 \subset S^3$

Rolfsen table

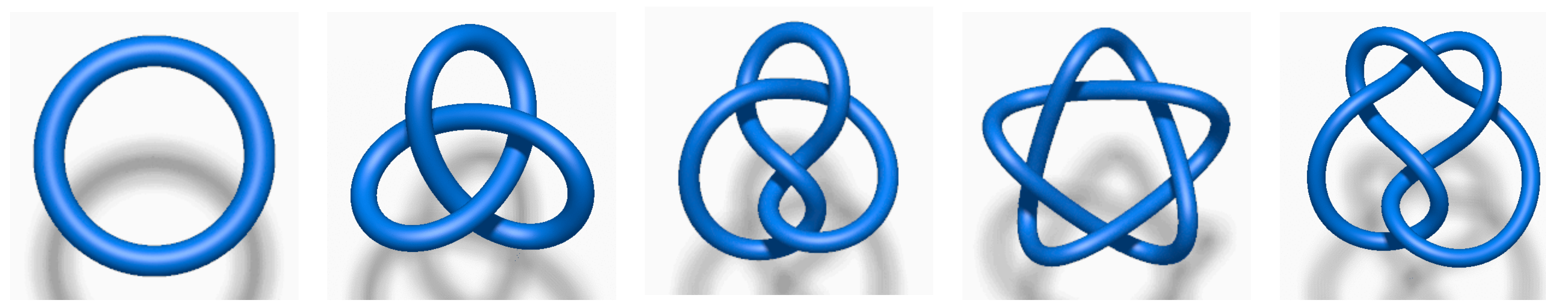




10_{161} ⟵ Perko pair (1973) ⟶ 10_{162}

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,



Jones polynomial: $J(K;q) = (-q^{\frac{3}{4}})^{w(K)} \dfrac{\langle K \rangle}{\langle \bigcirc \rangle}$

$$\langle \times \rangle = q^{\frac{1}{4}} \langle \asymp \rangle + \frac{1}{q^{\frac{1}{4}}} \langle\,)(\,\rangle$$

$w(K) = \text{overhand} - \text{underhand}$

$J(\bigcirc; q) = 1$

Jones (1985)

topological invariant: independent of how knot is drawn

Topological Invariants

- On a manifold $\mathcal{M}$ with metric $g_{\mu\nu}$, a topological invariant enjoys:

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = 0$$

- In Chern–Simons theory, the operators are Wilson loops

$$U_R(\gamma) = \mathrm{tr}_R \, \mathcal{P} \exp \left(i \oint_\gamma A \right)$$

- The colored Jones polynomial is a knot invariant in 3d:

$$J_n(K; q = e^{2\pi i/(k+2)}) = \frac{\int_{\mathcal{U}} [DA] \; U_n(K) e^{iS_{\mathrm{CS}}(A)}}{\int_{\mathcal{U}} [DA] \; U_n(0_1) e^{iS_{\mathrm{CS}}(A)}} = \frac{\langle U_n(K) \rangle}{\langle U_n(0_1) \rangle}$$

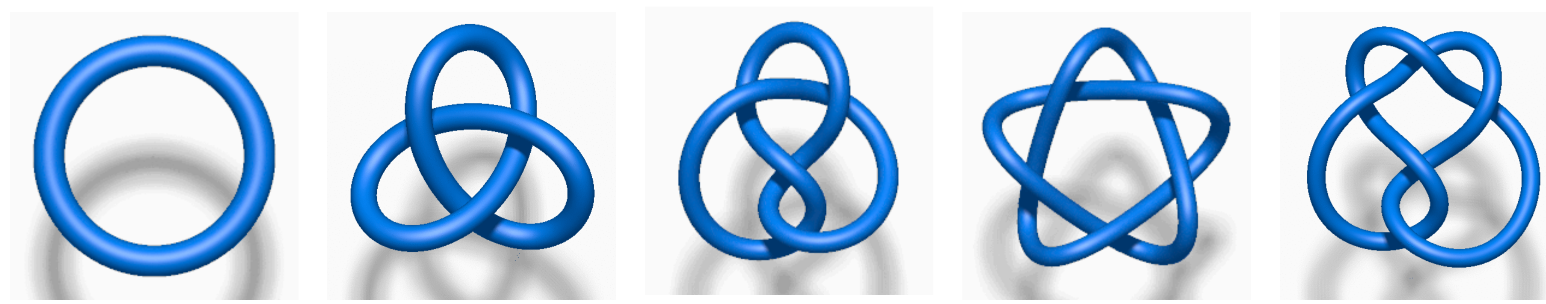
$$S_{\mathrm{CS}}(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) , \quad Z(\mathcal{M}) = \int_{\mathcal{U}} [DA] \; e^{iS_{\mathrm{CS}}(A)}$$

$$k \in \mathbb{Z}$$

Witten (1989)

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,



Jones polynomial: $J(K;q) = (-q^{\frac{3}{4}})^{w(K)} \dfrac{\langle K \rangle}{\langle \bigcirc \rangle}$

$$\langle \times \rangle = q^{\frac{1}{4}} \langle \asymp \rangle + \frac{1}{q^{\frac{1}{4}}} \langle)(\rangle$$

$w(K) = \text{overhand} - \text{underhand}$

vev of Wilson loop operator along K in $\square$ for $SU(2)$ Chern–Simons on S^3

Jones (1985)
Witten (1989)

$$J_2(4_1;q) = q^{-2} - q^{-1} + 1 - q + q^2 \ , \quad q = e^{\frac{2\pi i}{k+2}}$$

Open question: Is unknot unique knot for which $J_2(q) = 1$?

Open question: Unknot recognition problem — is this NP?

Dehn (1910)
Turing (1954)
Haken (1961)
Lackenby (2021)

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

Jones polynomial: $J(K;q) = (-q^{\frac{3}{4}})^{w(K)} \dfrac{\langle K \rangle}{\langle \bigcirc \rangle}$

$$\langle \times \rangle = q^{\frac{1}{4}} \langle \asymp \rangle + \frac{1}{q^{\frac{1}{4}}} \langle)(\rangle$$

$w(K) =$ overhand $-$ underhand

vev of Wilson loop operator along K in

$\square$ for $SU(2)$ Chern–Simons on S^3

Jones (1985)
Witten (1989)

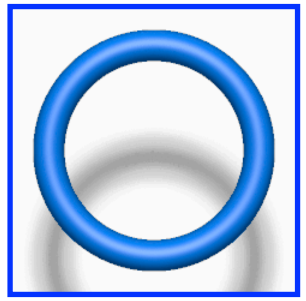
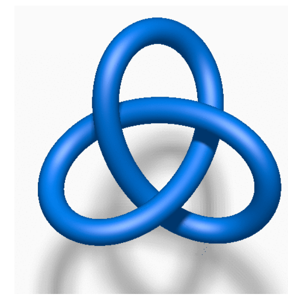
Hyperbolic volume: volume of $S^3 \backslash K$ is another knot invariant in 3d

computed from tetrahedral decomposition of knot complement

Thurston (1978)
Mostow (1968)

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

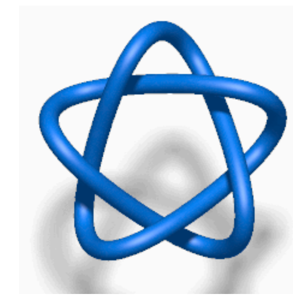
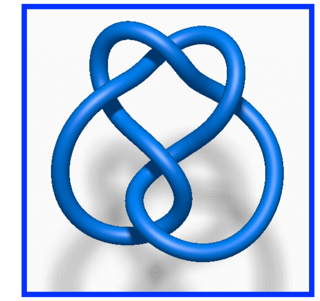



Table A1

# Crossings	# Knots	# Torus	# Sat.
0	1		
3a	1	1	
4a	1		
5a	2	1	
6a	3		
7a	7	1	
8a	18		
8n	3	1	
9a	41	1	
9n	8		
10a	123		
10n	42	1	
11a	367	1	
11n	185		
12a	1,288		
12n	888		
13a	4,878	1	
13n	5,110		2
14a	19,536		
14n	27,436	1	2
15a	85,263	1	
15n	168,030	1	6
16a	379,799		
16n	1,008,906	1	10

1701936 knots up to 16 crossings

All but 32 are hyperbolic

Hoste, Thistlewaite, Weeks (1998)

Conjecture [Adams]: proportion of hyperbolic knots approaches 1 as crossing number goes to ∞

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

Table A1			
# Crossings	# Knots	# Torus	# Sat.
0	1		
3a	1	1	
4a	1		
5a	2	1	
6a	3		
7a	7	1	
8a	18		
8n	3	1	
9a	41	1	
9n	8		
10a	123		
10n	42	1	
11a	367	1	
11n	185		
12a	1,288		
12n	888		
13a	4,878	1	
13n	5,110		2
14a	19,536		
14n	27,436	1	2
15a	85,263	1	
15n	168,030	1	6
16a	379,799		
16n	1,008,906	1	10

1701936 knots up to 16 crossings

All but 32 are hyperbolic

Hoste, Thistlewaite, Weeks (1998)

Conjecture [Adams]: proportion of hyperbolic knots approaches 1 as crossing number goes to ∞

This conjecture is false!

$$\lim_{n\to\infty} \sup \frac{S_n}{P_n} > \frac{1}{2 \cdot 10^{17}}$$

Malyutin (2018)
Belousov, Malyutin (2019)

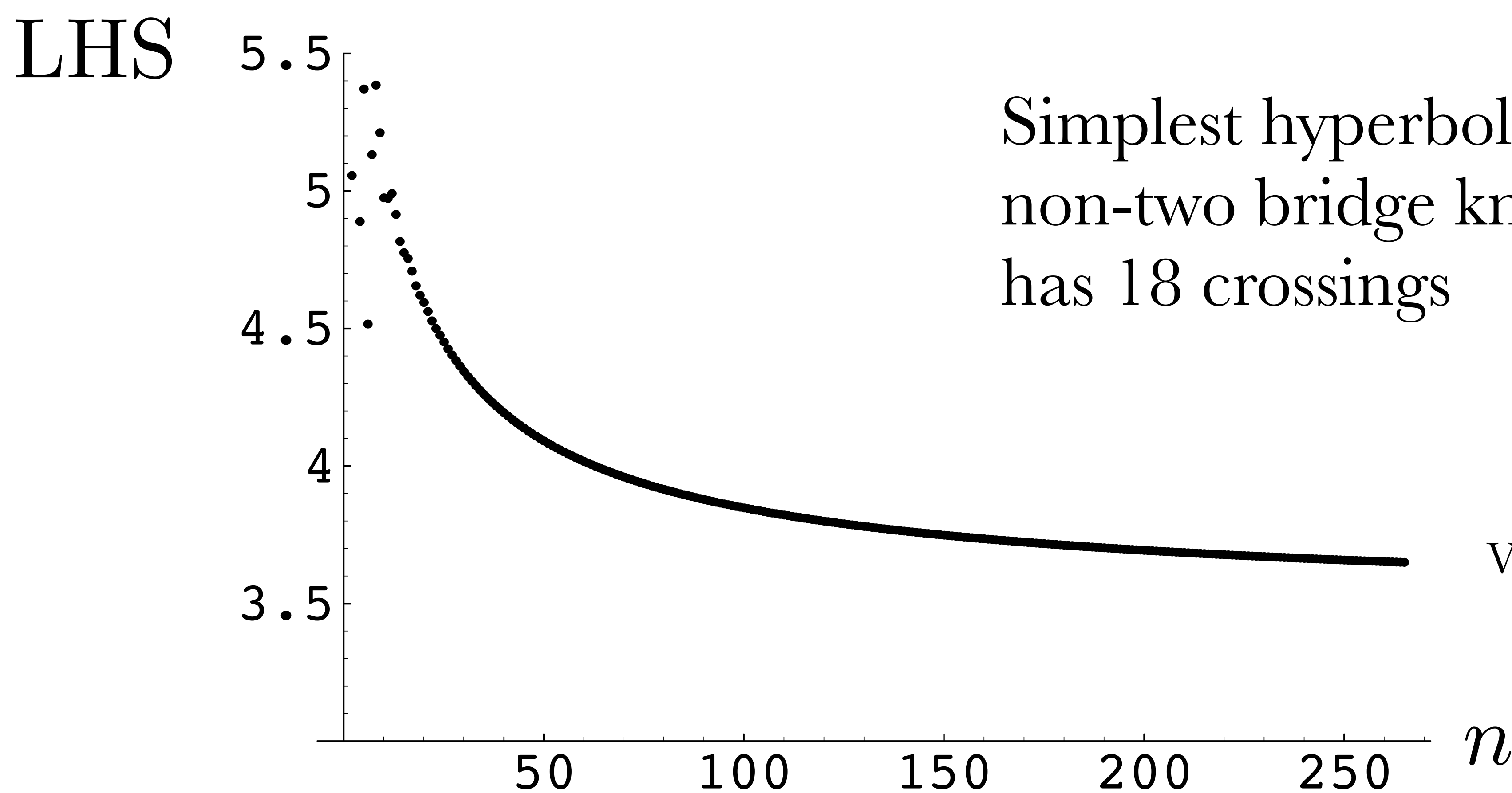
Dramatis Personae

Volume conjecture: $\lim_{n\to\infty} \frac{2\pi \log |J_n(K;\omega_n)|}{n} = \mathrm{Vol}(S^3 \setminus K)$

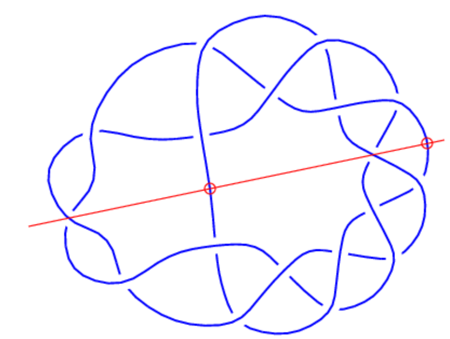
Kashaev (1997)
Murakami x 2 (2001)
Gukov (2005)

$\omega_n = e^{\frac{2\pi i}{n}}$

In fact, we take $n, k \to \infty$



Simplest hyperbolic non-two bridge knot, has 18 crossings



$\sigma_1^2(\sigma_1\sigma_2)^8$

$\mathrm{Vol}(S^3 \setminus K_0) = 3.474247\ldots$

Behavior is not monotonic!

Garoufalidis, Lan (2004)

Dramatis Personae

Volume conjecture:

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K;\omega_n)|}{n} = \mathrm{Vol}(S^3 \setminus K)$$

$$\omega_n = e^{\frac{2\pi i}{n}}$$

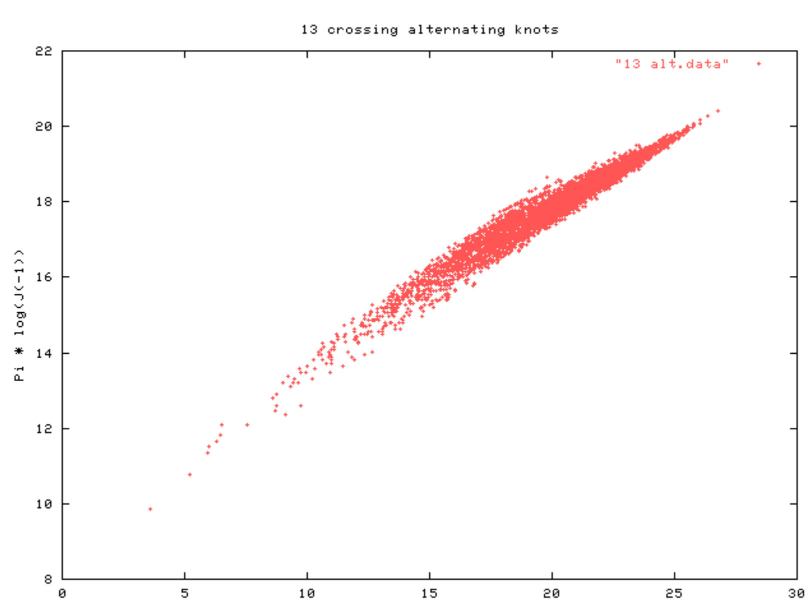
Kashaev (1997)
Murakami x 2 (2001)
Gukov (2005)

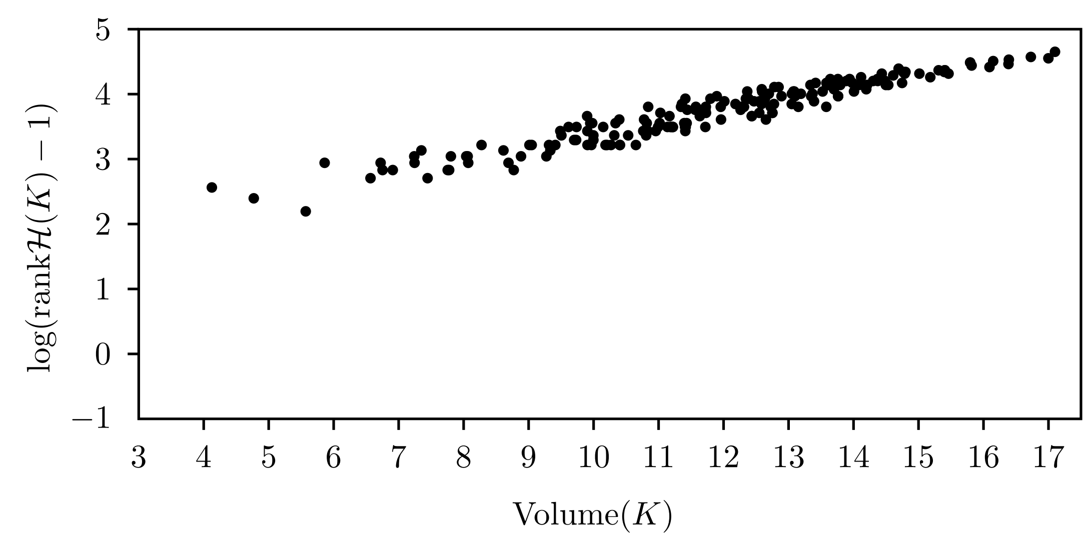
Khovanov homology: a homology theory $\mathcal{H}_K$ whose graded Euler characteristic is $J_2(K;q)$; explains why coefficients are integers

Khovanov (2000)
Bar-Natan (2002)

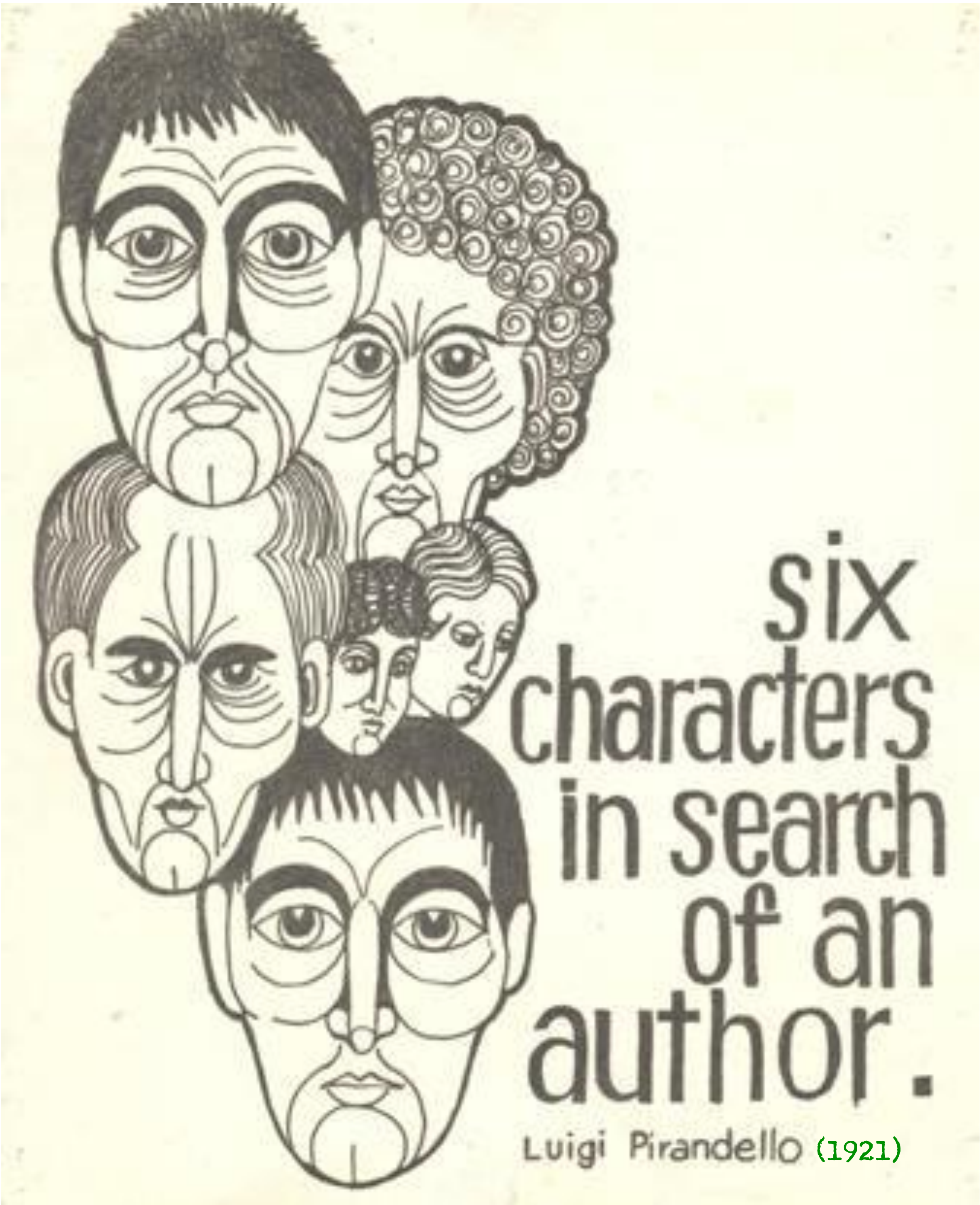
$$\log |J_2(K;-1)| \;,\;\; \log(\mathrm{rank}(\mathcal{H}_K) - 1) \propto \mathrm{Vol}(S^3 \setminus K)$$

Dunfield (2000)
Khovanov (2002)

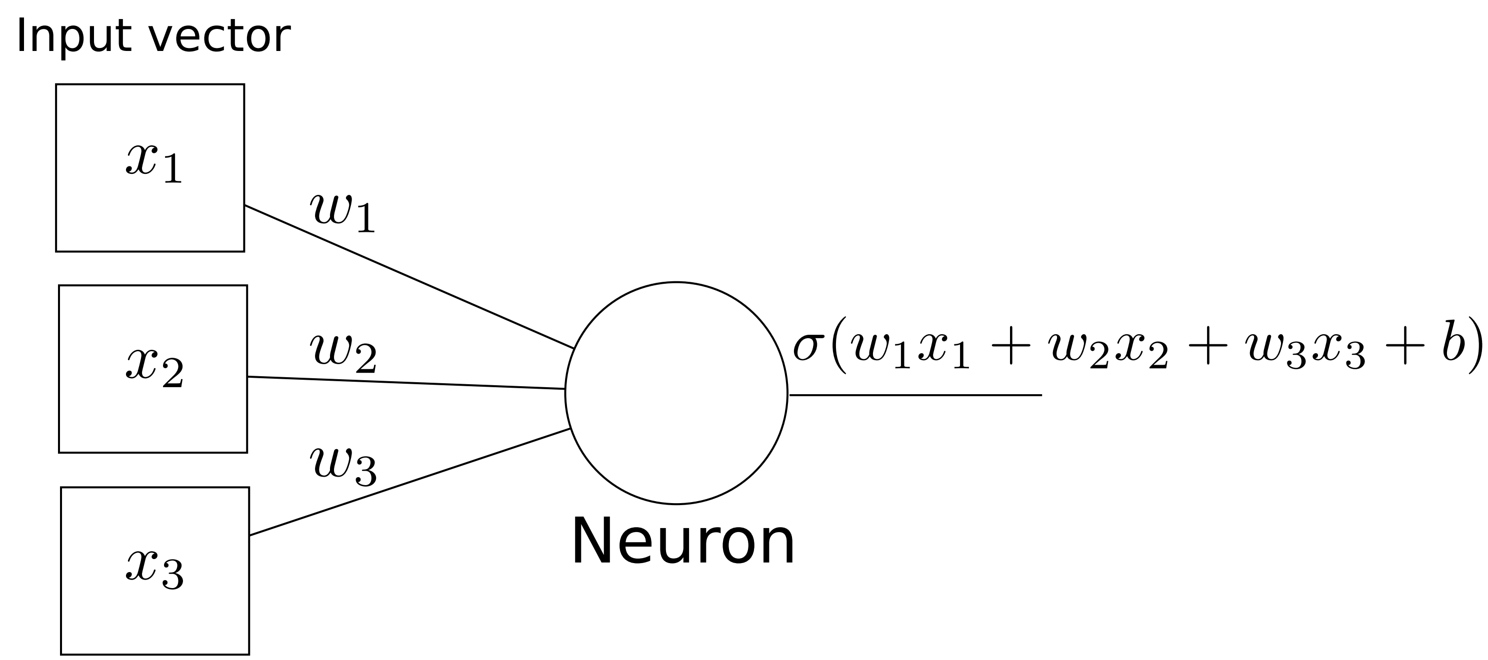




Dramatis Personae



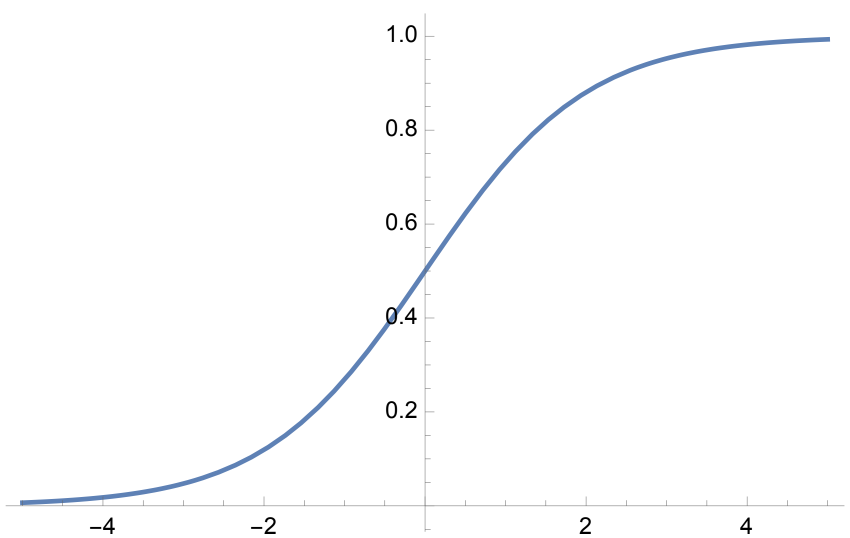
Feedforward Neural Networks



Rosenblatt (1957)

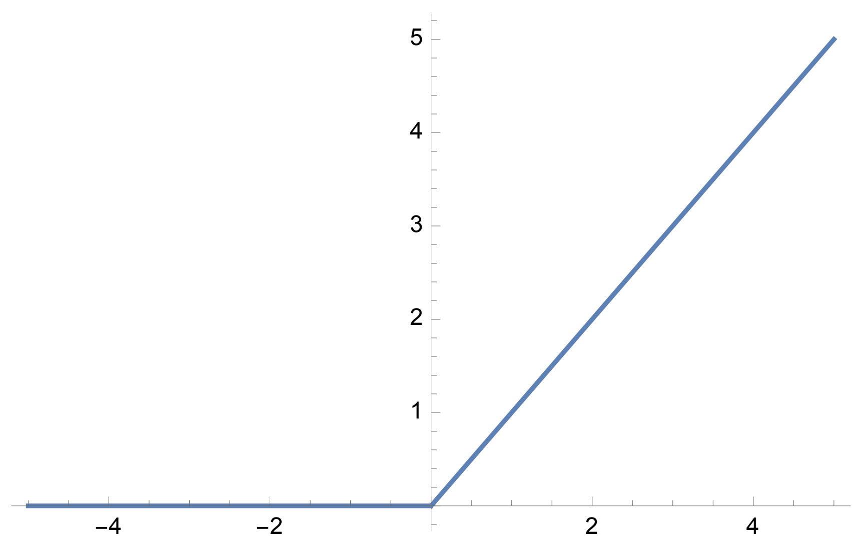
non-linearity

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



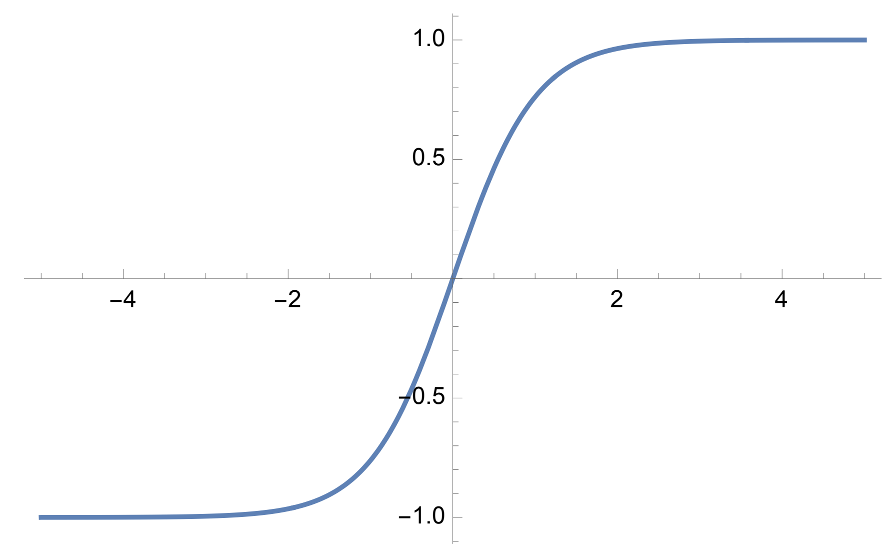
logistic sigmoid

$$\sigma(x) = \max(0, x)$$



ReLU

$$\sigma(x) = \tanh x$$

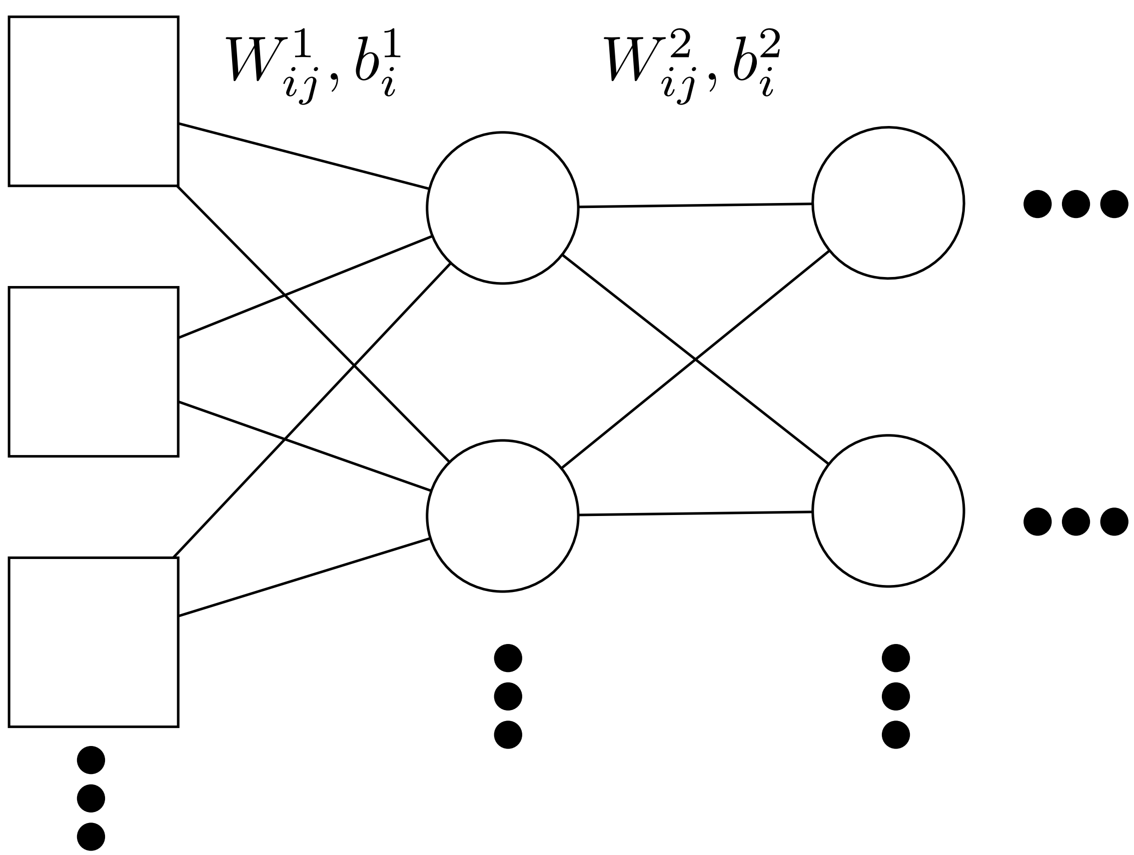


tanh

Feedforward Neural Networks

$$
\begin{aligned}
f_i : \quad \mathbb{R}^{n_{i-1}} &\to \mathbb{R}^{n_i} \\
\mathbf{v}_{i-1} &\mapsto \mathbf{v}_i = \sigma(\mathbf{v}'_i)
\end{aligned}
\qquad \mathbf{v}'_i = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i
$$

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \ , \qquad a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$



Hopfield (1982, 1984)
Hinton (1986)

Feedforward Neural Networks

$$f_i: \quad \mathbb{R}^{n_{i-1}} \quad \to \quad \mathbb{R}^{n_i}$$

number of neurons in i-th layer

$$\mathbf{v}_{i-1} \quad \mapsto \quad \mathbf{v}_i = \sigma(\mathbf{v}_i')$$

$$\mathbf{v}_i' = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i$$

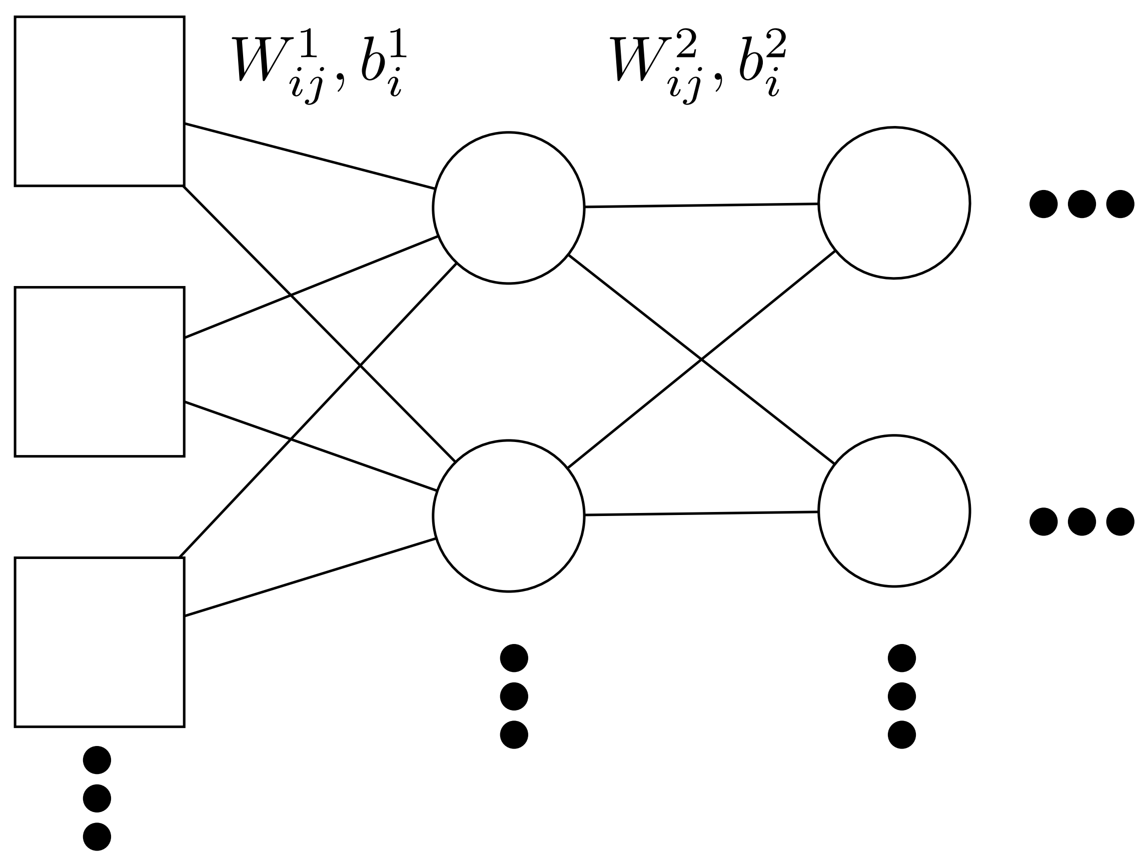
weight matrix

bias vector

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \ , \qquad a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$

number of layers

compare prediction
to ground truth



Feedforward Neural Networks

$$\begin{aligned} f_i : \quad \mathbb{R}^{n_{i-1}} &\quad \to \quad \mathbb{R}^{n_i} \\ \mathbf{v}_{i-1} &\quad \mapsto \quad \mathbf{v}_i = \sigma(\mathbf{v}'_i) \end{aligned}$$

number of neurons in i-th layer

$$\mathbf{v}'_i = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i$$

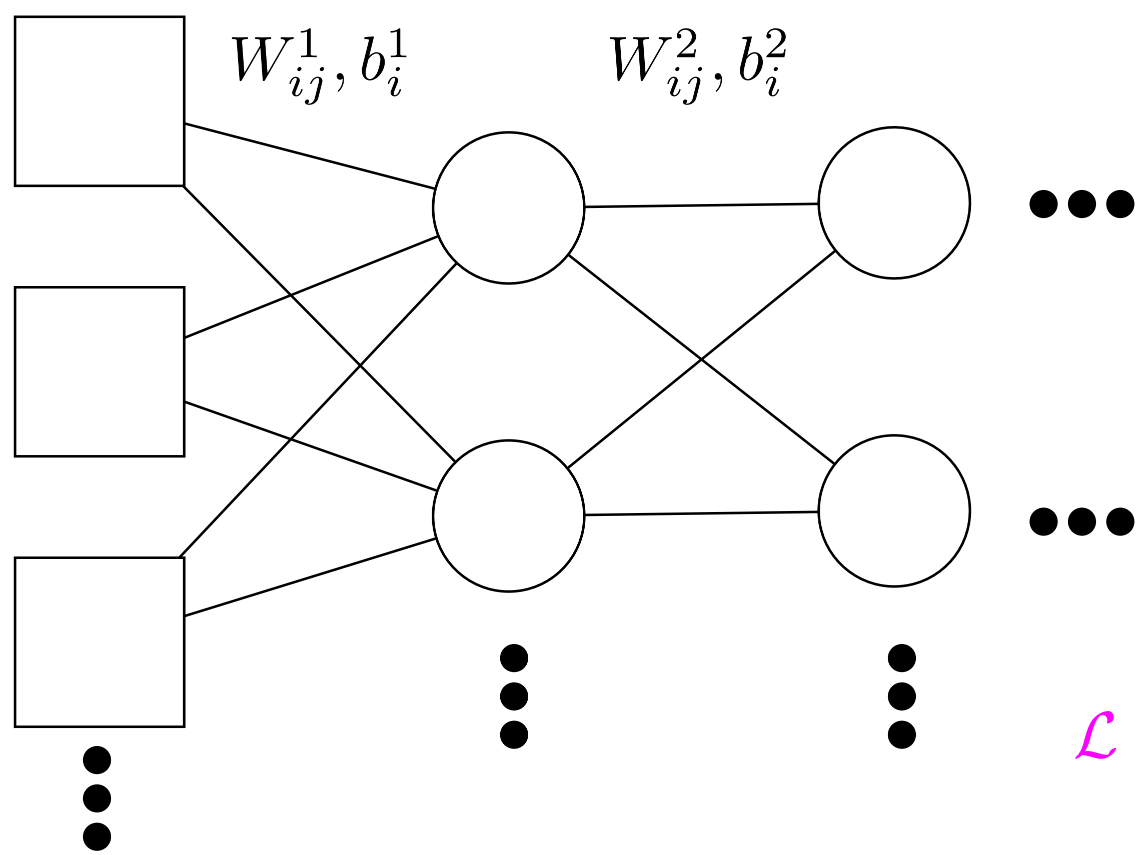
weight matrix

bias vector

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \ , \qquad a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$

number of layers

compare prediction
to ground truth

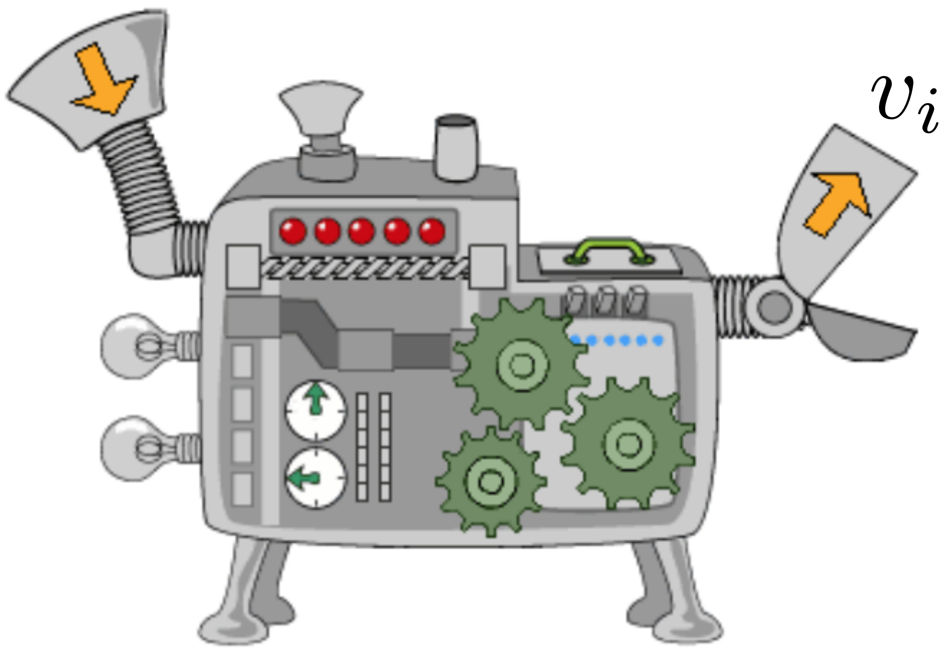


tune weights and biases
to minimize **loss function**
during **training**

$$\mathcal{L} = \sum_{\text{batch}} |\text{predicted} - \text{actual}|^2$$

Neural Network

J_i



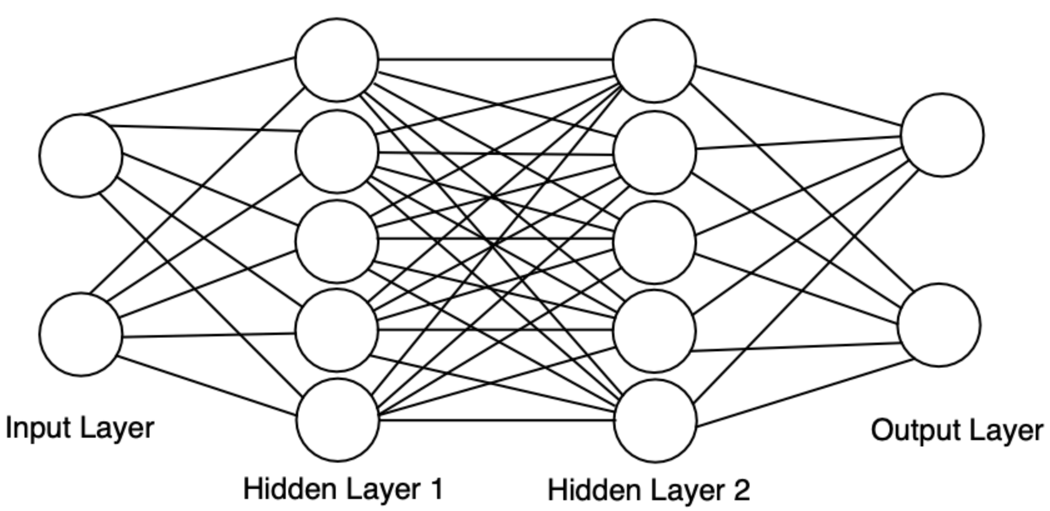
v_i

$$\{J_1, \ldots, J_n\} \longrightarrow \{v_1, \ldots, v_n\}$$

$$J_i \in T$$

$$\{J'_1, \ldots, J'_m\} \longrightarrow ???$$

$$J'_i \in T^c$$



$\vec{J}$ $\quad 100 \times 18 \quad 100 \times 100 \quad \sum_{a=1}^{100}$

12000 hyperparameters

Jones polynomials are represented as 18-vectors

$$\vec{J}_K = (\mathrm{min}, \mathrm{max}, \mathrm{coeffs}, 0, \ldots, 0)$$

Two layer neural network in Mathematica

$$f_\theta(\vec{J}_K) = \sum_a \sigma \left(W_\theta^2 \cdot \sigma(W_\theta^1 \cdot \vec{J}_K + \vec{b}_\theta^1) + \vec{b}_\theta^2 \right)^a$$

Logistic sigmoids for the hidden layers

Mean squared loss function

VJ, Kar, Parrikar (2019)

Mathematica Code

knot run.nb

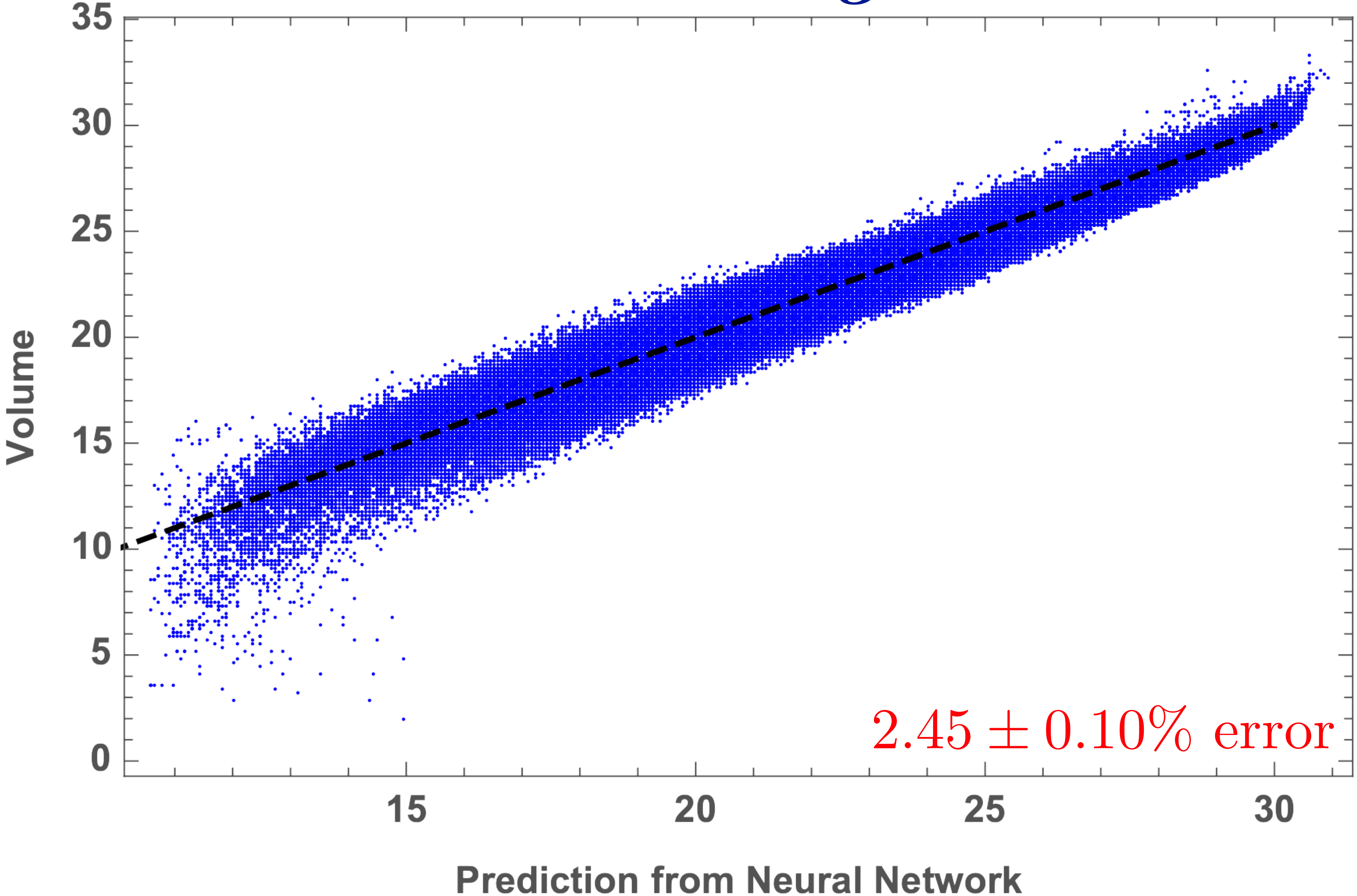
```
In[•]:= f[frac_] :=
  Module[{y, list, listcomp, joneslist, vollist, joneslistcomp, vollistcomp,
    test, aa, KnotNet0, TrainedKnotNet0},
   y = frac;
   list = Sort[RandomSample[Range[1, len], IntegerPart[len y]]];
   listcomp = Complement[Table[i, {i, 1, len}], list];
   joneslist = Table[joneses⟦list⟦i⟧⟧, {i, 1, Length[list]}];
   vollist = Table[volumes⟦list⟦i⟧⟧, {i, 1, Length[list]}];
   joneslistcomp = Table[joneses⟦listcomp⟦i⟧⟧, {i, 1, Length[listcomp]}];
   vollistcomp = Table[volumes⟦listcomp⟦i⟧⟧, {i, 1, Length[listcomp]}];
   KnotNet0 = NetChain[{LinearLayer[100], ElementwiseLayer[LogisticSigmoid],
      LinearLayer[100], ElementwiseLayer[LogisticSigmoid], SummationLayer[]},
     "Input" → {maxlen}];
   TrainedKnotNet0 = NetTrain[KnotNet0, joneslist → vollist] ;
   test = TrainedKnotNet0[joneses];
   Show[ListPlot[Table[{test⟦i⟧, volumes⟦i⟧}, {i, 1, len}],
     AxesLabel → {"Predicted Volume", "Volume"}],
    Plot[x, {x, 0, 40}, PlotRange → Full, PlotStyle → {Black, Dashed}]]
  ]
f[.1]
```

Mathematica Code

knot run.nb

In[40]:=
```
f[frac_] :=
 Module[{y, list, listcomp, joneslist, vollist, joneslistcomp, vollistcomp, test, aa,
   KnotNet0, TrainedKnotNet0},
  y = frac;
  list = Sort[RandomSample[Range[1, len], IntegerPart[len y]]];
  listcomp = Complement[Table[i, {i, 1, len}], list];
  joneslist = Table[joneses[[list[[i]]]], {i, 1, Length[list]}];
  vollist = Table[volumes[[list[[i]]]], {i, 1, Length[list]}];
  joneslistcomp = Table[joneses[[listcomp[[i]]]], {i, 1, Length[listcomp]}];
  vollistcomp = Table[volumes[[listcomp[[i]]]], {i, 1, Length[listcomp]}];
  KnotNet0 = NetChain[{LinearLayer[100], ElementwiseLayer[LogisticSigmoid],
     LinearLayer[100], ElementwiseLayer[LogisticSigmoid], SummationLayer[]}, "Input" → {maxlen}];
  TrainedKnotNet0 = NetTrain[KnotNet0, joneslist → vollist] ;
  test = TrainedKnotNet0[joneses];
  Show[ListPlot[Table[{test[[i]], volumes[[i]]}, {i, 1, len}],
    AxesLabel → {"Predicted Volume", "Volume"}],
   Plot[x, {x, 0, 40}, PlotRange → Full, PlotStyle → {Black, Dashed}]]
 ]
f[.1]
```

Volume from Jones



trained on 10% of the $313,209$ knots up to 15 crossings

VJ, Kar, Parrikar (2019)

Result

$$v_i = f(J_i) + \text{small corrections}$$

- J_i does not uniquely identify a knot

  e.g., 4_1 and K11n19 have same Jones polynomial, different volumes

- $174,619$ unique Jones polynomials

  2.83% average spread in volumes for a Jones polynomial

  intrinsic mitigation against overfitting

- Same applies to 1,701,903 hyperbolic knots up to 16 crossings

  841,139 unique polynomials

  (database compiled from **Knot Atlas** and SnapPy)

VJ, Kar, Parrikar (2019)

Result

$$v_i = f(J_i) + \text{small corrections}$$

- **Universal Approximation Theorem:** feedforward neural network, sigmoid activation function, single hidden layer with finite number of neurons can approximate continuous functions on compact subsets of $\mathbb{R}^n$

Cybenko (1989)
Hornik (1991)

- Surprise here is simplicity of architecture that does the job

- We want a **not** machine learning knot result

Entr'acte

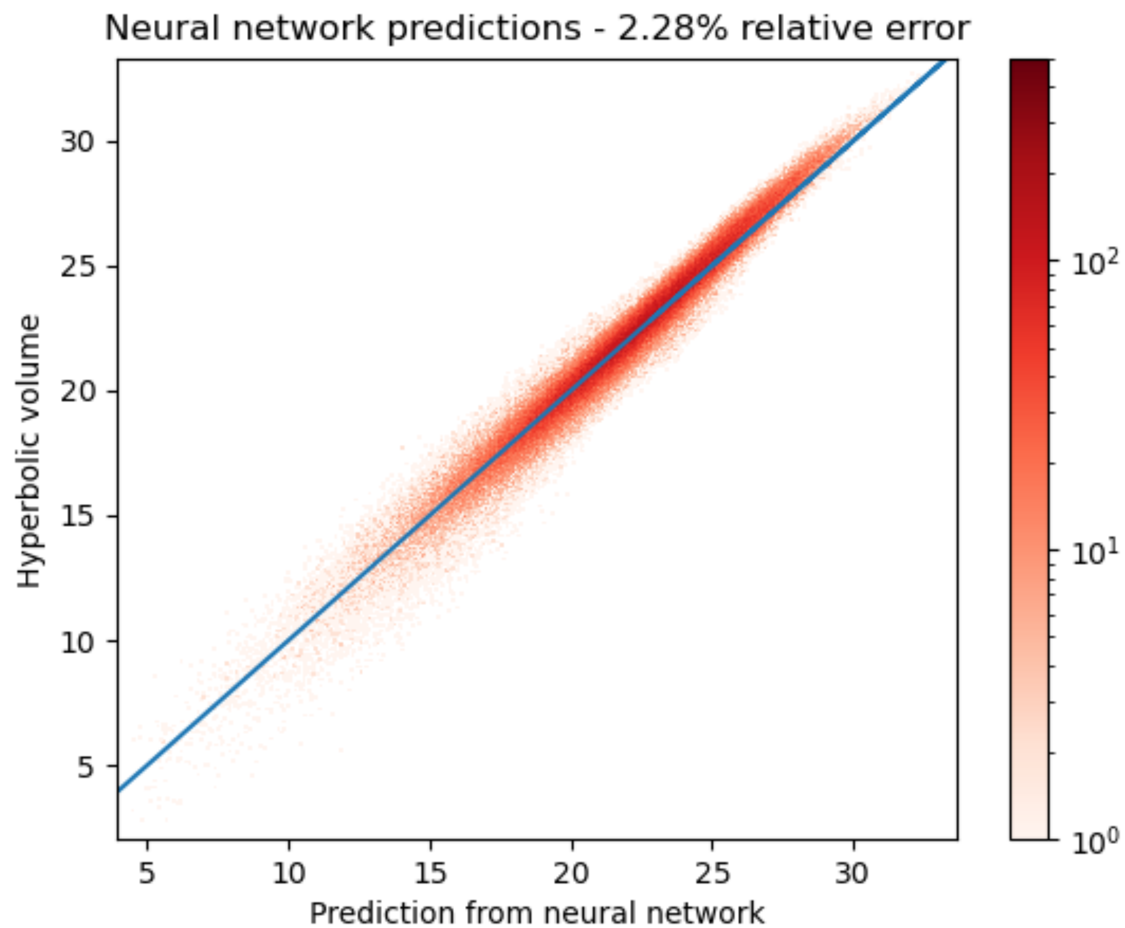
$$v_i = f(J_i) + \text{small corrections}$$

We seek to reverse engineer the neural network to obtain an analytic expression for the volume as a function of the Jones polynomial

To interpret the formula, we use machinery of analytically continued Chern–Simons theory

No Degrees Needed

- Suppose we drop the degrees and provide only the coefficients; Jones polynomial is no longer recoverable from the input vector
- Results are unchanged!

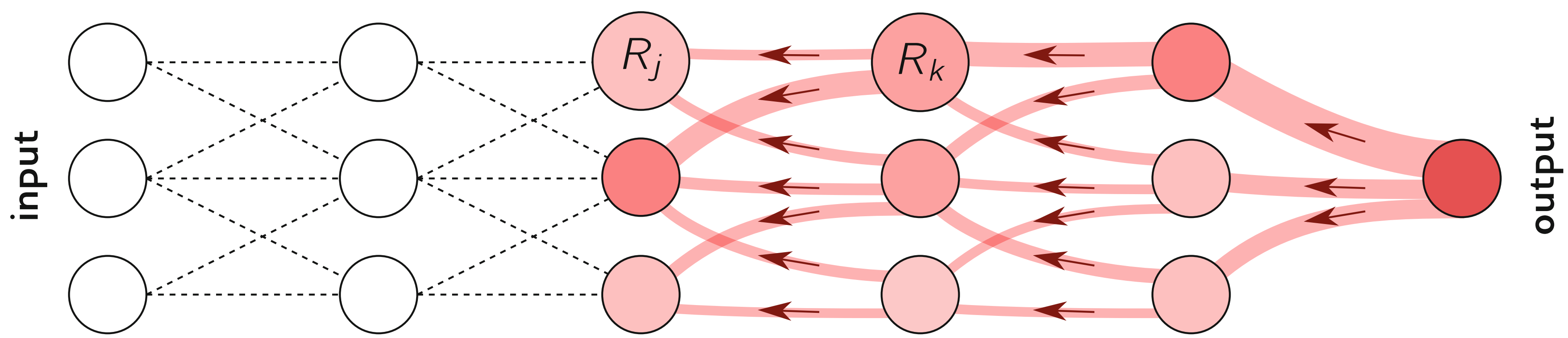


N.B.: we have switched to Python 3 using GPU-Tensorflow with Keras wrapper
two hidden layers, 100 neurons/layer, ReLu activation, mean squared loss, Adam optimizer

Craven, VJ, Kar (2020)

Layer-wise Relevance Propagation

- To determine which inputs carry the most weight, propagate backward starting from output layer employing a conservation property



Montavon et al. (2019)

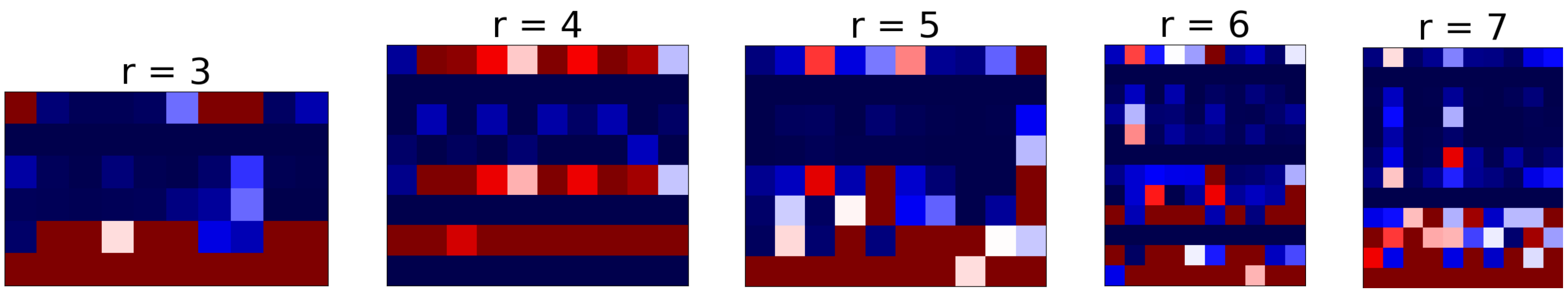
- Compute relevance score for a neuron using activations, weights, and biases

$$R_j^{m-1} = \sum_k \frac{a_j^{m-1} W_{jk}^m + N_{m-1}^{-1} b_k^m}{\sum_l a_l^{m-1} W_{lk}^m + b_k^m} R_k^m \quad , \quad \sum_k R_k^m = 1$$

j^{th} neuron in layer $m-1$

Craven, VJ, Kar (2020)

Layer-wise Relevance Propagation



- Each column is a single input corresponding to evaluations of the Jones polynomial at phases $e^{\frac{2\pi i p}{r+2}}$, $0 \leq 2p \leq r+2$, $p \in \mathbb{Z}$

- Ten different knots

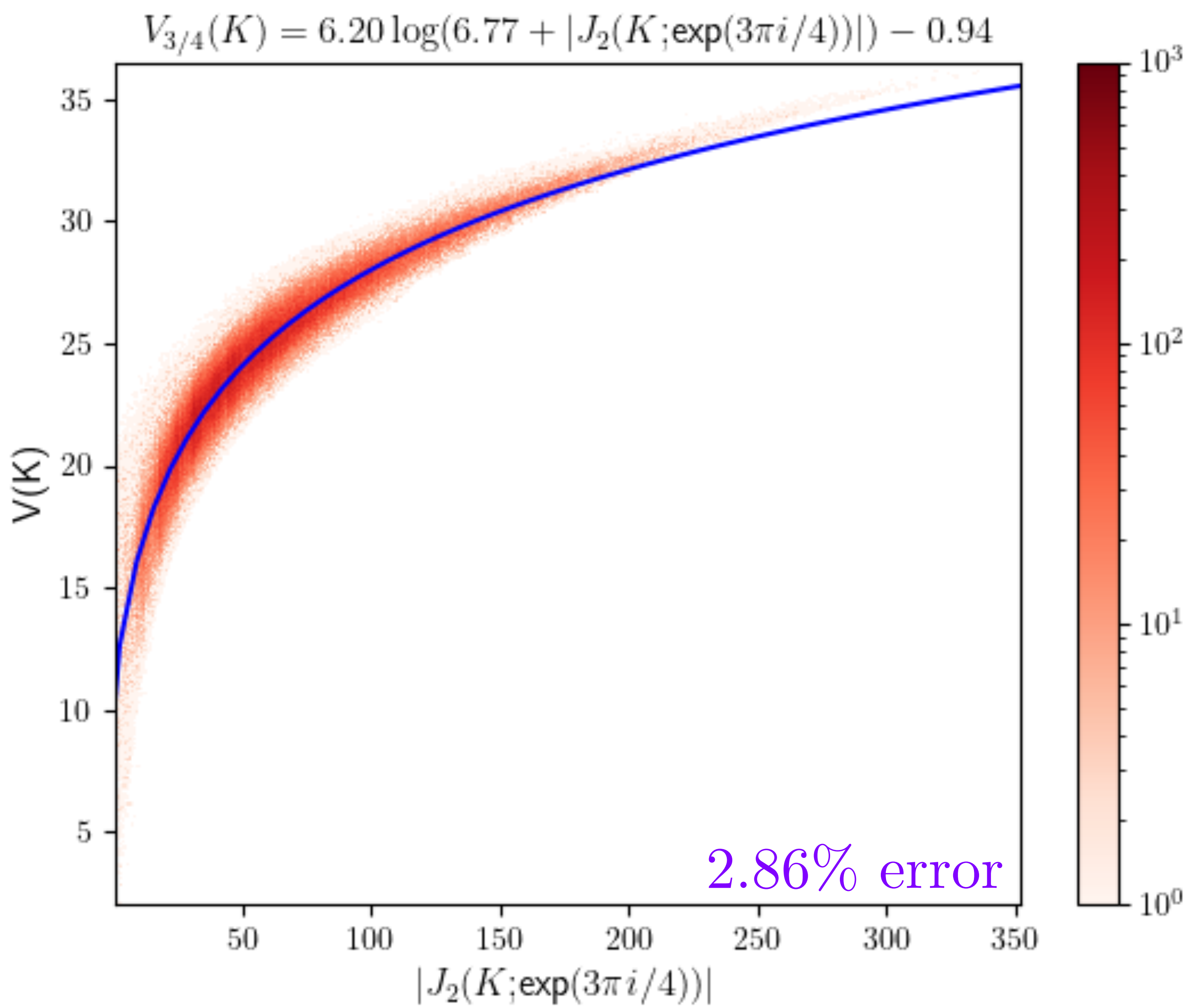
- We show the relevances (red is most relevant) and notice that the same input features light up

Craven, VJ, Kar (2020)

Phenomenological Function

$$\omega = e^{\frac{3\pi i}{4}}$$

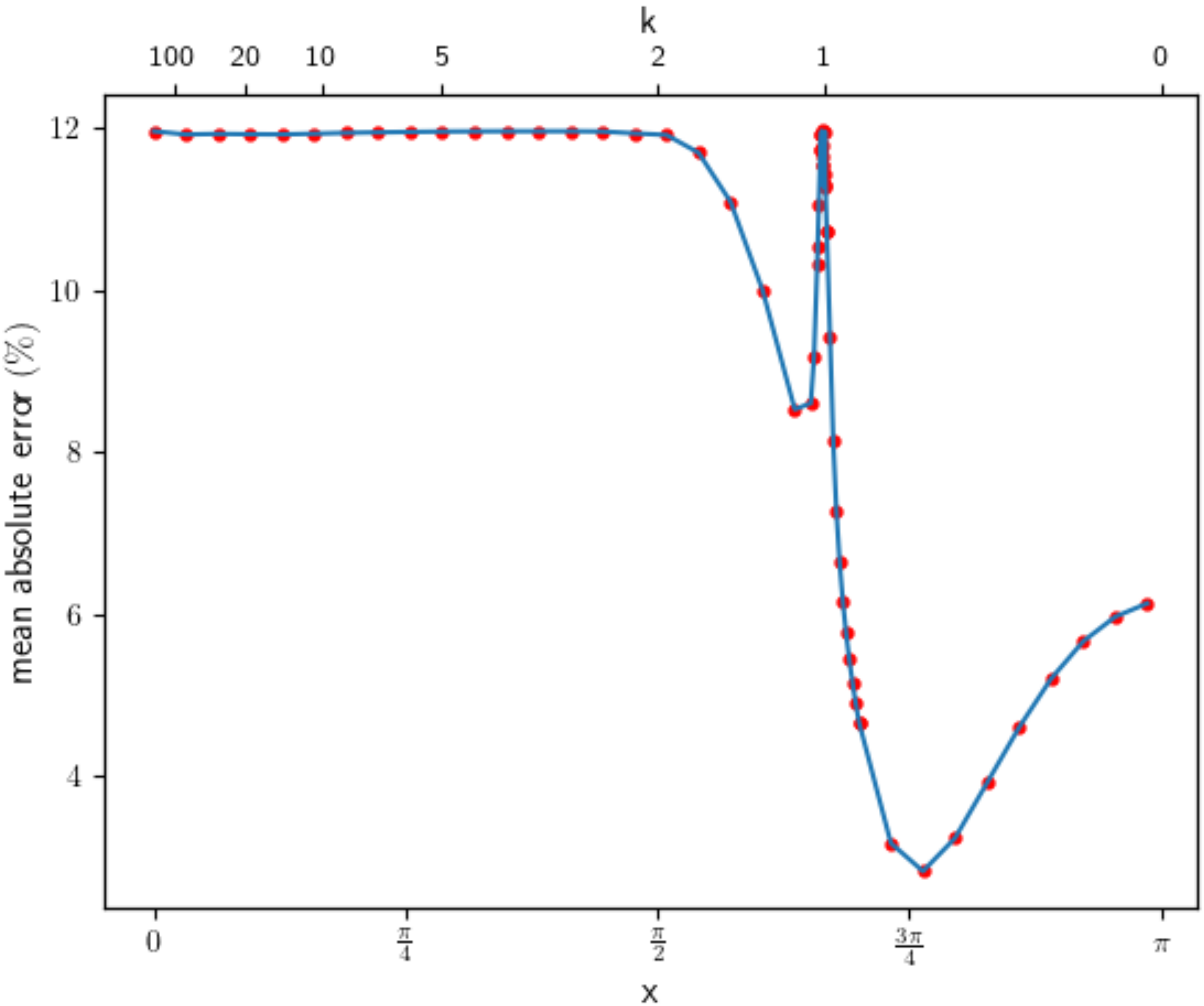
$$k = \frac{2}{3}$$



- Use layerwise relevance propagation to determine best phase
- Approximation formula works almost as well as neural network

Craven, VJ, Kar (2020)

The Shape of Things



Analytic Continuation

- We can analytically continue the level

- Appeal to Morse theory; the integration cycle $\mathcal{C}$ used to compute path integral is decomposed in terms of Lefschetz thimbles

- Path integral evaluated on flat connections depends on one parameter,

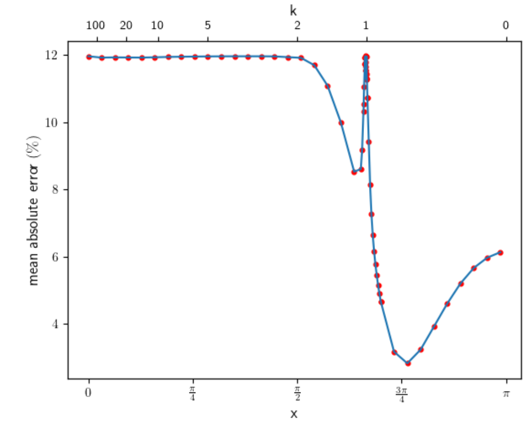
  $\gamma \equiv \dfrac{n-1}{k}$; the semiclassical limit is $n, k \to \infty$

  N.B.: for the Jones polynomial, $n = 2$, $\gamma = k^{-1}$

- As γ is varied, analytically continued integration cycle can pick up contributions from new critical points or lose current ones; these are Stokes phenomena that occur along Stokes lines in complex γ plane

Witten (2010)

The Shape of Things



Plateau: $k > 2$

$$\mathrm{Vol}(S^3 \setminus K) = \langle \mathrm{Vol}(S^3 \setminus K) \rangle$$

this gives 11.97% error for knots up to 16 crossings

corresponds to latent correlations in the dataset

Minimum: near $k = \dfrac{2}{3}$ or $\gamma = \dfrac{3}{2}$

Lefschetz thimbles contain geometric conjugate $SL(2, \mathbb{C})$ connection we expect in semiclassical limit for most knots

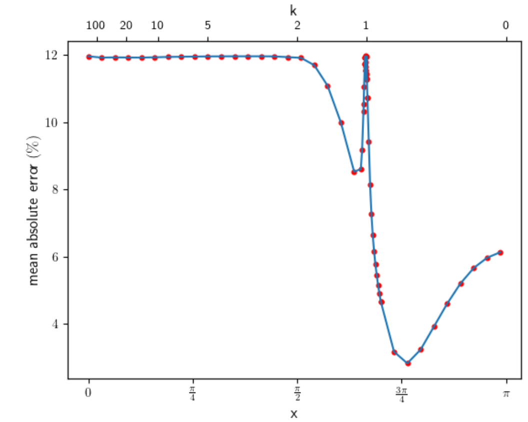
Dip: $0 < k < \dfrac{2}{3}$

knots retain geometric conjugate connection even as $k \ll 1$ or $\gamma \gg 1$

this is explanation for observation that $\log |J_2(K; -1)| \propto \mathrm{Vol}(S^3 \setminus K)$

Dunfield (2000)

The Shape of Things



Ramp: $\frac{2}{3} < k < 2$ interpolating regime

knots lose access to geometric conjugate connection

at $k = \frac{3}{2}$ or $\gamma = \frac{2}{3}$, the geometric conjugate connection of 4_1 enters

Witten (2010)

Spike: near $k = 1$

at integer values of level with $k + 1 \geq n$, the path integral receives contributions only from $SU(2)$ valued critical points

i.e., no analytic continuation is necessary

because we lose knowledge of the geometric conjugate connection, the error becomes high

Conclusion: geometric conjugate connection is crucial to success of approximation formula

Explanation

The approximation formula works well for levels k for which $\mathcal{A}_+$ makes a contribution to the Chern–Simons path integral, and its accuracy increases with fraction of knots in dataset that receive such a contribution

$$Z \sim e^{iS(\mathcal{A}_+)}\left(1 - e^{2\pi i k}\right)$$

Craven, VJ, Kar (2020)

Polynomials from Braids

- Braid group has generators that satisfy algebra

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \ , \quad 1 \leq i < n-1 \ ;$$
$$[b_i, b_j] = 0 \ \text{ if } \ |i-j| > 1$$

- Two closed braids equivalent iff related by Markov moves

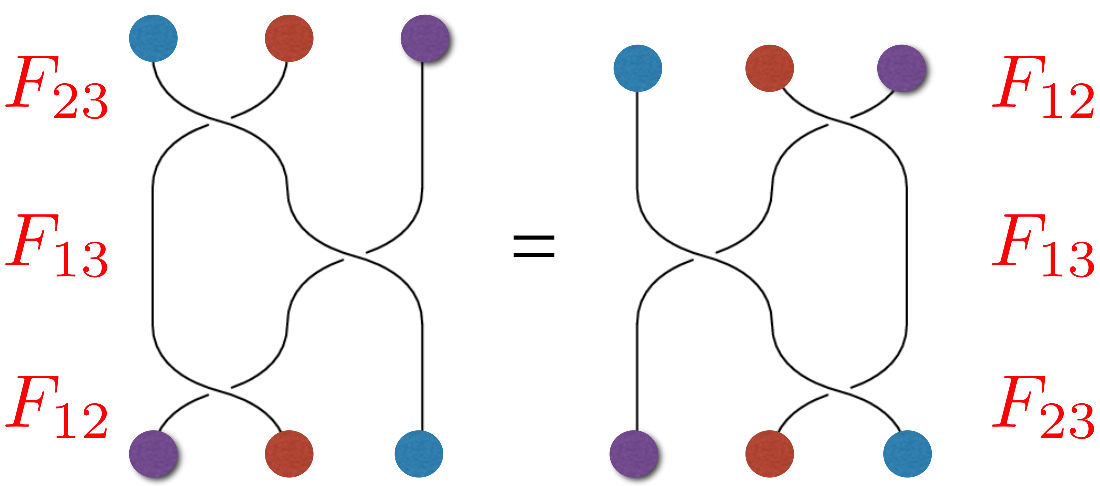
$$AB \to BA \ \text{ for } \ A, B \in B_n \ ;$$
$$A \to Ab_n^{\pm 1} \ \text{ for } \ A \in B_n \ , \quad b_n \in B_{n+1} \quad \text{and inverse move}$$

- Obtain invariant polynomials by constructing representation of braid group and then defining Markov move independent polynomials

- Yang–Baxter equation as braid relation

$$F : X \times X \to X \times X$$

$$F_{12} \circ F_{13} \circ F_{23} = F_{23} \circ F_{13} \circ F_{12}$$



Exactly Solvable Models

- Let G_i be representation of $b_i \in B_n$

- Suppose we have $\phi(AB) = \phi(BA)$ for $A, B \in B_n$;

  $\phi(AG_n) = \tau\phi(A)$, $\phi(AG_n^{-1}) = \overline{\tau}\phi(A)$ for $A \in B_n$, $G_n \in B_{n+1}$

  $\tau = \phi(G_i)$, $\overline{\tau} = \phi(G_i^{-1})$ $\forall\ i$

- Invariant polynomial is $\alpha(A) = (\tau\overline{\tau})^{-(n-1)/2} \left(\frac{\overline{\tau}}{\tau}\right)^{e(A)/2} \phi(A)$

  $A \in B_n$, $e(A)$ is sum of exponents of b_i in A

- Can compute these in N-state vertex model

$$\phi(A) = \mathrm{Tr}(H \cdot A) \qquad h = \tau\,\mathrm{diag}(1, q, \ldots, q^{N-1})$$

$$\tau(q) = \Big(\sum_{m=0}^{N-1} q^m\Big)^{-1} , \quad \overline{\tau} = \tau(q^{-1}) \qquad H = \underbrace{h \otimes \cdots \otimes h}_{n}$$

$$G_i = \mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{i-1} \otimes R_{i,i+1} \otimes \mathbf{1}_{i+1} \otimes \cdots \otimes \mathbf{1}_n$$

Akutsu, Wadati (1987)

R-matrix

- $N = 2$, six-vertex model computes Jones polynomials

- $N = 3$, nineteen-vertex model computes adjoint Jones polynomials

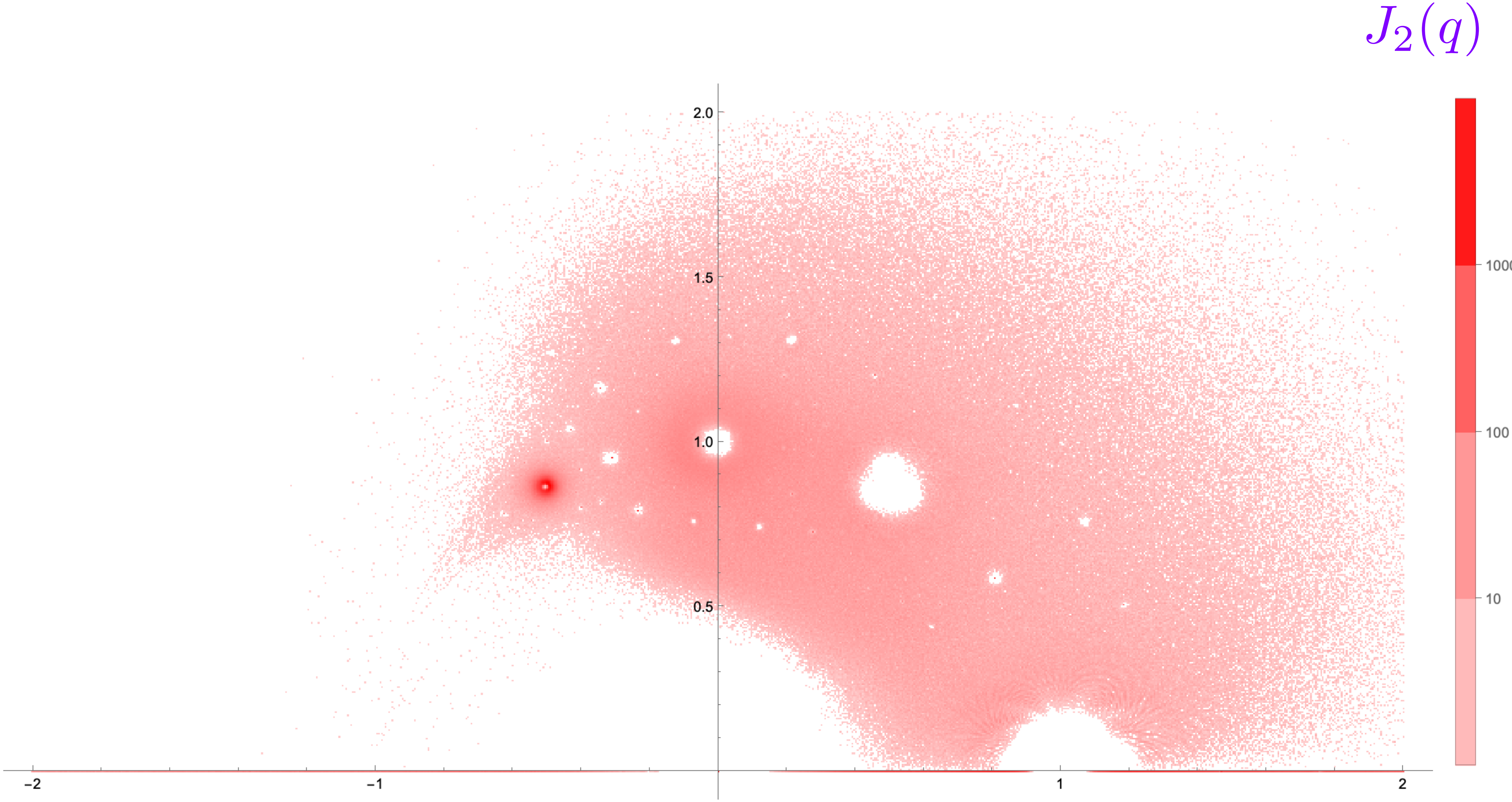
- n strand braid corresponds to a $N^n \times N^n$ R-matrix satisfying Yang–Baxter equation

- We have computed:

11,941	$J_3(q)$	up to 13 crossings (92%)
18,353	$J_3(q)$	at 14 crossings (39%)
147,022	$J_3(q)$	at 15 crossings (58%)

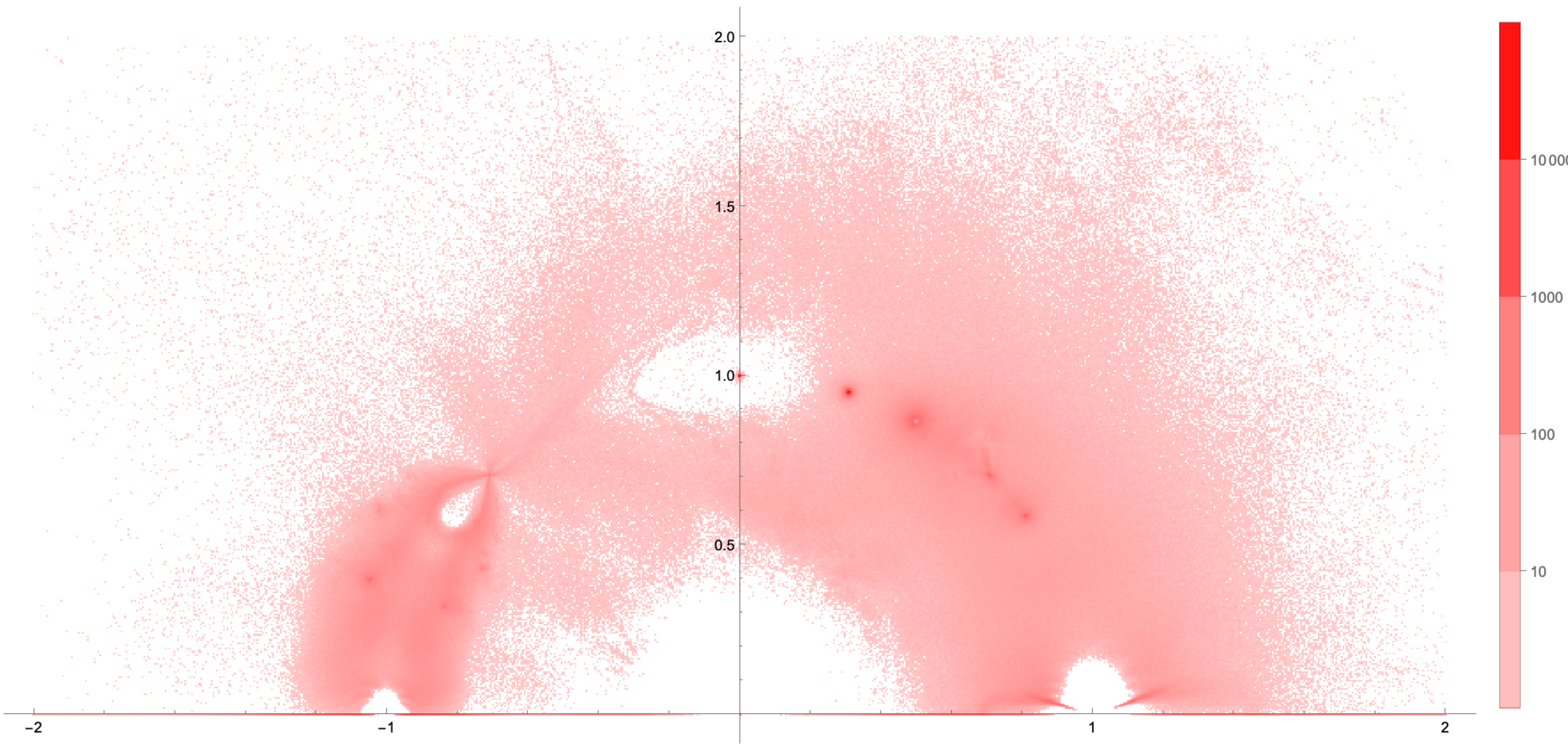
- calculations up to 7 strands

Zeroes

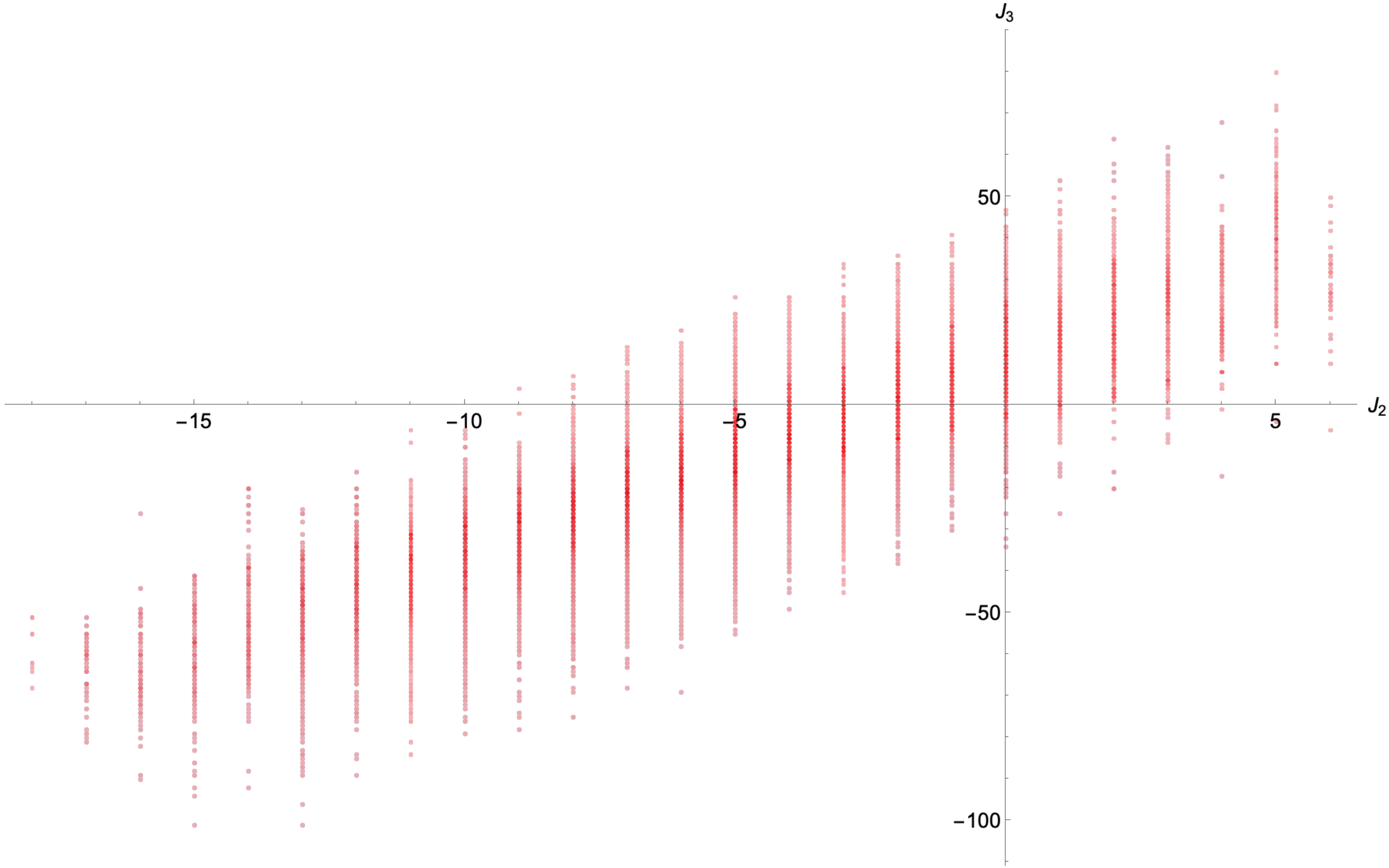


Zeroes

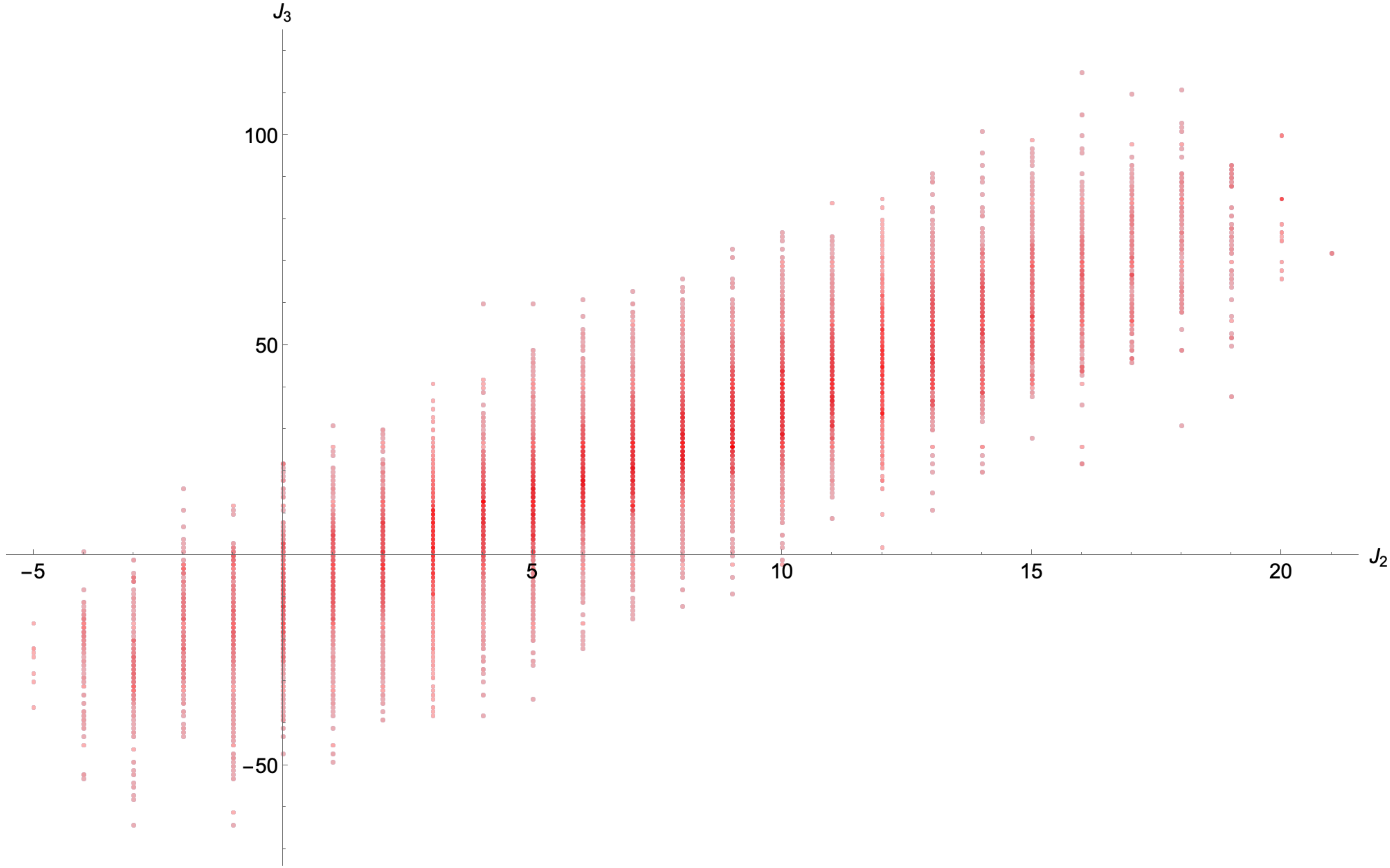
$J_3(q)$



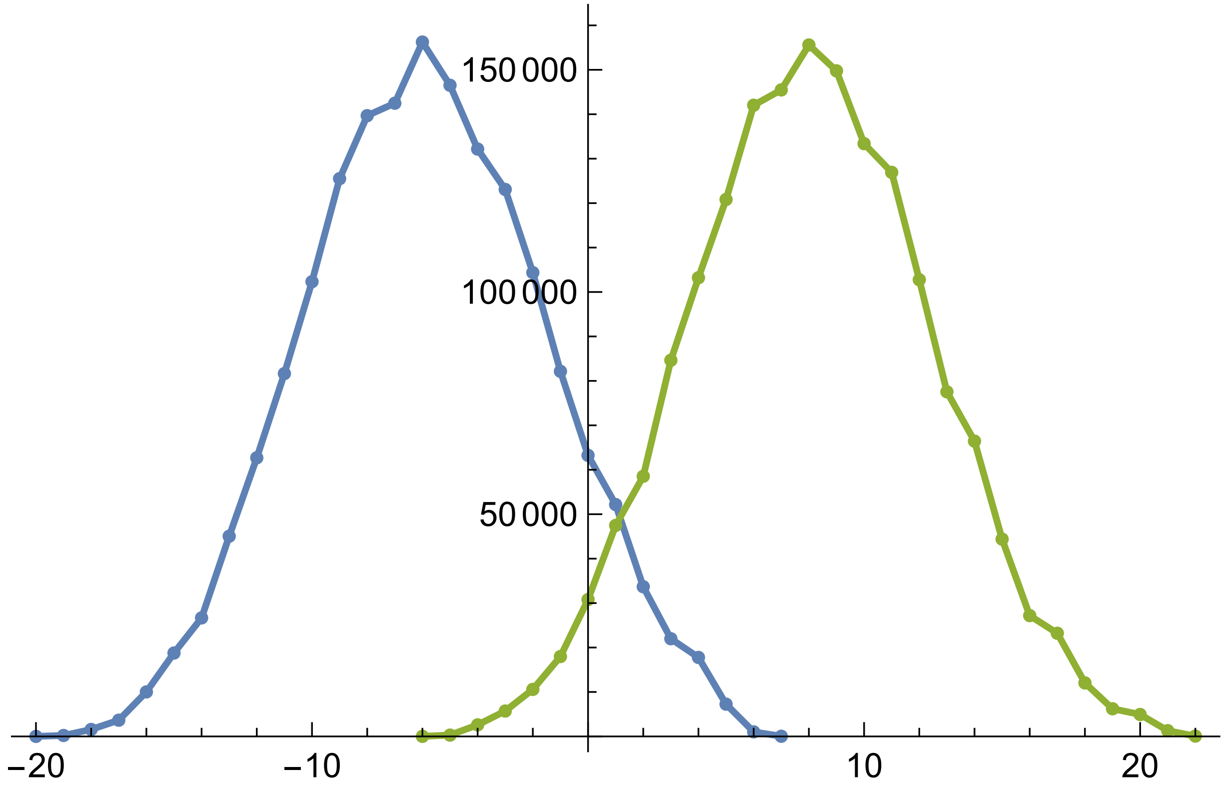
Minimum Degree



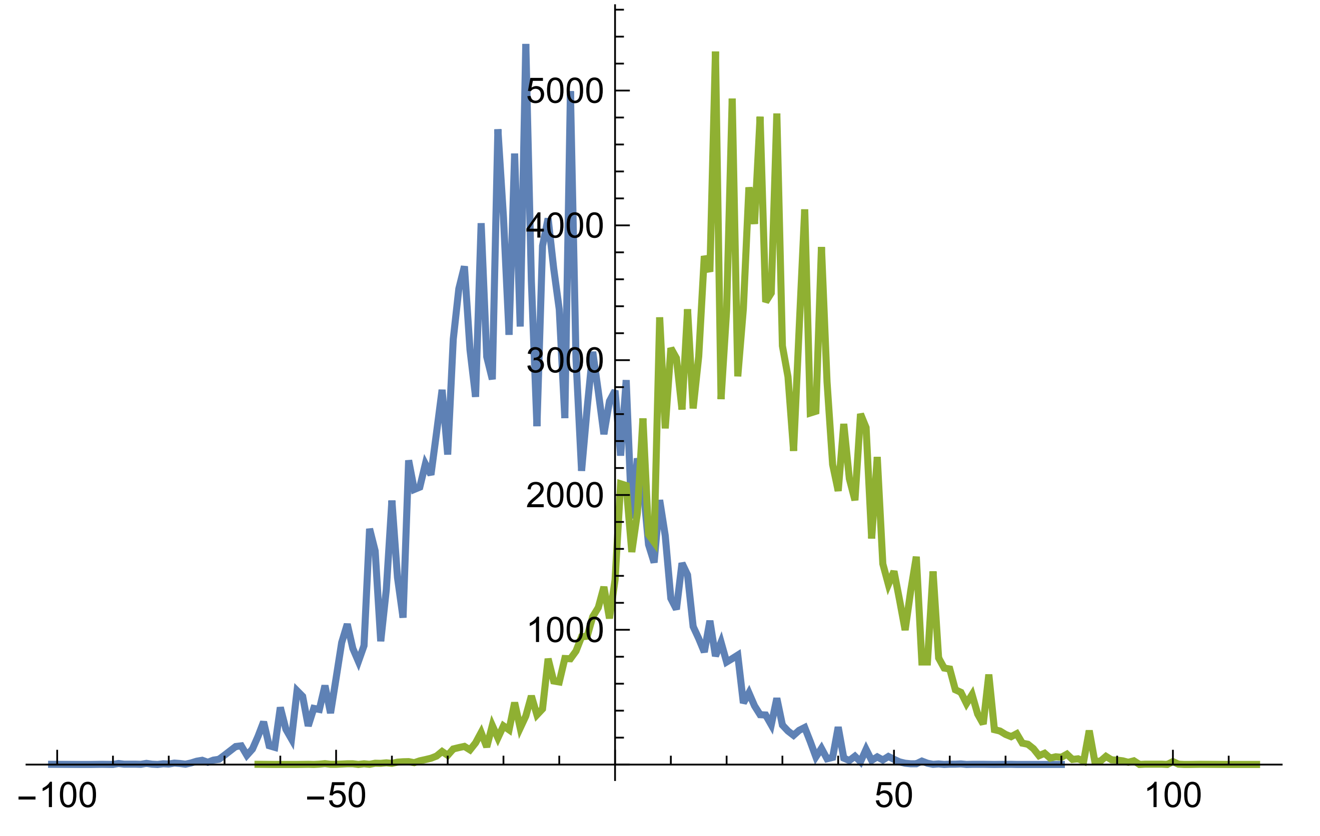
Maximum Degree



Degrees

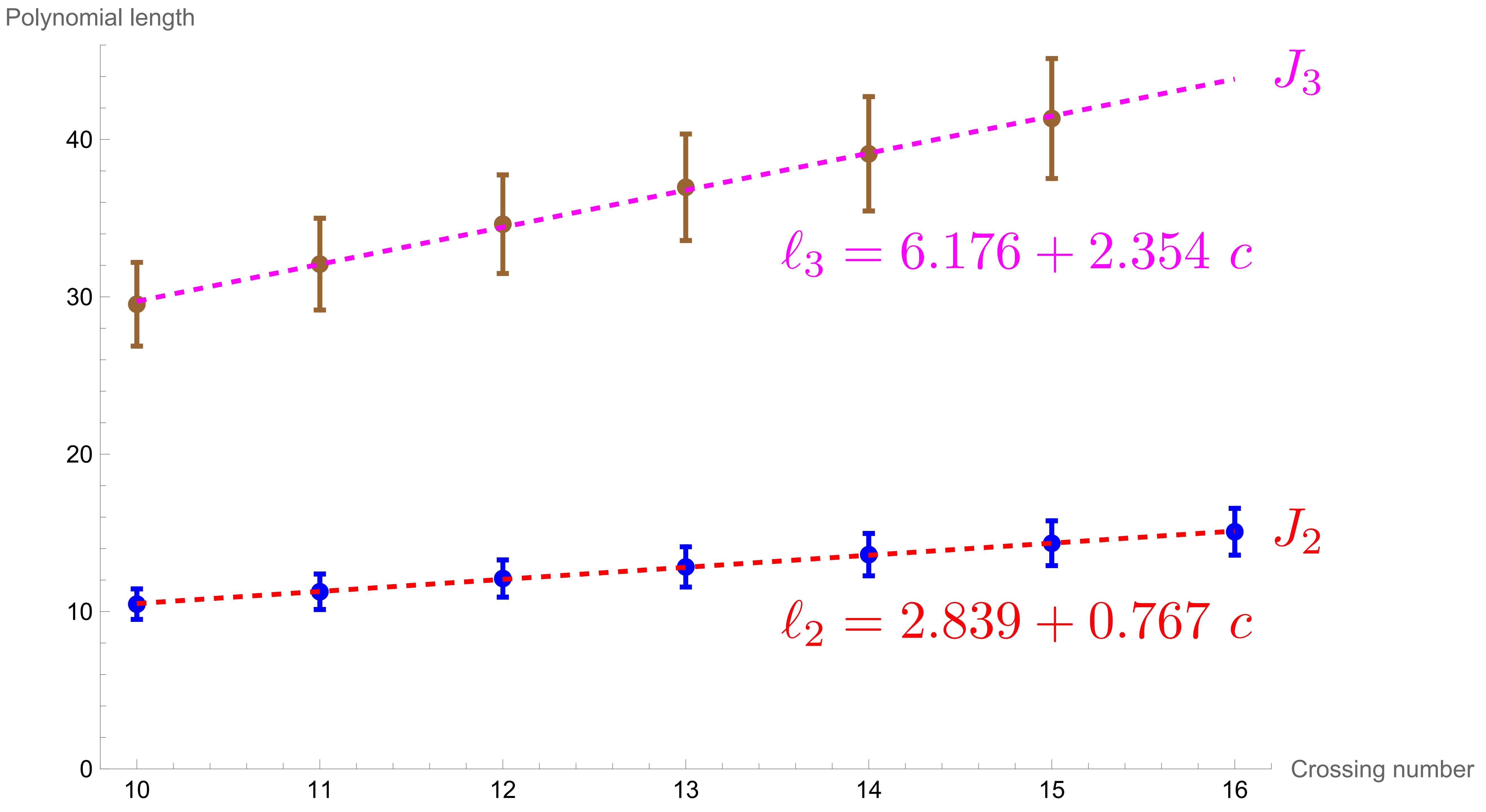


$J_2(q)$



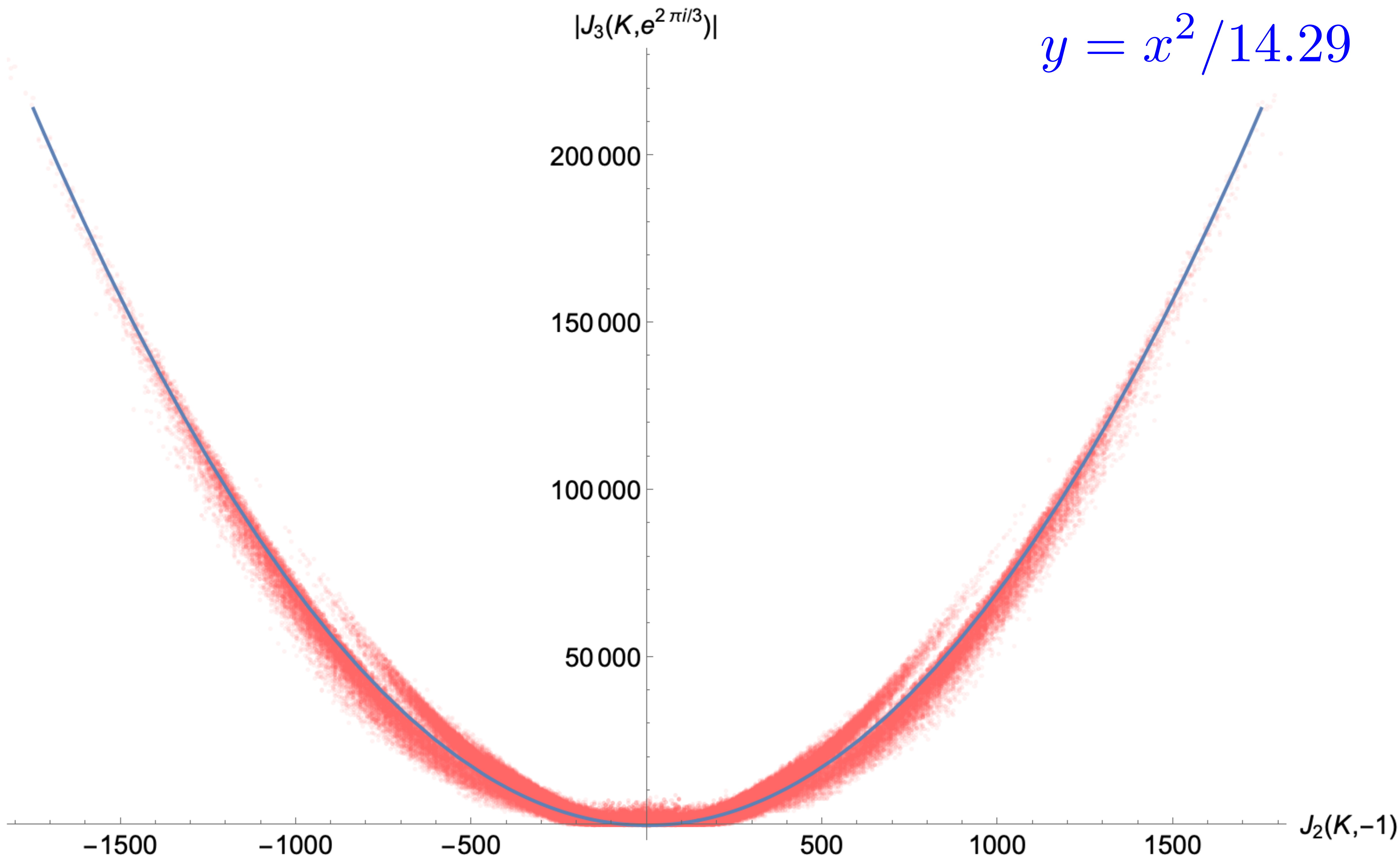
$J_3(q)$

Length

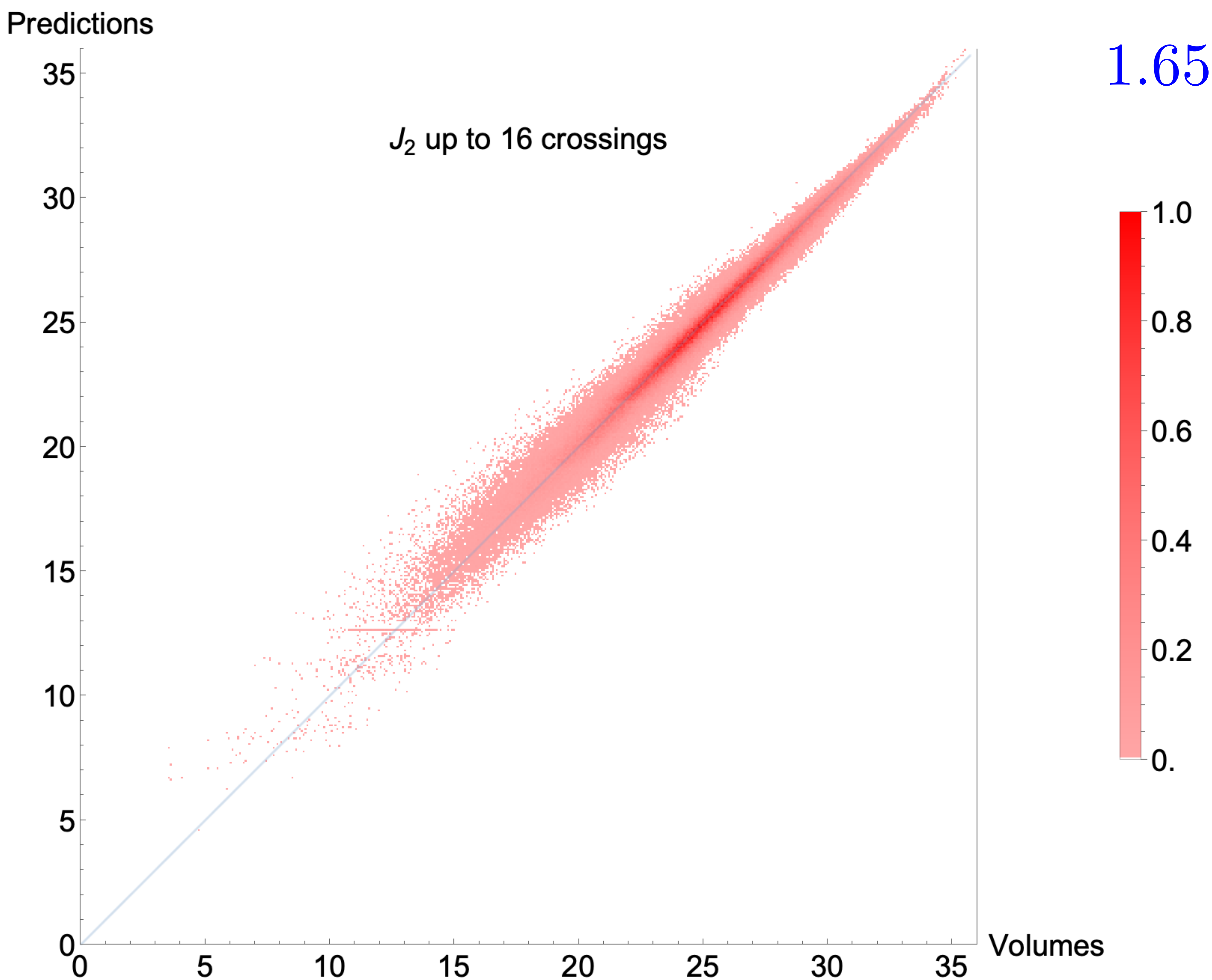


Polynomial length
40
30
20
10
0
10
11
12
13
14
15
16
Crossing number
J_3
$\ell_3 = 6.176 + 2.354\ c$
J_2
$\ell_2 = 2.839 + 0.767\ c$

$|J_3(e^{2\pi i/3})|$ vs. $J_2(-1)$

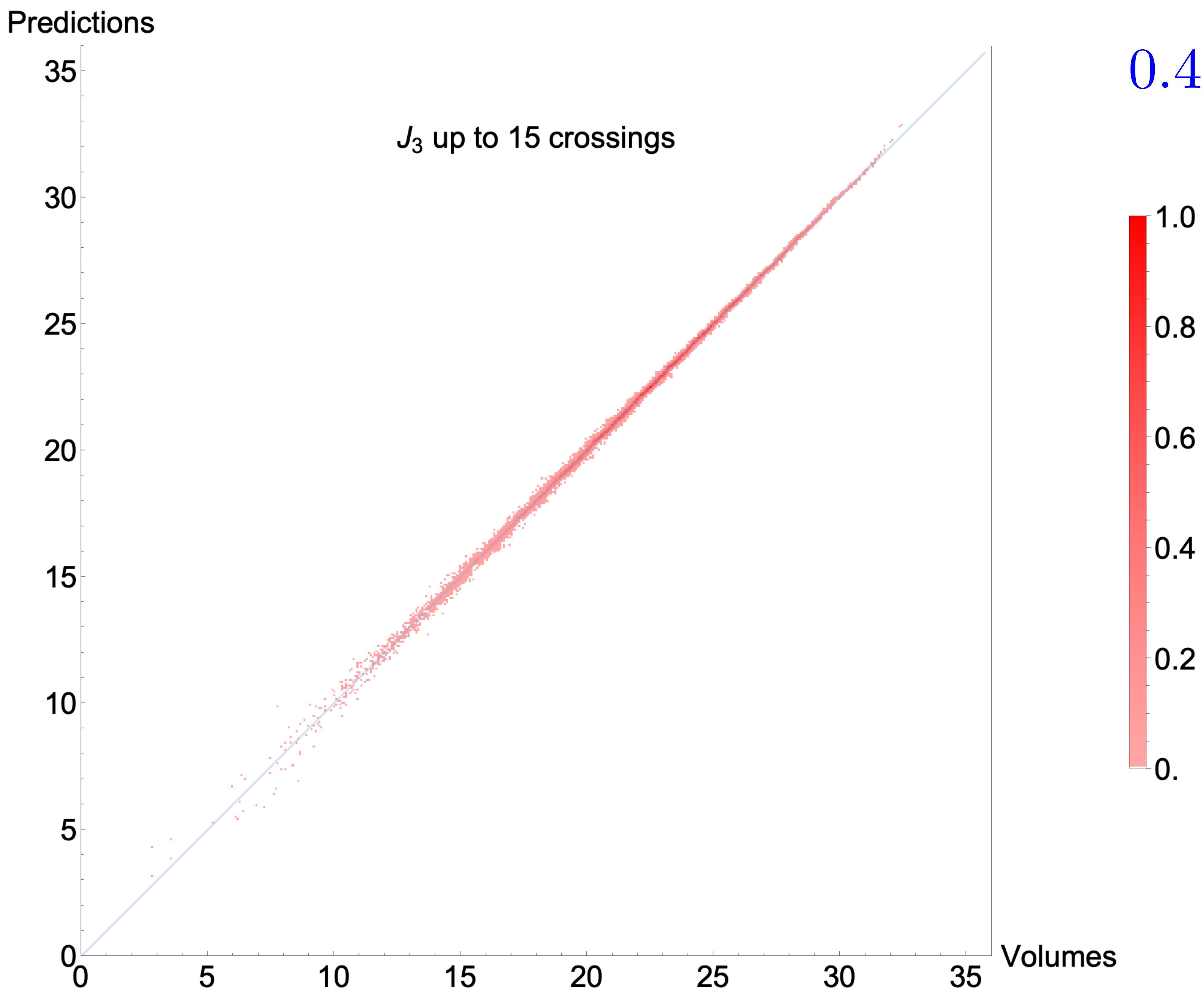


Jones to Volume



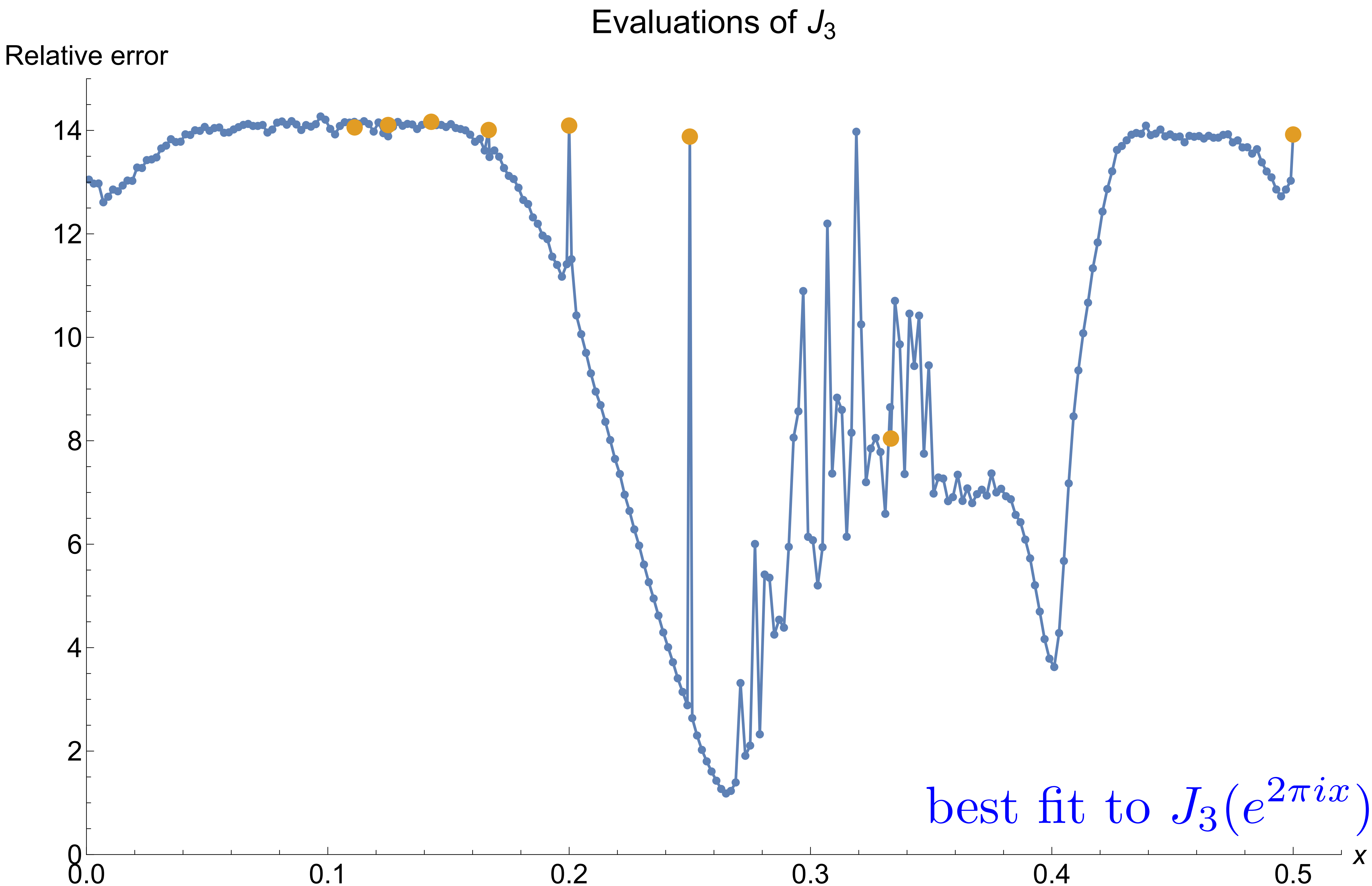
1.65% error

Adjoint Jones to Volume



0.40% error

Phase Evaluations



Best Phase

- The best phase for evaluating the adjoint Jones polynomial is $e^{8\pi i/15}$
- This corresponds to Chern–Simons level $k = \dfrac{7}{4}$ (or $\gamma = \dfrac{8}{7}$)

Best Phase Conjecture

- The best phase for evaluating the adjoint Jones polynomial is $e^{8\pi i/15}$
- This corresponds to Chern–Simons level $k = \frac{7}{4}$ (or $\gamma = \frac{8}{7}$)
- Since we have two data points, we can guess a formula for higher colors

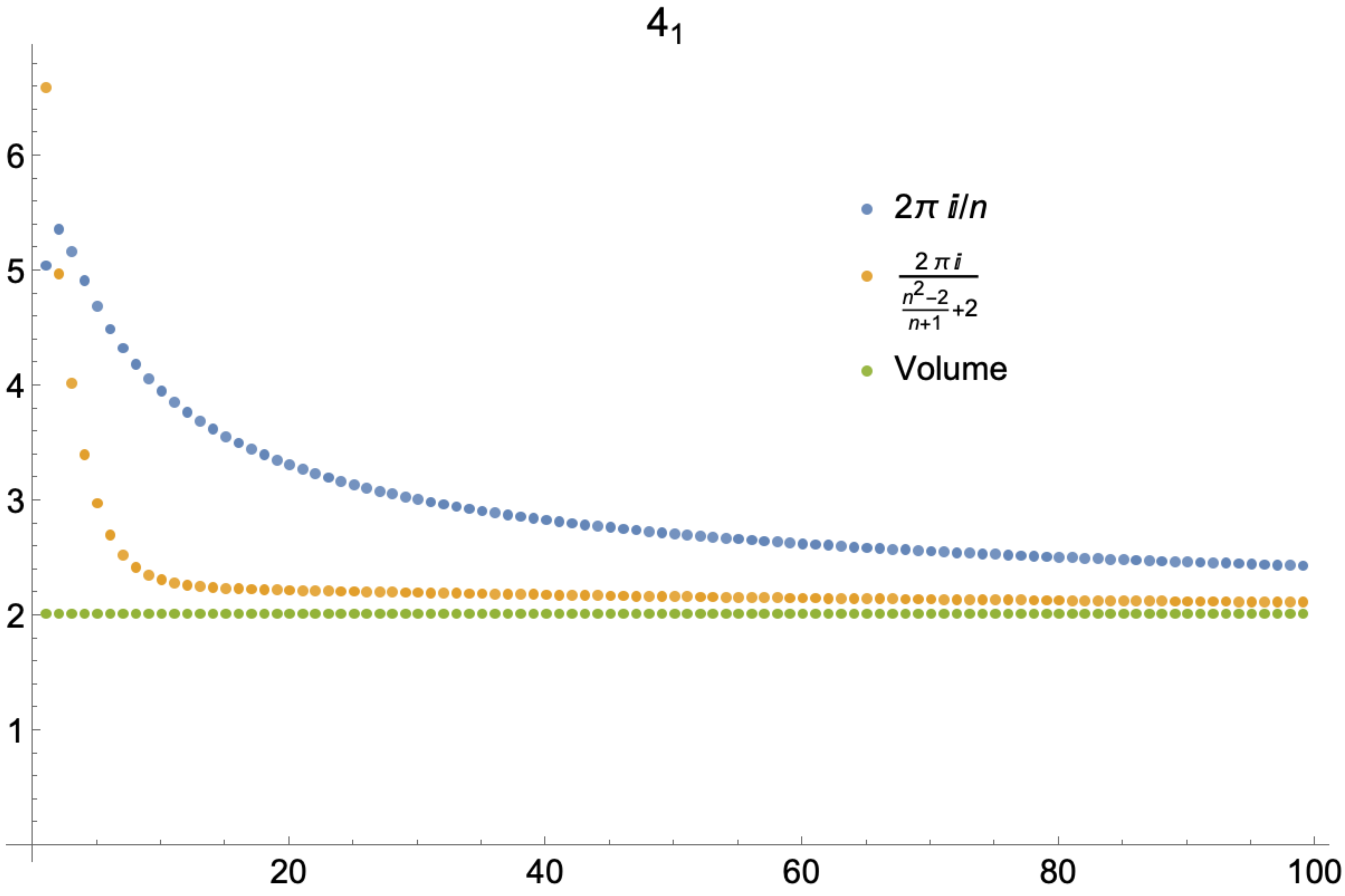
$$\frac{x}{x_{\text{vc}}} = \frac{n+1}{n+2} \qquad n \to \infty : \qquad \frac{x}{x_{\text{vc}}} \to 1 - \frac{1}{n}$$

$$q = \exp\left(2\pi i \frac{n+1}{n(n+2)}\right) \qquad q \to \exp\left(2\pi i \left(\frac{1}{n} - \frac{1}{n^2}\right)\right)$$

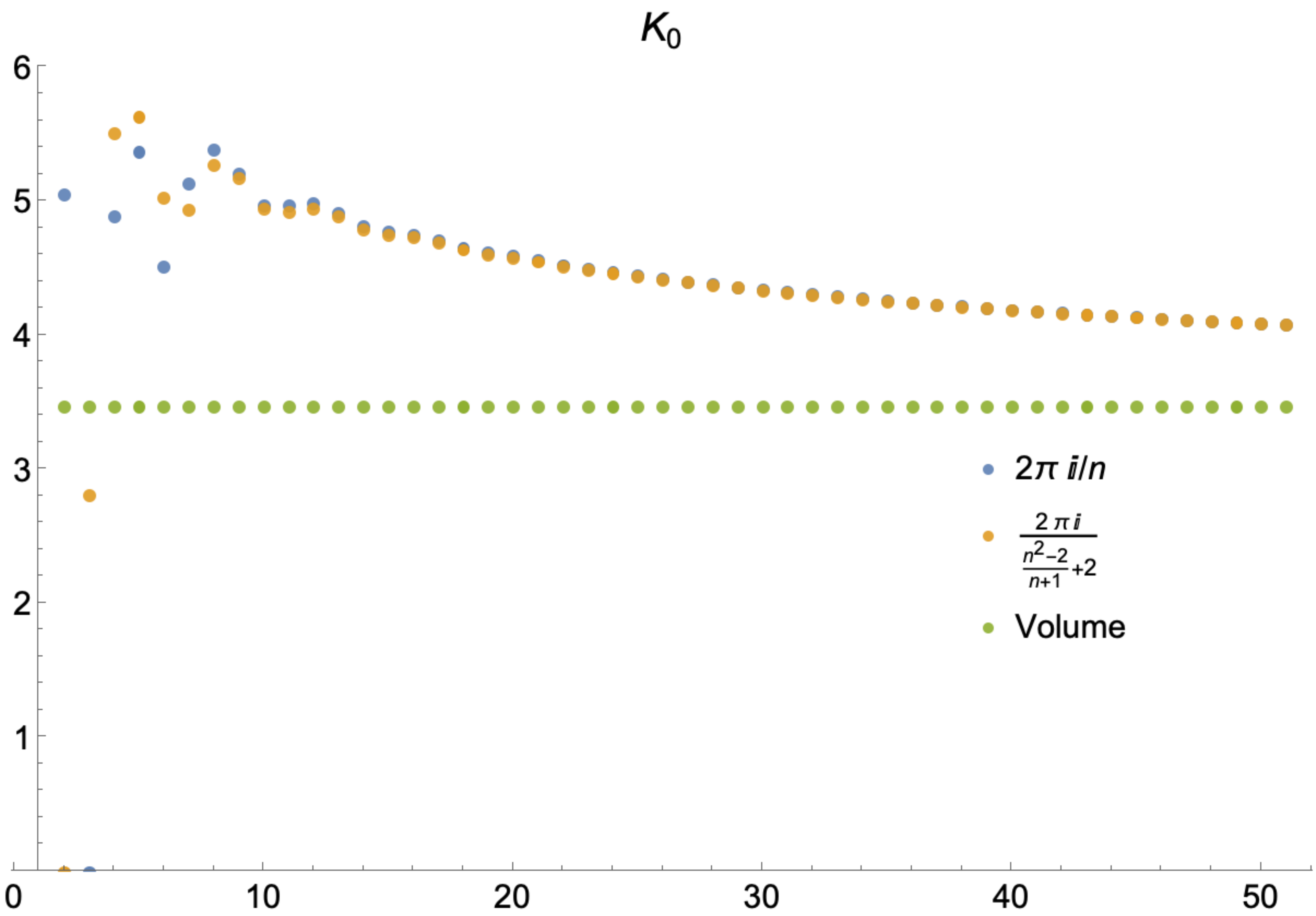
$$k = \frac{n^2-2}{n+1} \qquad k \to (n-1) - \frac{1}{n}$$

$$\gamma = \frac{n^2-1}{n^2-2} \qquad \gamma \to 1 + \frac{1}{n^2}$$

Best Phase Conjecture



Best Phase Conjecture



Summary

- Calculated a large (though incomplete) dataset of adjoint Jones polynomials

- Consistent with the naïve expectation of the volume conjecture, adjoint polynomials predict the volume better than fundamental polynomials

- Non-trivial because convergence at large color not necessarily monotonic

- Increasing crossing number improves prediction

- Conjectured color dependent best phase for prediction

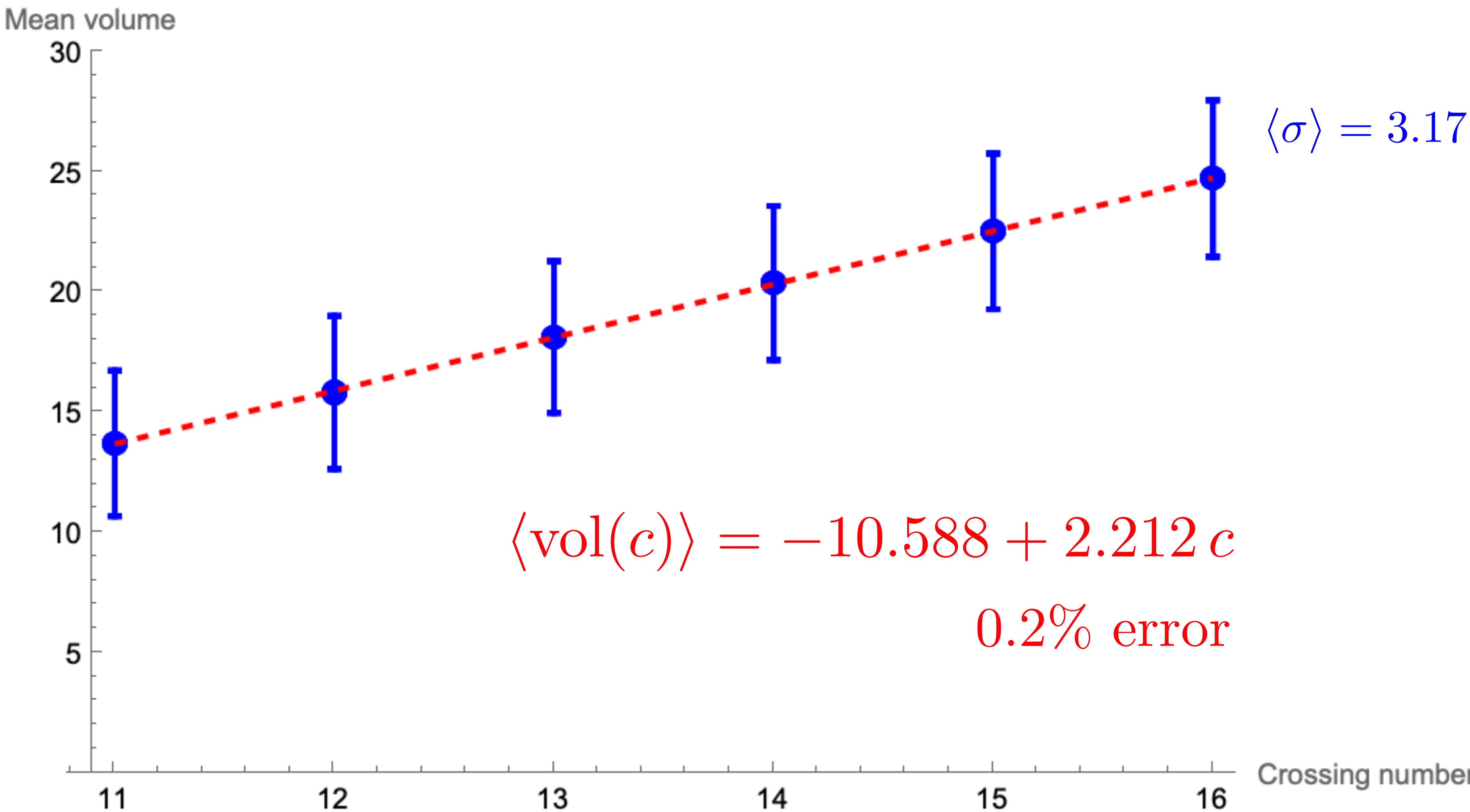
Prospectus

- Complete calculation of adjoint polynomials
- Calculate datasets of polynomials for higher colors
- Check volume conjecture: do predictions hold?

  is convergence monotonic?
- A better organizing principle for (hyperbolic) knots

  e.g., minimum number of ideal tetrahedra needed to compute volume

Organizing Principle

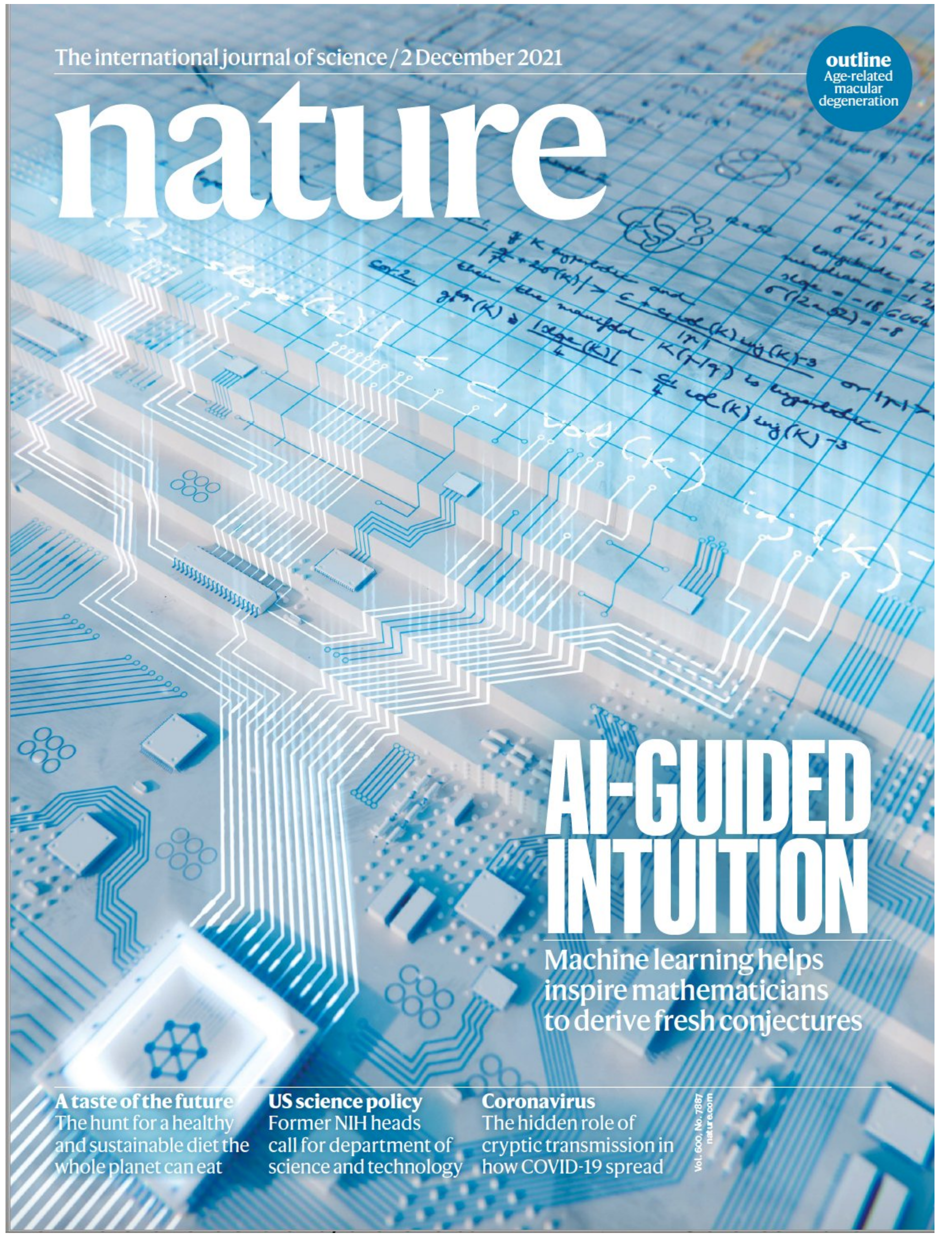


$\langle \sigma \rangle = 3.17$

$$\langle \mathrm{vol}(c) \rangle = -10.588 + 2.212\, c$$

0.2% error

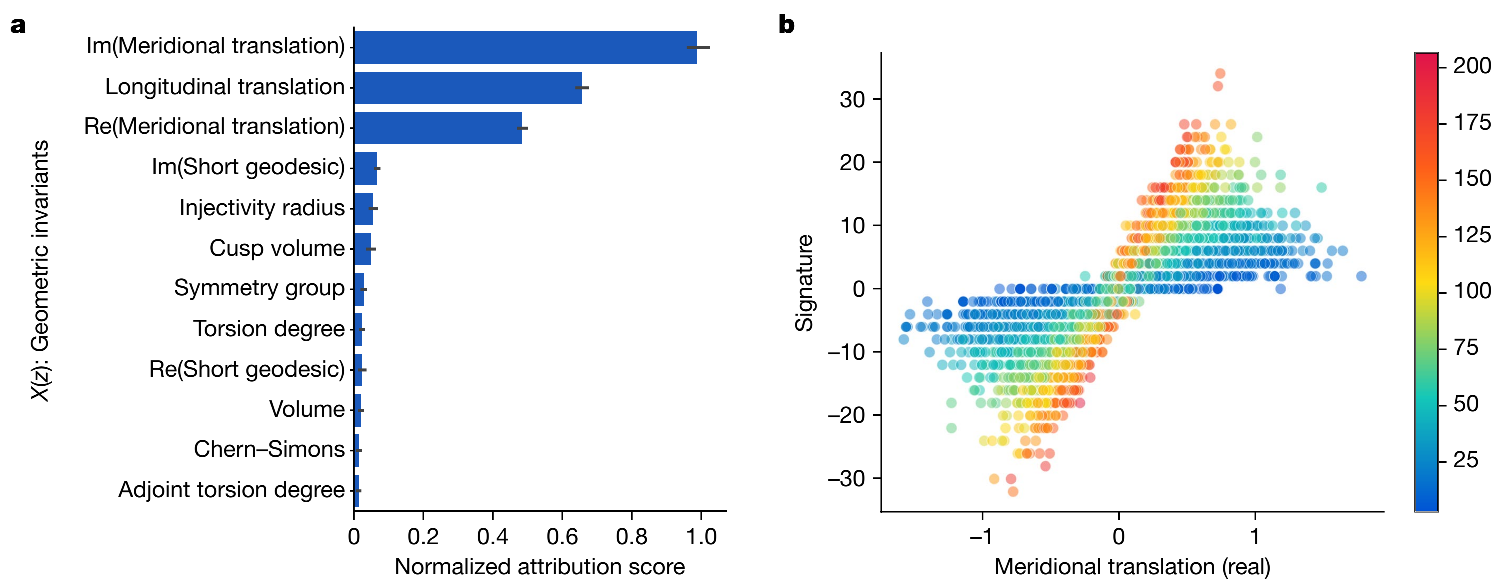
Other Experiments



- Obtain a theorem in knot theory from saliency analysis of ML experiments

Theorem: There exists a constant c such that for every hyperbolic knot,

$$|2\sigma(K) - \mathrm{slope}(K)| \leq c\,\mathrm{Vol}(K)\,\mathrm{inj}(K)^{-3}$$



Davies, et al. (2021)

- See also: Hughes (2016)

  Levit, Hajij, Sazdanovic (2019)

  Gukov, Halverson, Ruehle, Sulkowski (2020)

  Craven, Hughes, VJ, Kar (2021, 2022)

  Gukov, Halverson, Manolescu, Ruehle (2023)

Poincaré Conjecture

- A homotopy *n*-sphere has the same homotopy and homology groups as S^n

- **Conjecture:** Any homotopy *n*-sphere is homeomorphic, PL-isomorphic, diffeomorphic to S^n

  — homeomorphic is always true

  — piecewise linear isomorphic is true for $n \neq 4$

  — $n = 1$ is trivial

  — $n = 2$ proved by Poincaré (1907) and Koebe (1907)

  — $n \geq 5$ wins Fields Medal for Smale (1966); *cf.* Newman (1966)

  — $n = 4$ wins Fields Medal for Freedman (1986)

  — $n = 3$ wins Fields Medal for Perelman (2006, declined)

SPC4

- **Conjecture:** Any manifold homotopy equivalent to S^n is also diffeomorphic to S^n

isomorphism of smooth manifolds, *i.e.*, f, f^{-1} continuously differentiable

- False generally, but true if $n = 1, 2, 3, 5, 6, 12, 56, 61$

- $n = 7$: 28 different smooth structures on the sphere

Milnor (1956)

Remark: These are interpreted as gravitational instantons

Witten (1985)

dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
# structures	1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

Observation: spikes at $n = 3 \bmod 4$

Kervaire, Milnor (1963)

SPC4

- **Conjecture:** Any manifold homotopy equivalent to S^n is also diffeomorphic to S^n

  isomorphism of smooth manifolds, *i.e.*, f, f^{-1}continuously differentiable

- False generally, but true if $n = 1, 2, 3, 5, 6, 12, 56, 61$

- $n = 7:$ 28 different smooth structures on the sphere Milnor (1956)

  Remark: These are interpreted as gravitational instantons Witten (1985)

- $n = 4:$ answer unknown; equivalent to PL-isomorphic statement

  [also unknown for $n = 126$, conjectured to be false for even dimensions $n \geq 64$]

- If certain knots have certain topological invariants, this supplies counterexamples to SPC4

$$S^3 = \partial B_4 \ , \quad K = \partial \Sigma$$

$$K \subset S^3 \ , \quad \Sigma \subset B_4$$

SPC4: Any smooth manifold B with $\partial B = S^3$ homotopy equivalent to B_4 is also diffeomorphic to B_4

THANK YOU!

Department:
Science and Innovation
REPUBLIC OF SOUTH AFRICA

NRF
National
Research
Foundation