Non-perturbative Phase Diagram of $2d\ \mathcal{N} = (2,2)$ Super Yang-Mills Theory

ANOSH JOSEPH
University of the Witwatersrand

1st Joint India/South Africa String Meeting
14th Joburg Workshop on String Theory
University of the Witwatersrand
December 15 - 20, 2024

Based on

Phys. Rev. D **110** (2024) 5, 054507

2312.04980 [hep-lat]

PHYSICAL REVIEW D **110,** 054507 (2024)

Nonperturbative phase diagram of two-dimensional $\mathcal{N} = (2,2)$ super-Yang-Mills theory

Navdeep Singh Dhindsa,[1,2,*] Raghav G. Jha,[3,†] Anosh Joseph,[4,‡] and David Schaich[5,§]

Lattice as a Machinery for Non-perturbative Physics

Holographic conjecture has **non-perturbative physics** in it

A version of the conjecture:

Weakly coupled
gravitational theories
in $D+1$ dimensions

Strongly coupled $\mathrm{SU}(N)$
super Yang-Mills
in D dimensions

Lattice as a Machinery for Non-perturbative Physics

How to study strongly coupled SYM?

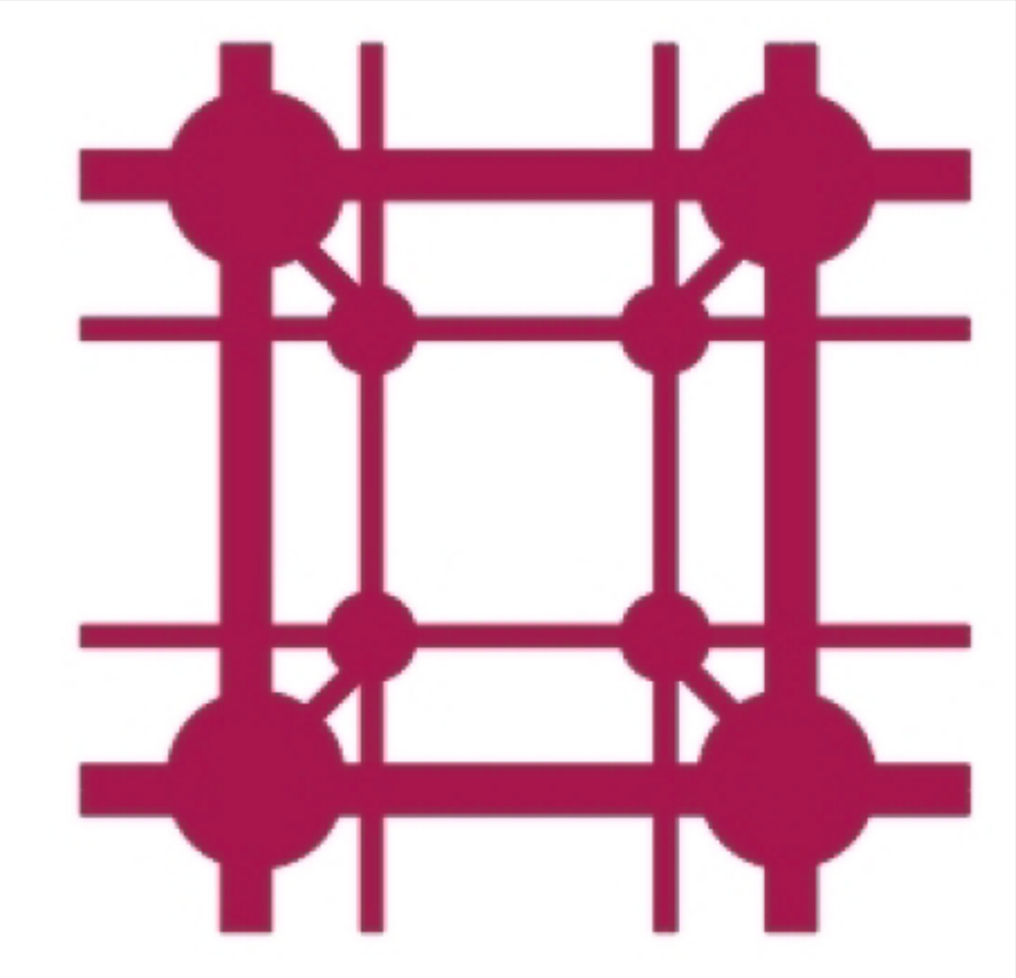
At large and finite N?

Really hard!

Use spacetime lattice as a tool

Use **lattice gauge theory**

Lattice and SUSY



Naive discretization breaks SUSY

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu$$

Appears that SUSY and lattice are not compatible

There exists a clever way to bring them together

Combine internal symmetry with spacetime symmetry [Witten 1988]

Lattice and SUSY

Resulting lattice is supersymmetric

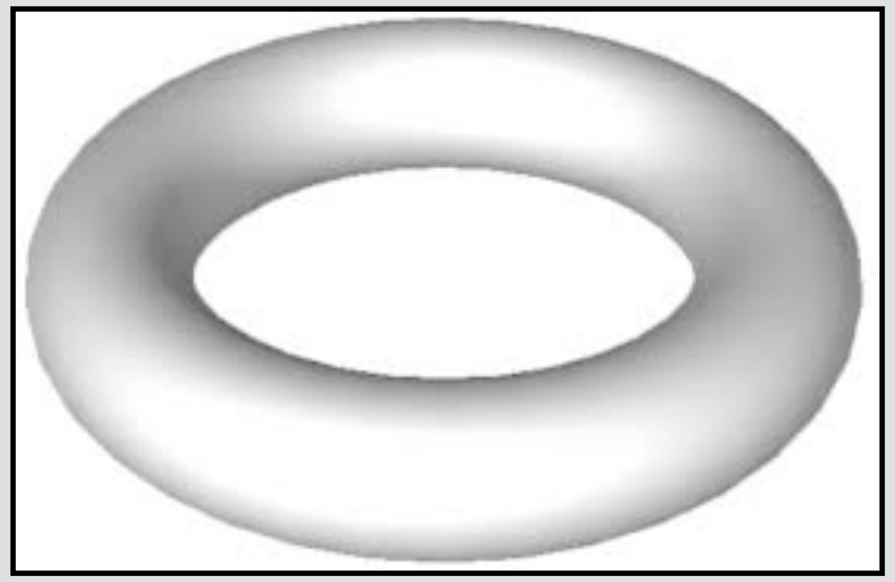
Can preserve a subset of SUSY at finite lattice spacing

Requires 2^d supercharges in d spacetime dimensions

This condition is satisfied for many interesting theories:

- $\mathcal{N} = 4$ SYM in $4d$
- Dimensional reductions of this theory
- Includes $\mathcal{N} = (8,8)$ in $2d$

$\mathcal{N} = (8{,}8)$ SYM in $2d$



Has a well defined holographic dual

Can consider holography at finite T

At low T, gravitational system contains various types of black solutions

- Homogeneous $\mathrm{D1}$ (black string) solutions
- They wrap around spatial circle
- Localized $\mathrm{D0}$ (black hole) solutions

$\mathcal{N} = (8,8)$ SYM in $2d$

At low T

First-order **Gregory-Laflamme** phase transition between two solutions

Transition is captured in the dual gauge theory

A “spatial deconfinement” transition

Magnitude of **spatial Wilson line** serves as **order parameter**

$\mathcal{N} = (2,2)$ SYM in $2d$

This work

Focus on a **gauge theory** system with **lower number of supercharges**

Holographic **dual** has **not** been **constructed** yet

How much does it resemble to its 16 supercharge counterpart?

How does reduction in SUSY affect **holographic features**?

$\mathcal{N} = (2,2)$ SYM in $2d$

Obtained from $\mathcal{N} = 1$ SYM in $4d$

$$S = \int d^4x \; \mathcal{L} = \int d^4x \; \mathrm{Tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\lambda} \gamma^{\mu} D_{\mu} \lambda \right)$$

$\mathcal{N} = (2{,}2)$ SYM in $2d$

Symmetries of the $4d$ theory:

$\mathrm{SO}(4)_E \times \mathrm{U}(1)$

Euclidean rotational symmetry

Chiral $\mathrm{U}(1)$ R symmetry: $\lambda \;\; \rightarrow \;\; e^{-i\theta\gamma_5}\lambda$

$\mathcal{N} = (2,2)$ SYM in $2d$

Dimensionally reduce the $4d$ theory to $2d$

Symmetry group becomes

$$\mathrm{SO}(2)_E \times SO(2)_{R_1} \times SO(2)_{R_2}$$

Chiral symmetry of $4d$ theory

$2d$ Lorentz symmetry

Internal symmetry

$\mathcal{N} = (2,2)$ SYM in $2d$

Twist the theory

Just a change of variables in flat Euclidean spacetime

Define a **new rotation group**

$$\mathrm{SO}(2)' \equiv \mathrm{diag}\left(\mathrm{SO}(2)_E \times \mathrm{SO}(2)_{R_1}\right)$$

Twisted rotation group

$\mathcal{N} = (2,2)$ SYM in $2d$

Supercharges of the theory decomposes into

$$Q,\ Q_a,\ Q_{ab}$$

Fermions

$$\eta,\ \ \psi_a,\ \ \chi_{ab}$$

Bosons

$$A_a,\ \ X_1,\ \ X_2\ \ \ \to\ \ \ \mathcal{A}_a = A_a + iX_a$$

$\mathcal{N} = (2,2)$ SYM in $2d$

After the twist, action can be written as

$$S = Q\Psi$$

$$\Psi = \frac{N}{4\lambda}\int d^2x \ \mathrm{Tr}\left(\chi_{ab}\mathcal{F}_{ab} + \eta[\bar{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d\right)$$

$\mathcal{N} = (2,2)$ SYM in $2d$

SUSY transformations:

$$Q\mathcal{A}_a = \psi_a$$

$$Q\psi_a = 0$$

$$Q\chi_{ab} = -\bar{\mathcal{F}}_{ab}$$

$$Q\bar{\mathcal{A}}_a = 0$$

$$Q\eta = d$$

$$Qd = 0$$

d: Auxiliary field

$$d = \sum_a [\bar{\mathcal{D}}_a, \mathcal{D}_a]$$

$\mathcal{N} = (2,2)$ SYM in $2d$

Action:

$$S = \frac{N}{4\lambda}\int d^2x \ \mathrm{Tr} \ \left(-\bar{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}[\bar{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\bar{\mathcal{D}}_a\psi_a\right)$$

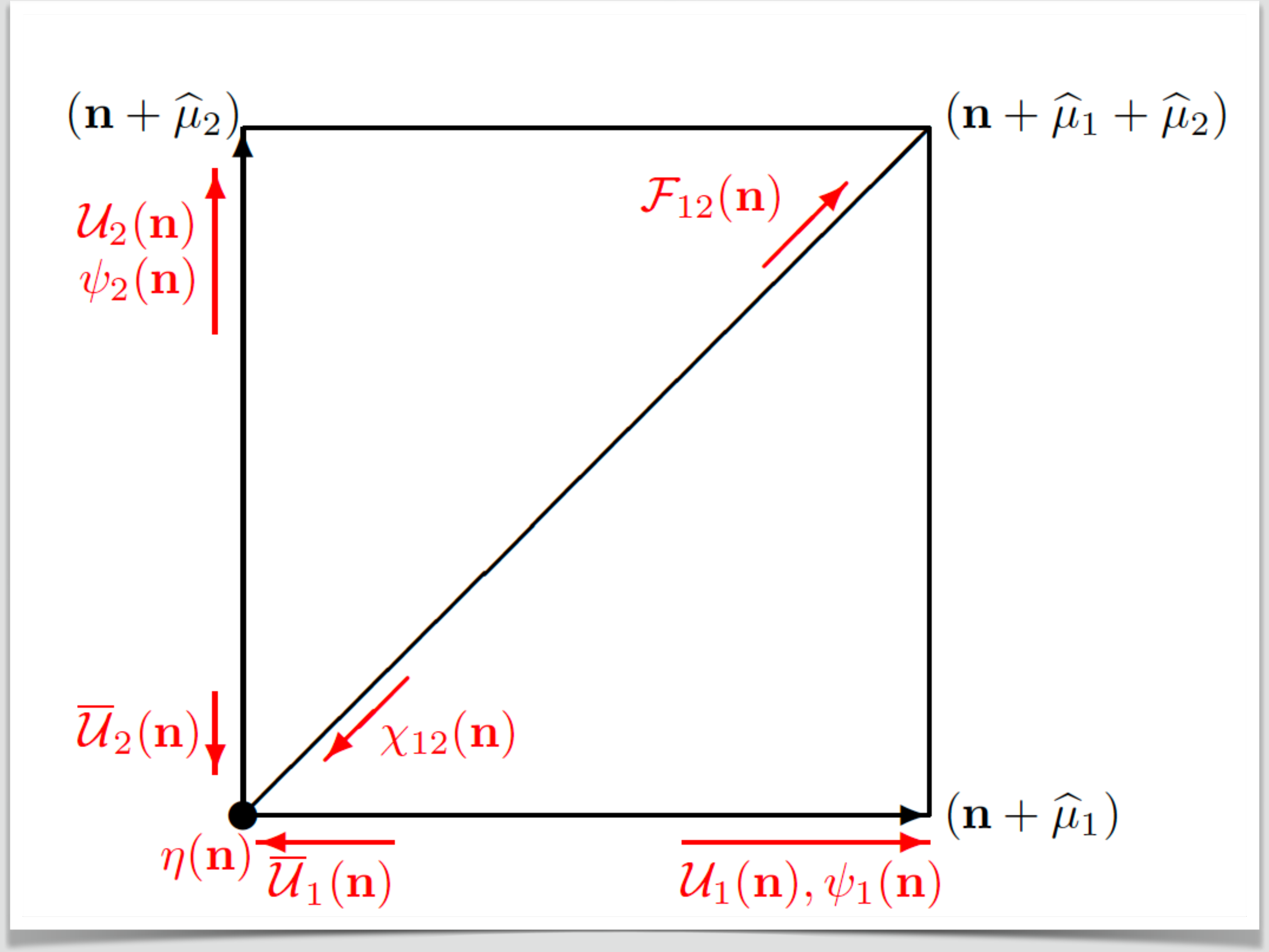
Action is invariant under Q

$$QS = Q^2\Psi = 0$$

Lattice Action

Discretize on a square lattice

$$\mathscr{A}_a \quad \rightarrow \quad \mathscr{U}_a$$



Lattice Action

Discretize on a square lattice

$$S = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[-\bar{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) + \frac{1}{2}\left(\bar{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n)\right)^2 - \chi_{ab}(n)\mathcal{D}_{[a}^{(+)}\psi_{b]}(n) - \eta(n)\bar{\mathcal{D}}_a^{(-)}\psi_a(n) \right]$$

Supersymmetric

Local

Gauge invariant

Free from fermion doublers

All nice properties!

Lattice Action

Impose periodic/anti-periodic boundary conditions

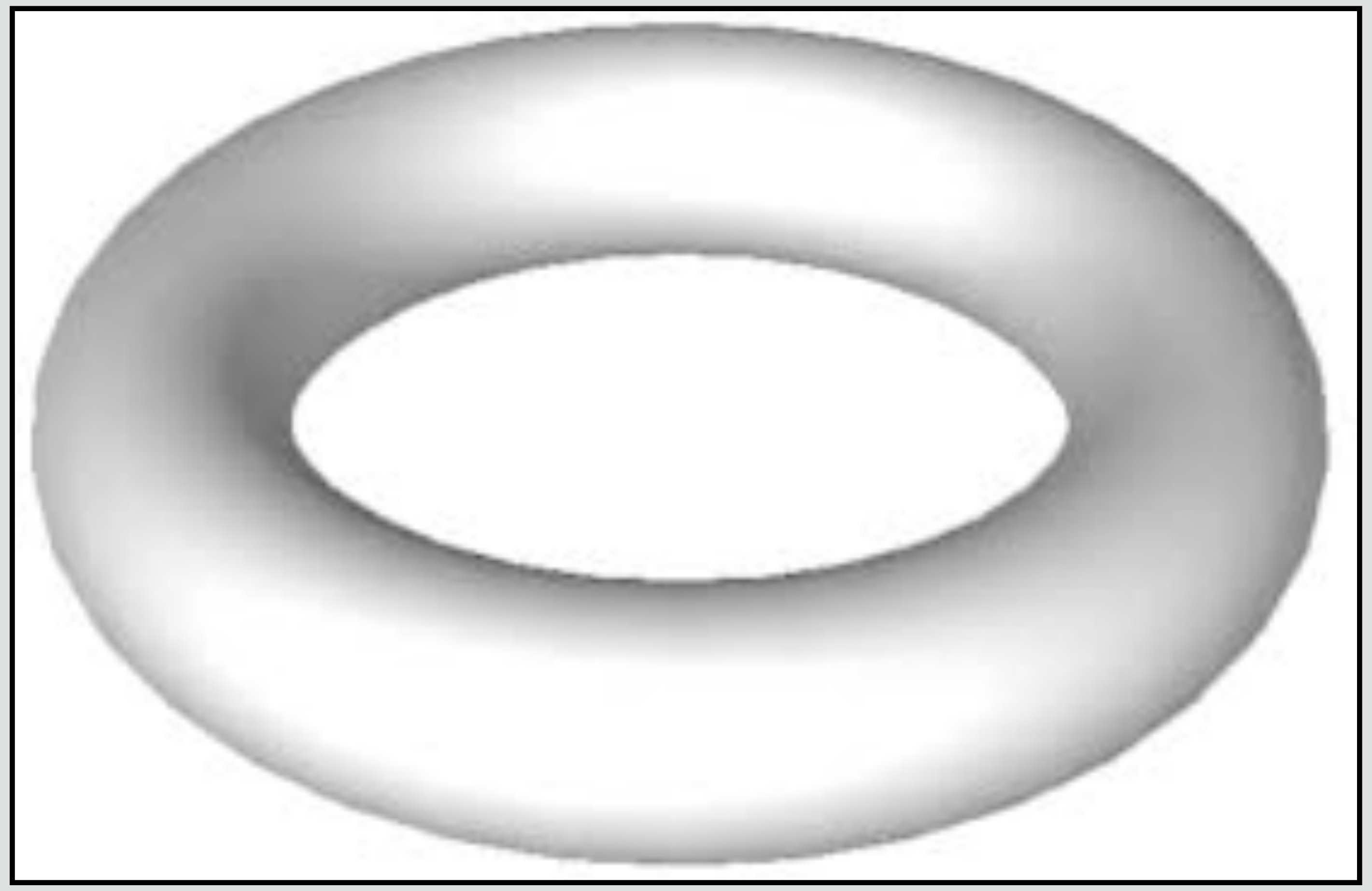
Periodic on spatial circle

Anti-periodic on temporal circle

Two extents:

$$L = aN_x \qquad \beta = aN_\tau$$

a: Lattice spacing



Lattice Action

Dimensionless couplings:

$$r_\tau \equiv \beta\sqrt{\lambda} = N_\tau\sqrt{\lambda_{\text{lat}}} = 1/t \qquad\qquad r_x \equiv L\sqrt{\lambda} = N_x\sqrt{\lambda_{\text{lat}}}$$

$$\lambda_{\text{lat}} = \lambda a^2$$

Another parameter: aspect ratio α

$$\alpha = \frac{L}{\beta} = \frac{r_x}{r_\tau} = \frac{N_x}{N_\tau}$$

Perform simulations at fixed r_τ and r_x

How Do We Simulate This Theory?

Use path integral Monte Carlo

$$\langle O \rangle = \int [D \text{ all fields}] \; O \; e^{-S}$$

This integral has a large dimensionality

Implement this integral on a lattice

How Do We Simulate This Theory?

A simple example:

Consider path integral of pure Yang-Mills with $\mathrm{SU}(3)$ gauge group

$$Z = \int DU \; e^{-\beta S[U]}$$

On a hyper-cubic lattice with 10 lattice sites in each direction

4 links field U_μ per site

Each link field is determined by 8 parameters of $\mathrm{SU}(3)$

Dimensionality of the integral will be:

$$(10 \times 10 \times 10 \times 10) \times 4 \times 8 = 320{,}000$$

How Do We Simulate This Theory?

Large numbers!

They are telling us that

We need some kind of statistical / sampling method

Use **Monte Carlo** sampling

We will use a sophisticated version of this:

Rational Hybrid Monte Carlo (RHMC) algorithm

Our code is publicly available on **GitHub**

SUSY LATTICE

Spatial Deconfinement Transition

Coming back to our theory, $\mathcal{N} = (2{,}2)$ SYM ...

Spatial deconfinement transition signals **topology changing transition**

Between **black-string** and **black-hole** geometries

Observable: Unitarized spatial Wilson line

$$W^u \equiv \frac{1}{NN_\tau} \sum_{t=0}^{N_\tau - 1} \mathrm{Tr} \left[\prod_{x=0}^{N_x - 1} U_x(x, t) \right]$$

Spatial Deconfinement Transition

Unitarized link field $U_a(n)$ is extracted from

$$\mathcal{U}_a(n) = e^{X_a(n)} U_a(n)$$

Also monitor unitarized **Polyakov loop**

$$P^u \equiv \frac{1}{NN_x} \sum_{x=0}^{N_x - 1} \mathrm{Tr} \left[\prod_{t=0}^{N_\tau - 1} U_t(x, t) \right]$$

Spatial Deconfinement Transition

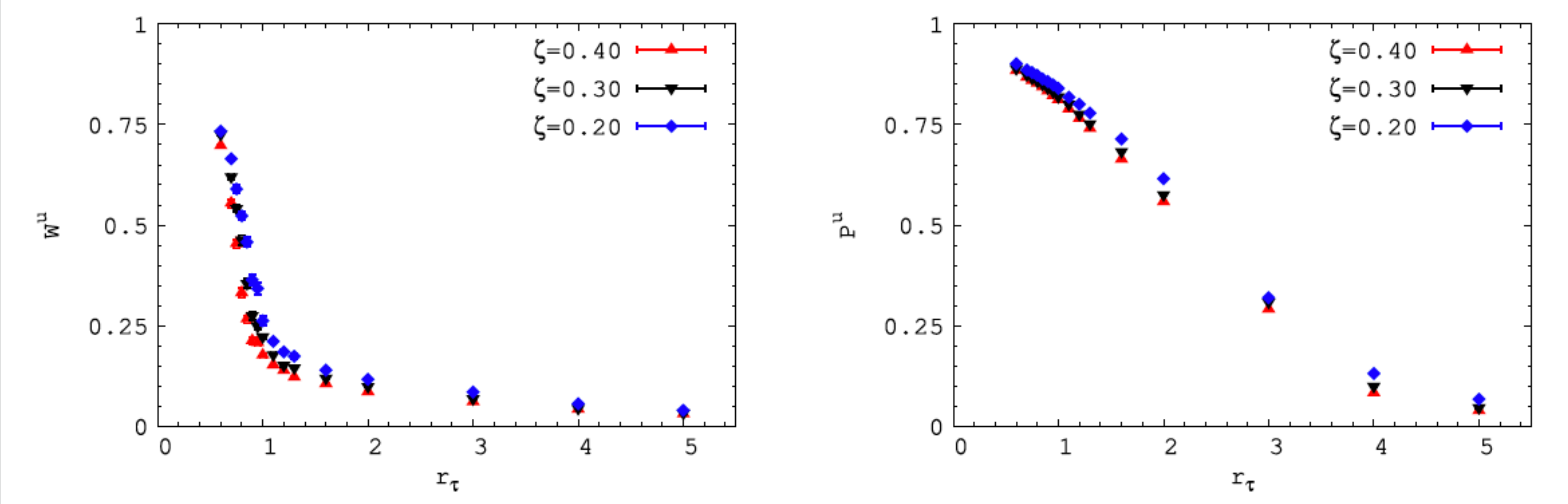
Calculations need to remain in the **deconfined phase** for P^u

$$0.5 \lesssim \langle\, | P^u | \,\rangle \leq 1$$

To admit a holographic dual interpretation in terms of black-hole geometry

Spatial Deconfinement Transition

Results from our work...



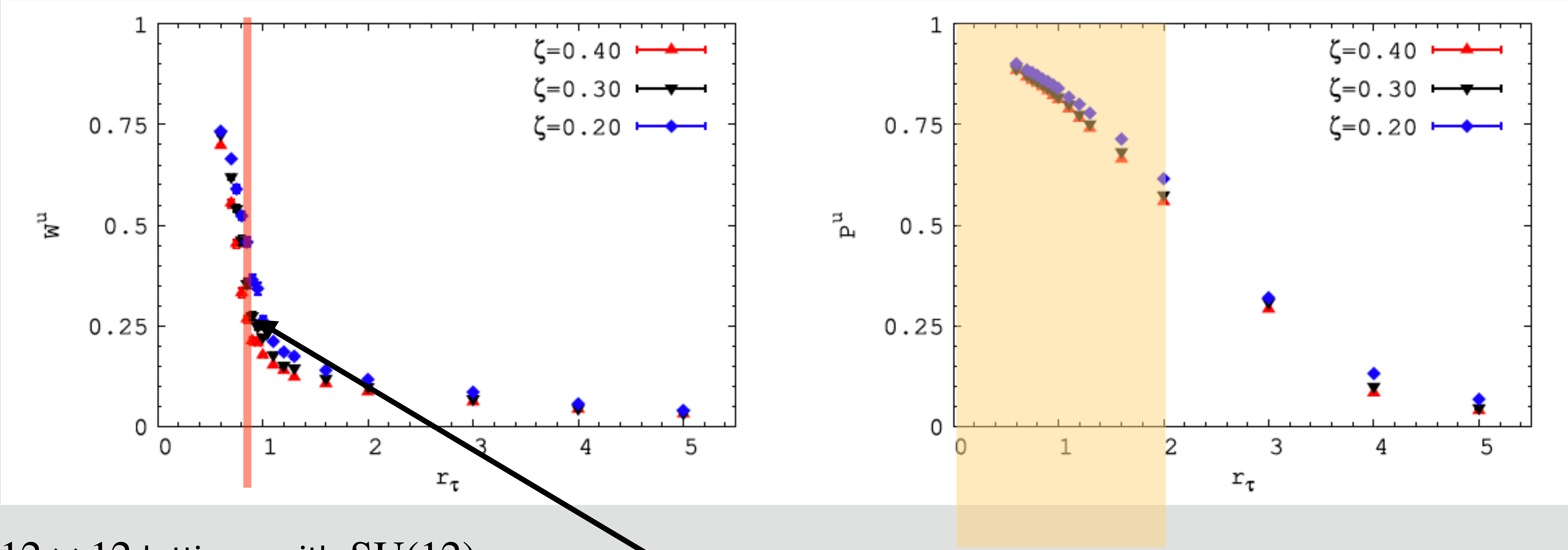
12×12 lattices with $\mathrm{SU}(12)$

ζ : a parameter

To control flat directions

Spatial Deconfinement Transition

Results from our work...



12×12 lattices with $\mathrm{SU}(12)$

$r_\tau \approx 0.85$

Thermally deconfined

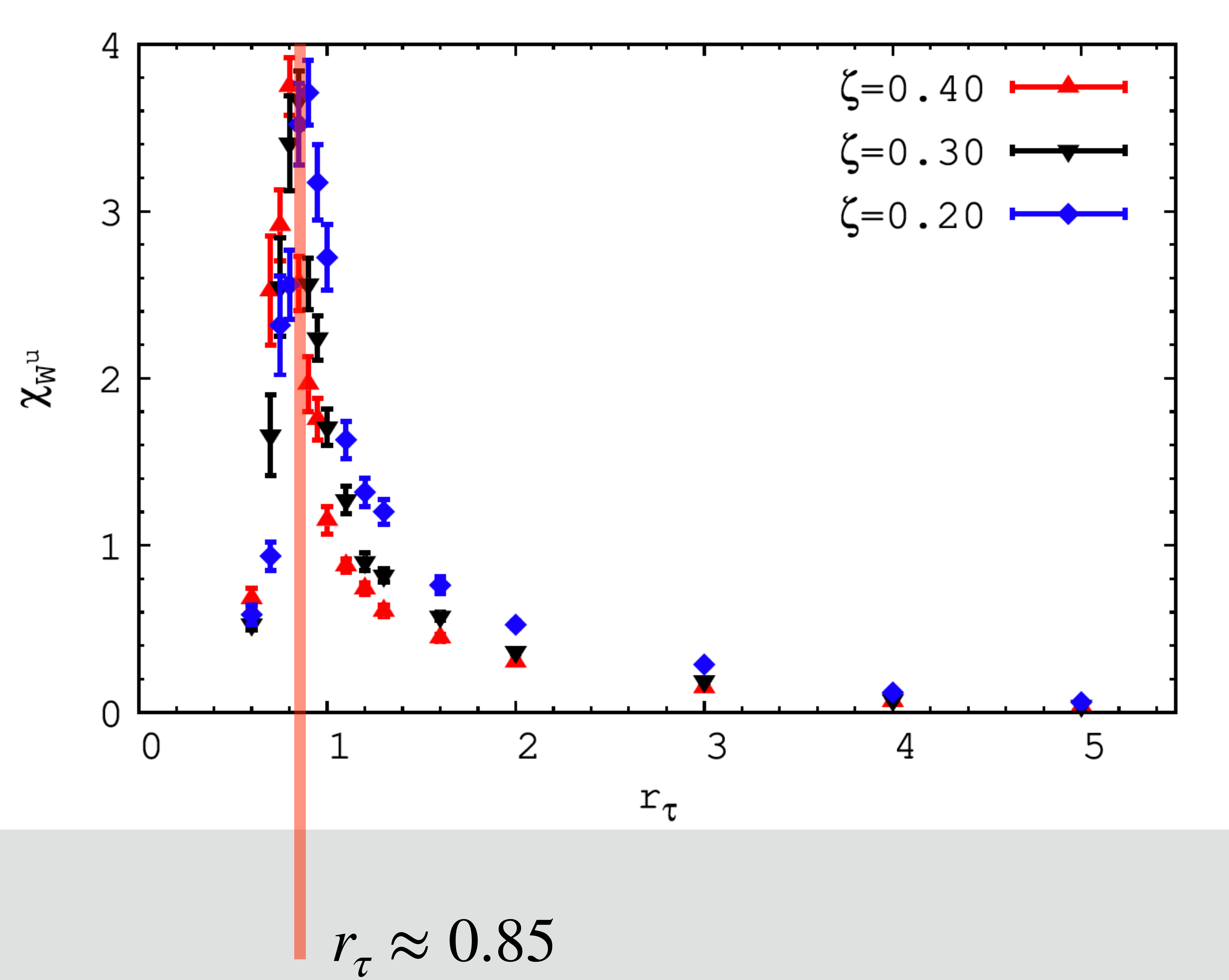
Spatial Deconfinement Transition

Need to pinpoint the transition

Look at Polyakov loop susceptibility

$$\chi_{W^u} \equiv N^2 \left(\langle |W^u|^2 \rangle - \langle |W^u| \rangle^2 \right)$$

Again, clear that $r_\tau \approx 0.85$



How to Find Order of the Transition?

Find the N dependance of $\chi_{\max}$

$$\chi_{\max} = CN^{2b}$$

For first order transition:

$$\chi_{\max} \propto N^2$$

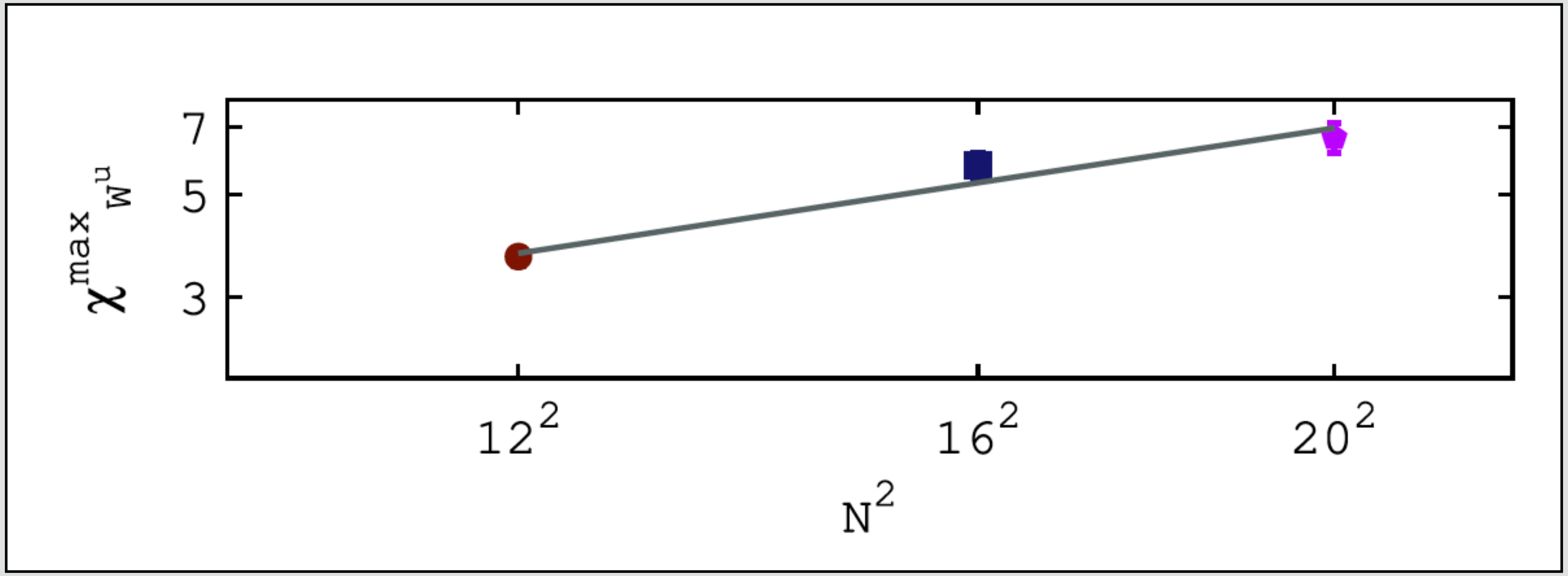
For a crossover, peak height is independent of N

For a continuous second-order transition $0 < b < 1$

Order of the transition

Fit gives $b = 0.61(8)$

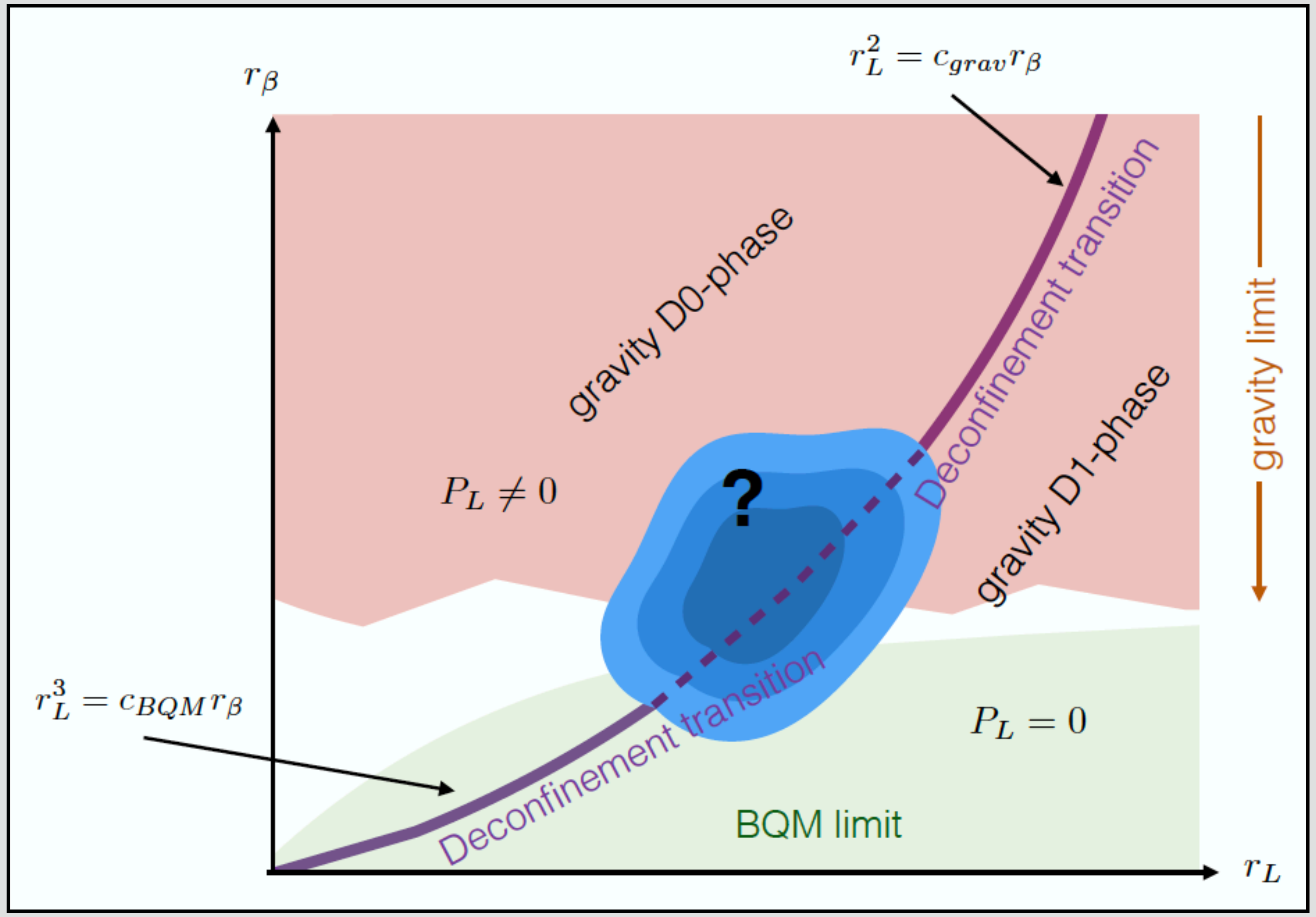
A continuous 2nd order transition in $\mathcal{N} = (2,2)$ SYM



Note: In $\mathcal{N} = (8,8)$ SYM the transition is 1st order!

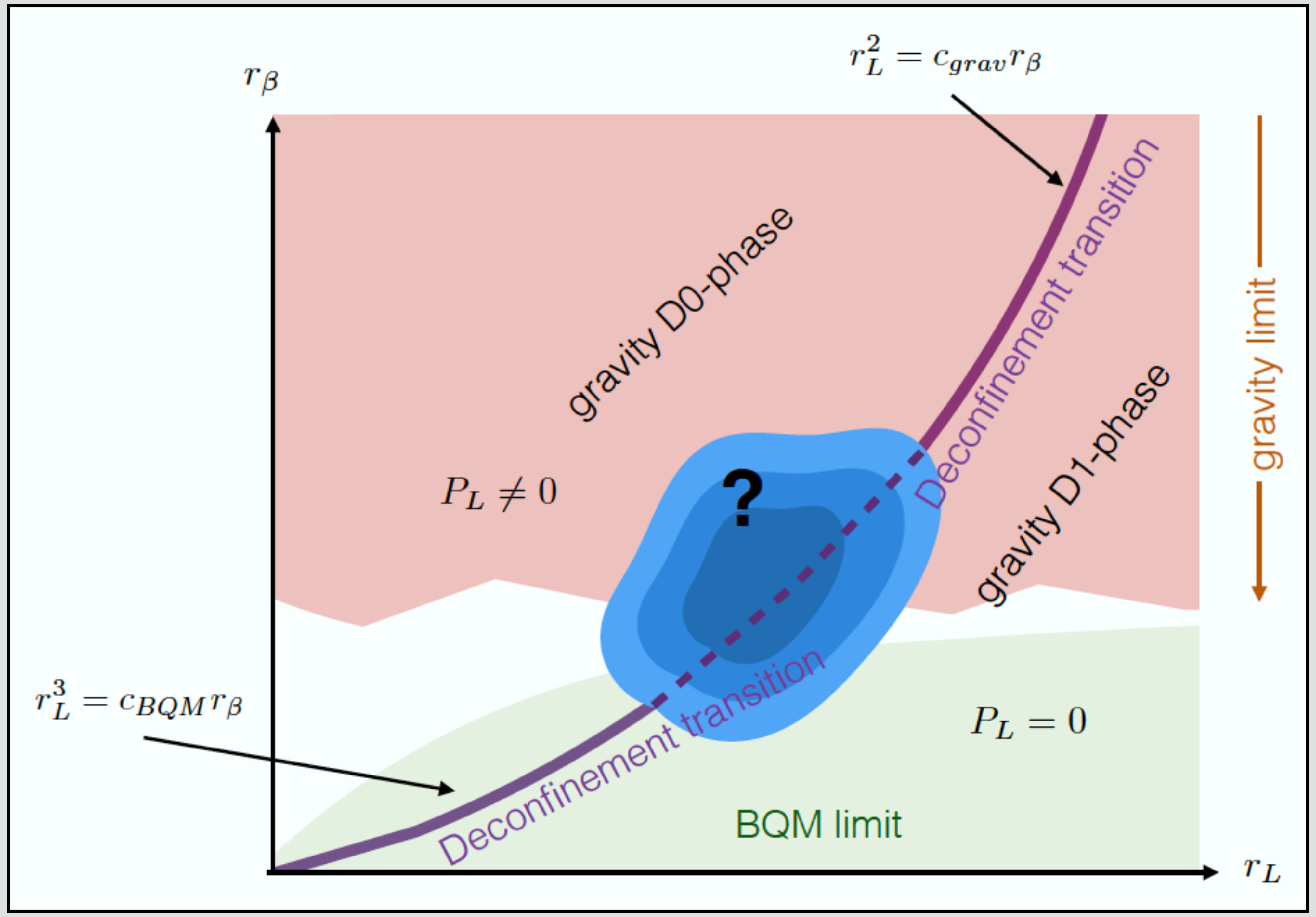
Phase Diagram in (r_x, r_τ) Plane

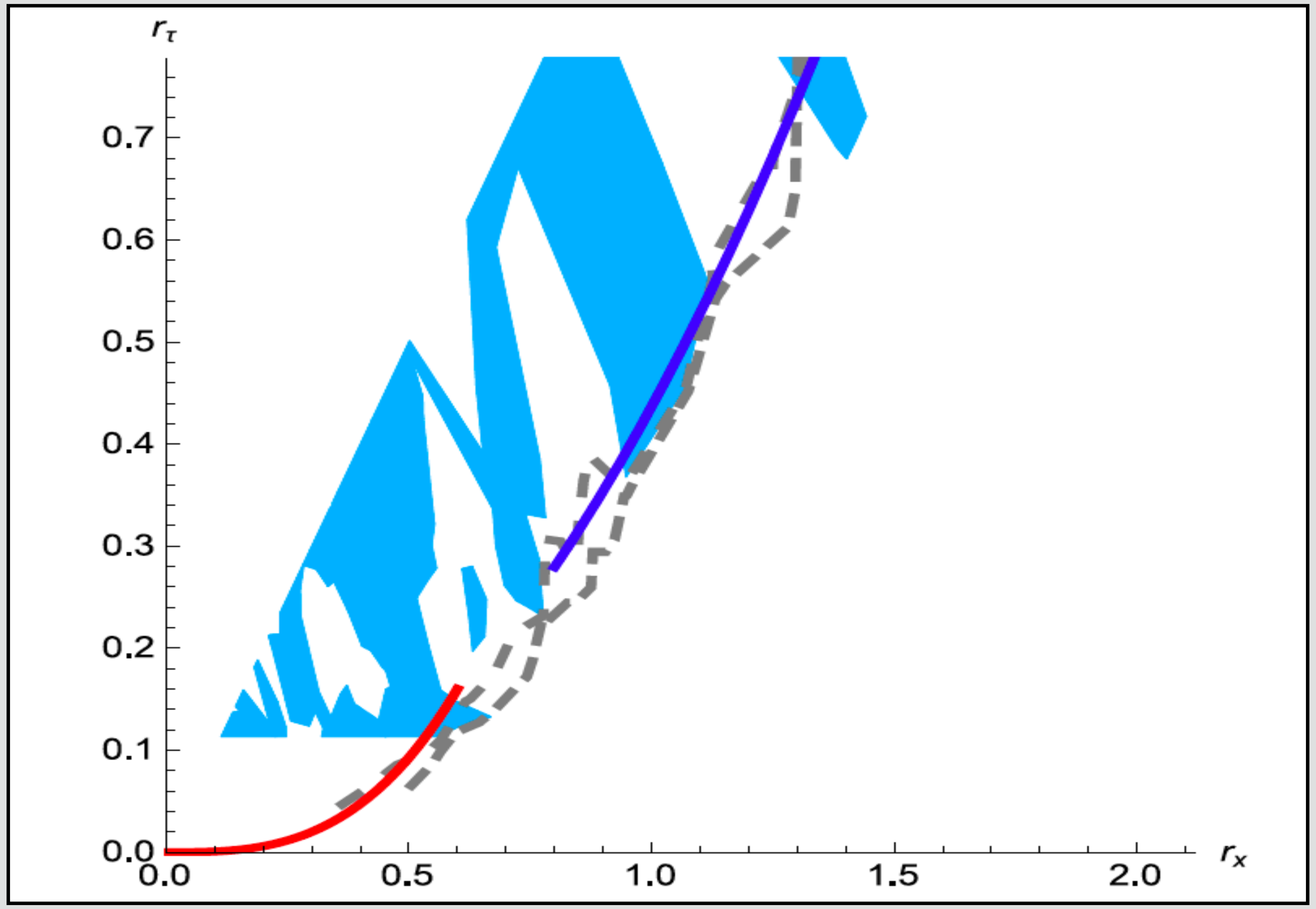
What is known in $2d\ \mathcal{N} = (8{,}8)$ SYM theory?



Phase Diagram in (r_x, r_τ) Plane

What is known in $2d$ $\mathcal{N} = (8,8)$ SYM theory?

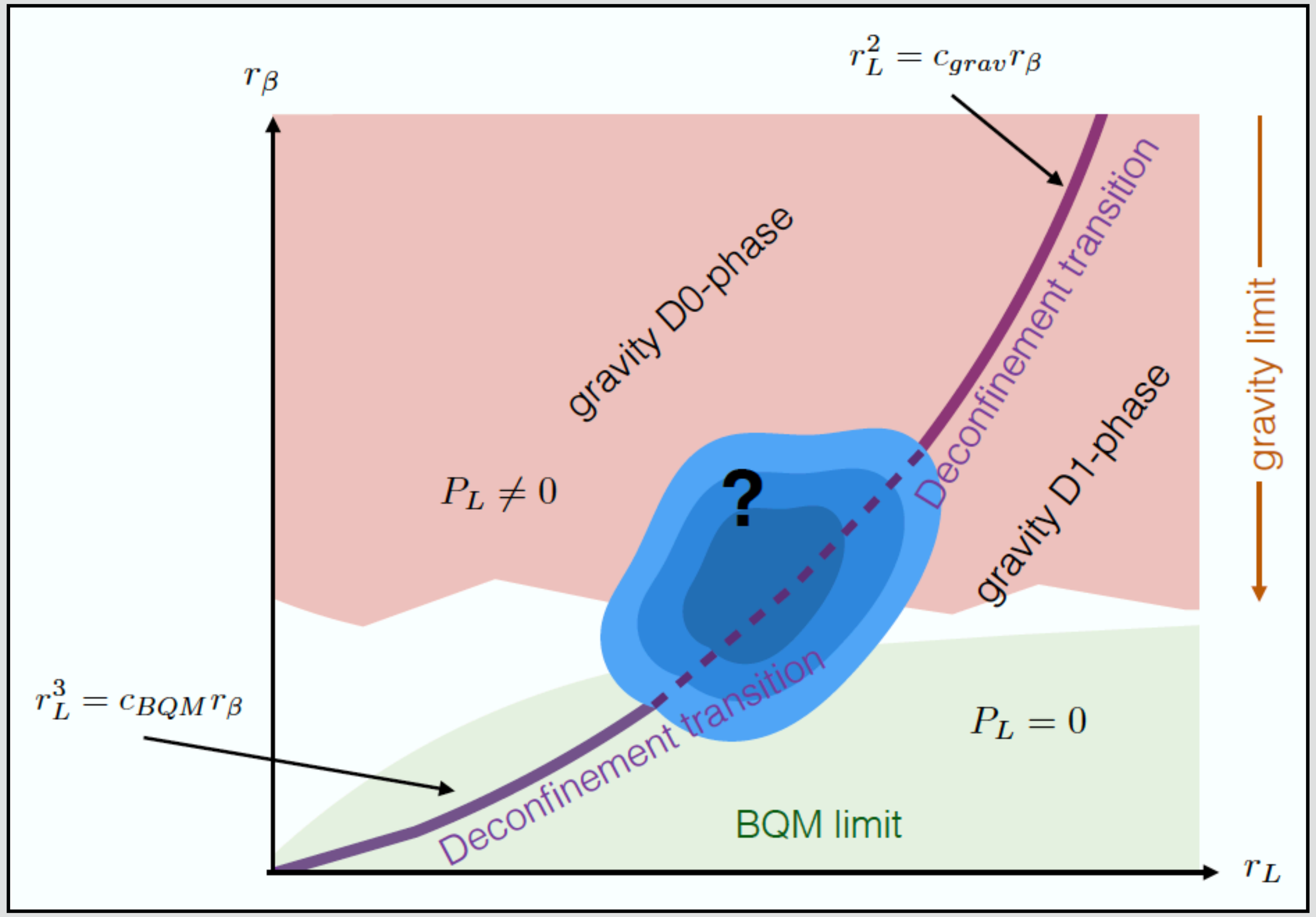


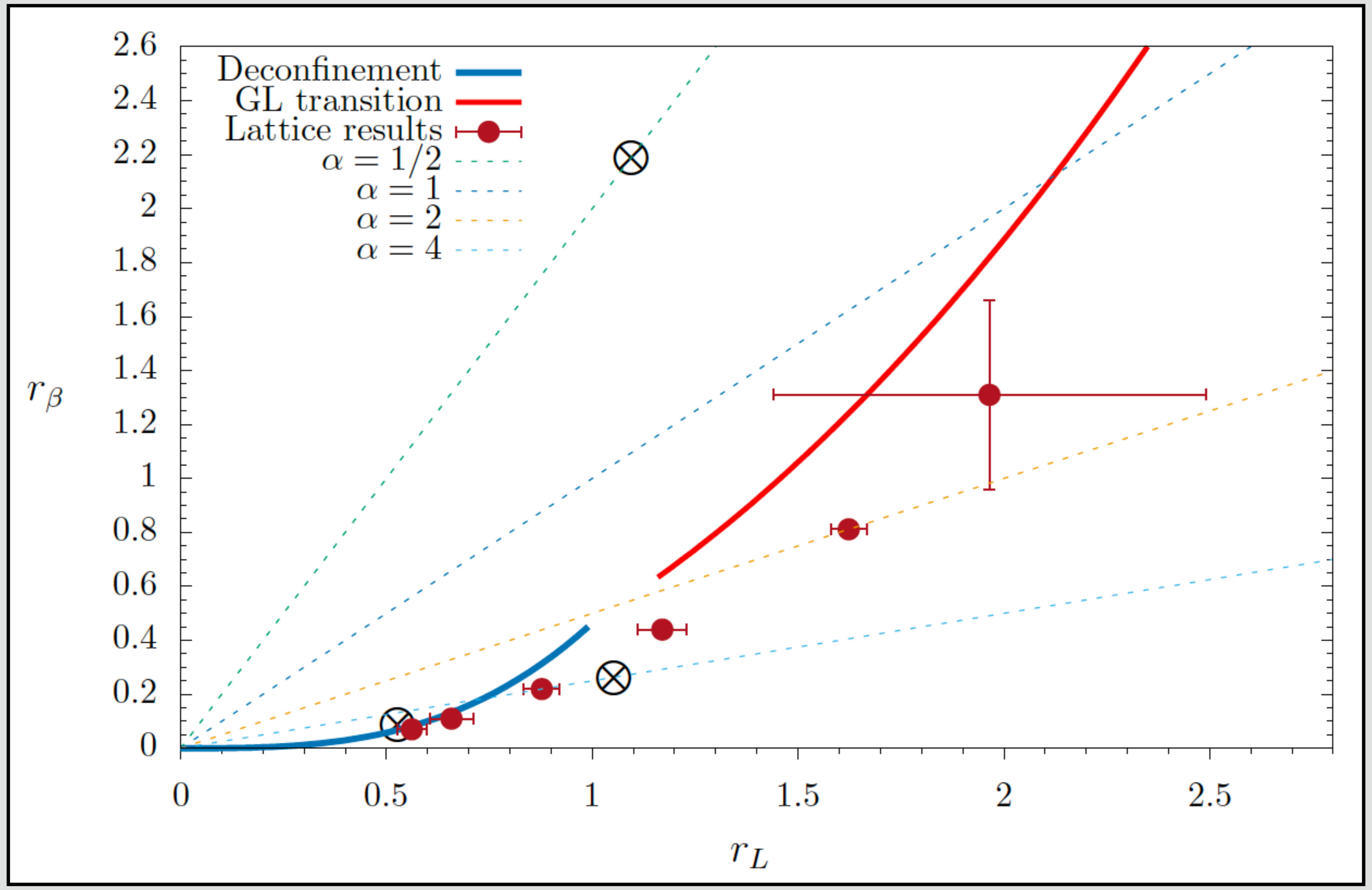


Catterall, Joseph, Wiseman
JHEP **12** (2010) 022

Phase Diagram in (r_x, r_τ) Plane

What is known in $2d$ $\mathcal{N} = (8,8)$ SYM theory?



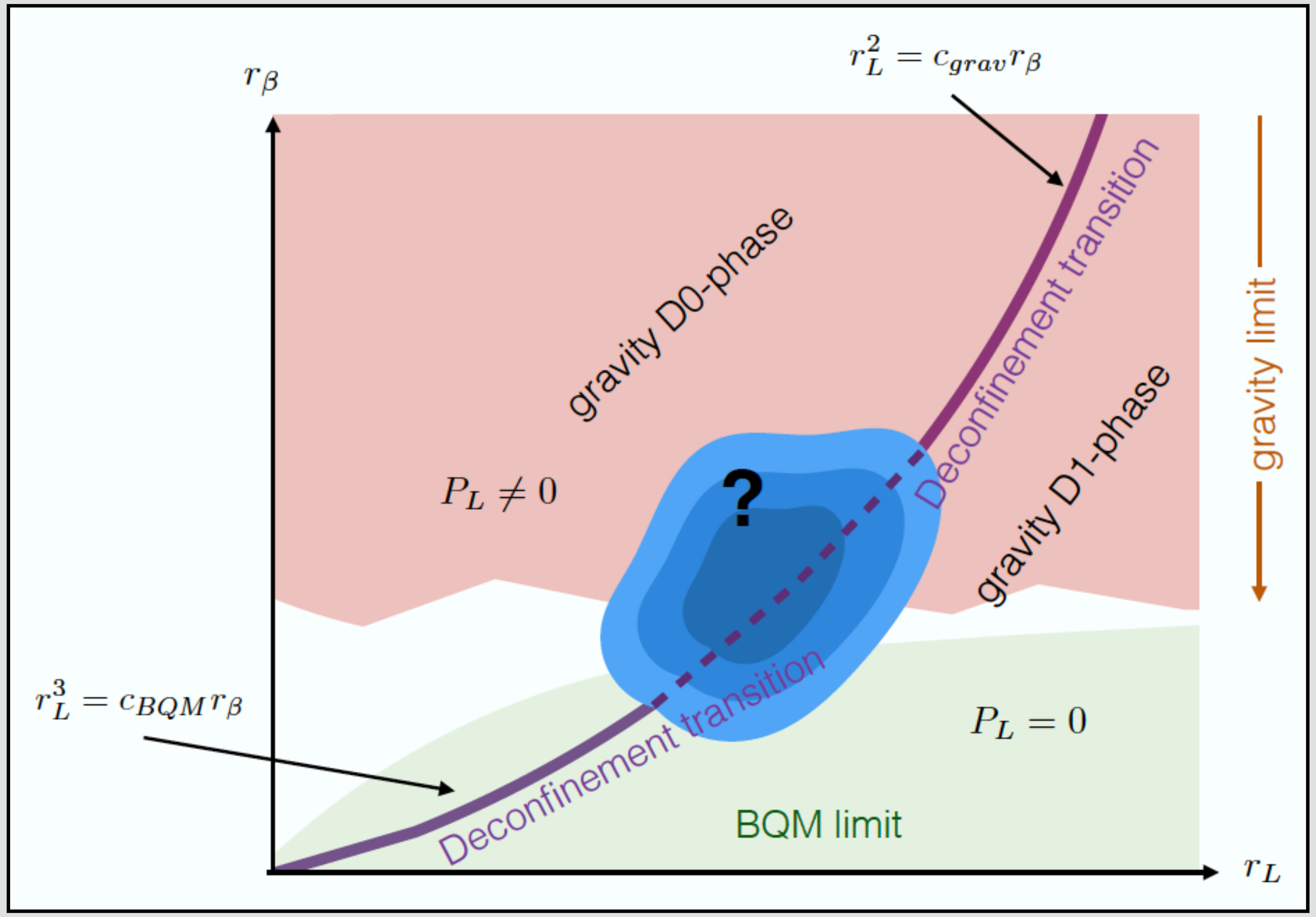


Catterall, Jha, Schaich, Wiseman

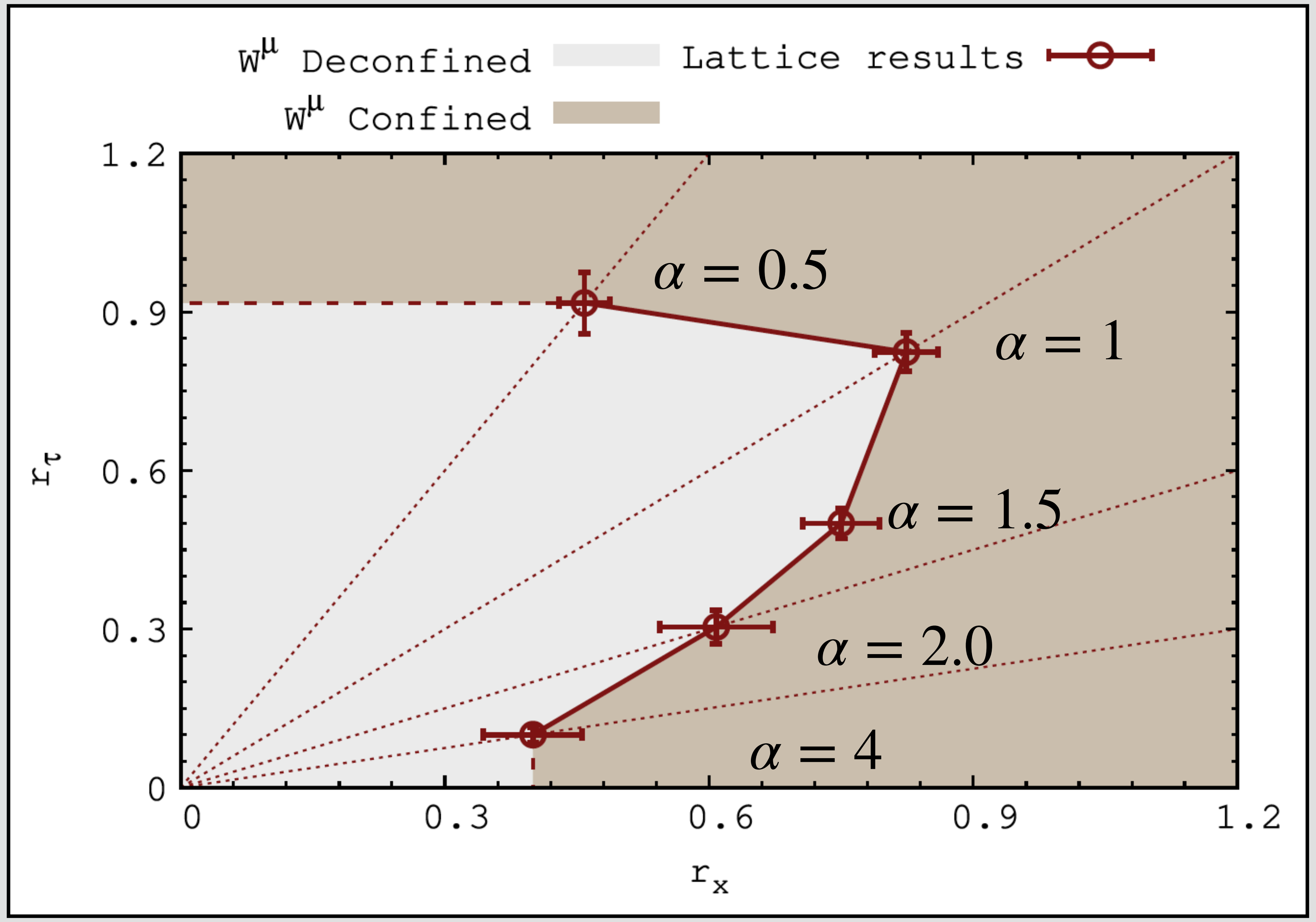
Phys. Rev. D **97** (2018) 8 086020

Phase Diagram in (r_x, r_τ) Plane

This work: $2d\ \mathcal{N} = (2,2)$ SYM theory



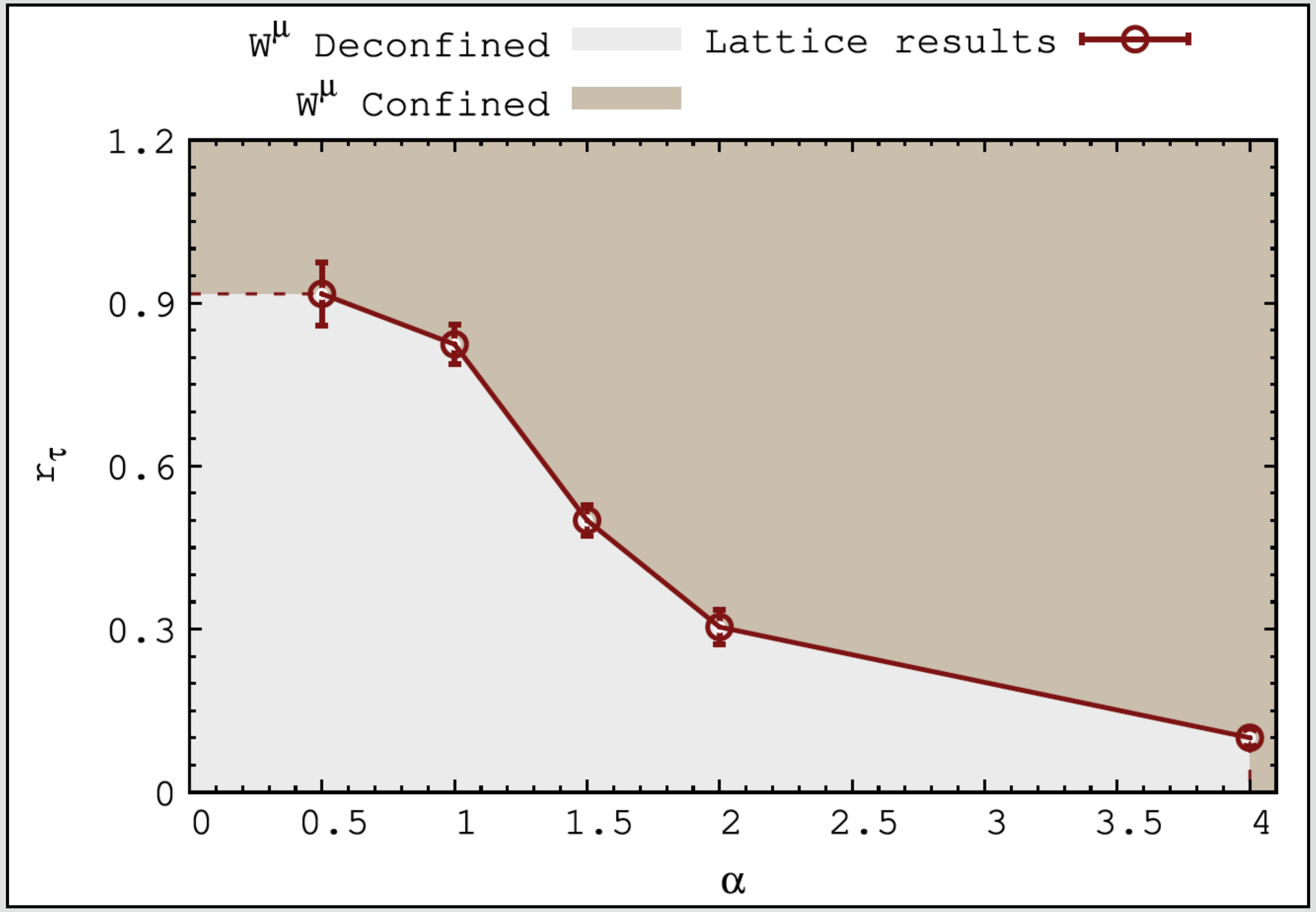
$2d\ \mathcal{N} = (8,8)$ SYM theory



$2d\ \mathcal{N} = (2,2)$ SYM theory
This work

Phase Diagram in (α, r_x) Plane

Another look:



This work

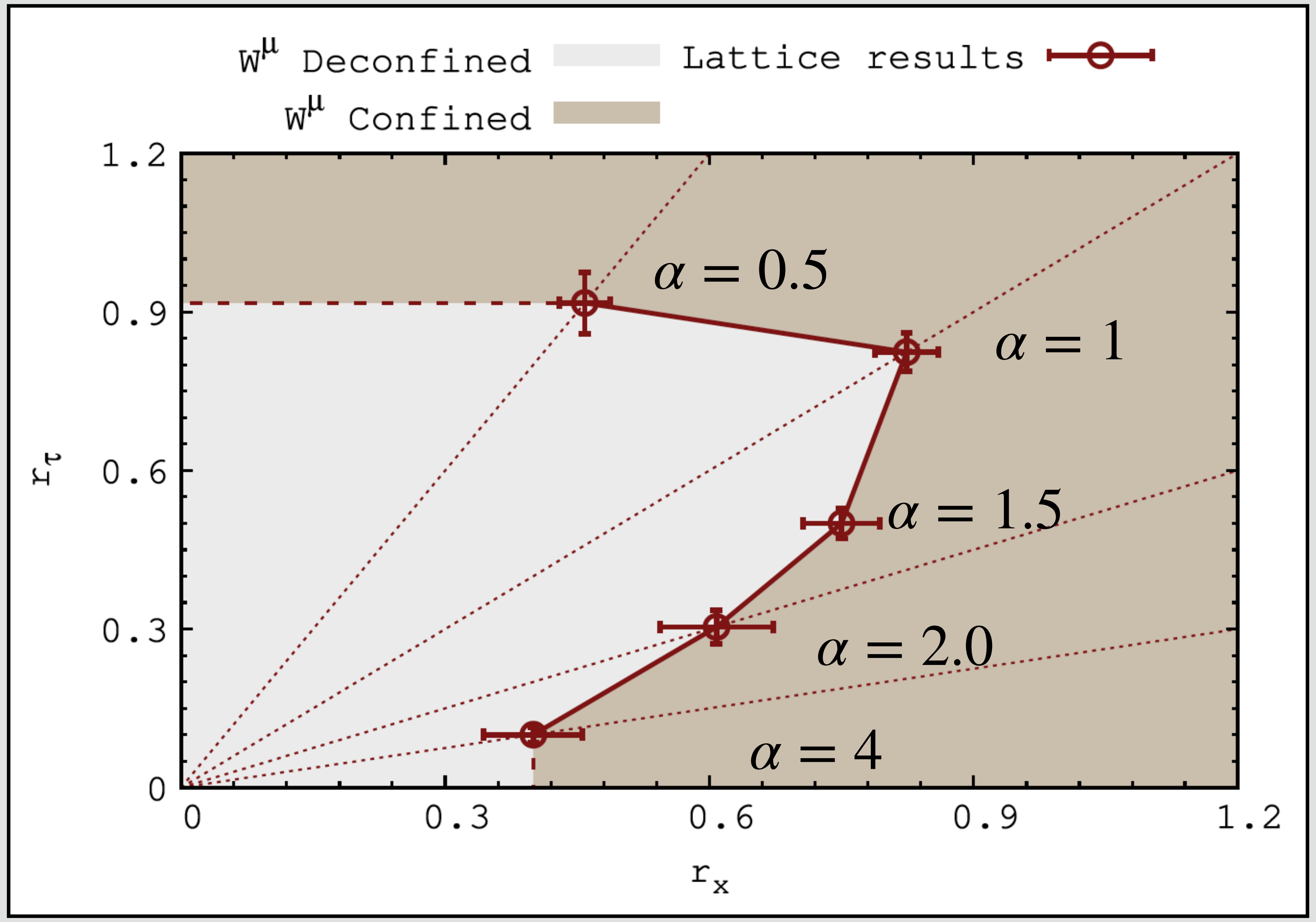
For smaller $\alpha \leq 1$, we have $r_\tau^{(c)}$ roughly constant, $r_\tau^{(c)} \approx 1$

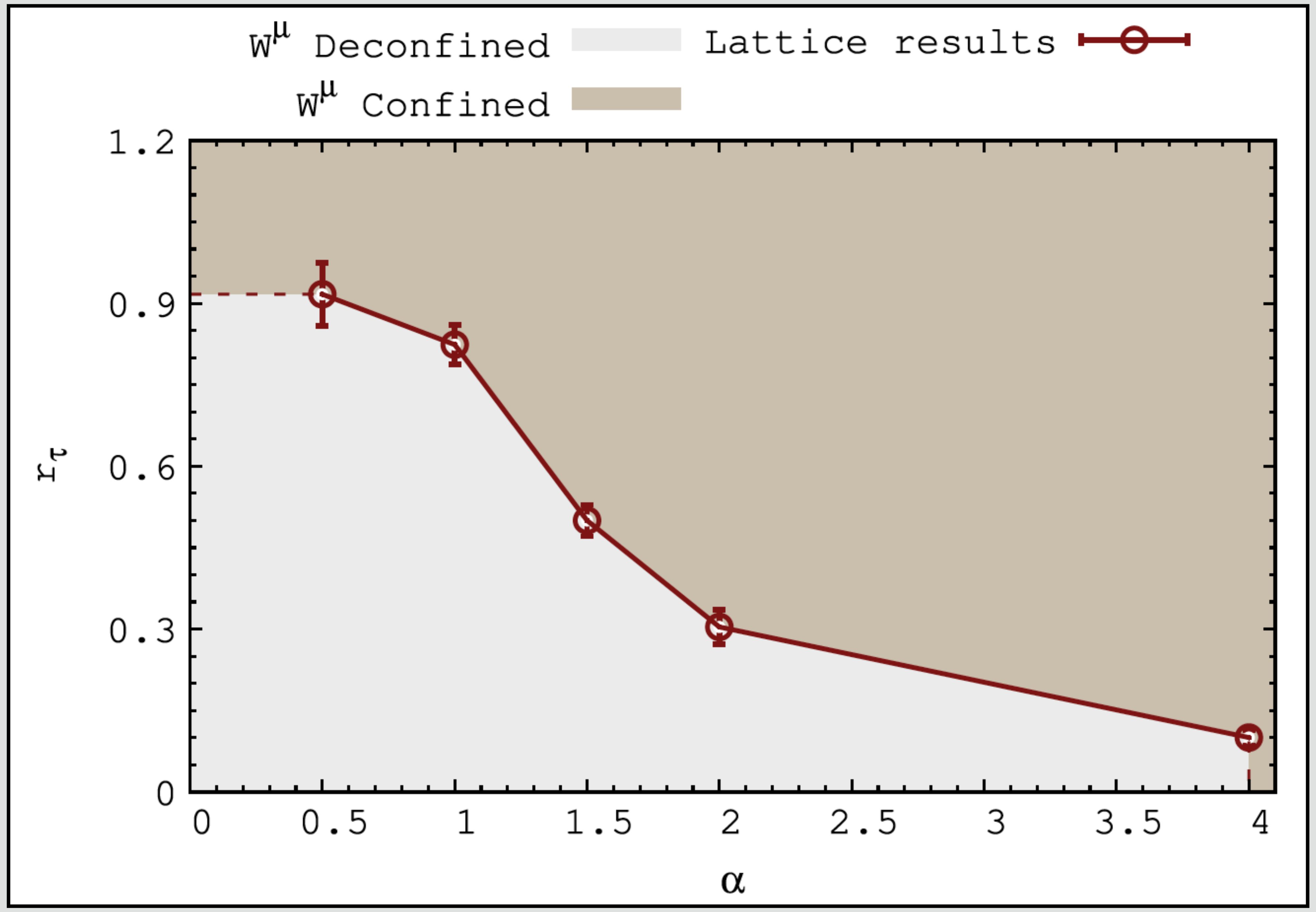
Order of the transition

Absence of spatial deconfinement transition, where holography is valid, $r_\tau \gg 1$

Does not rule out existence of gravity dual.

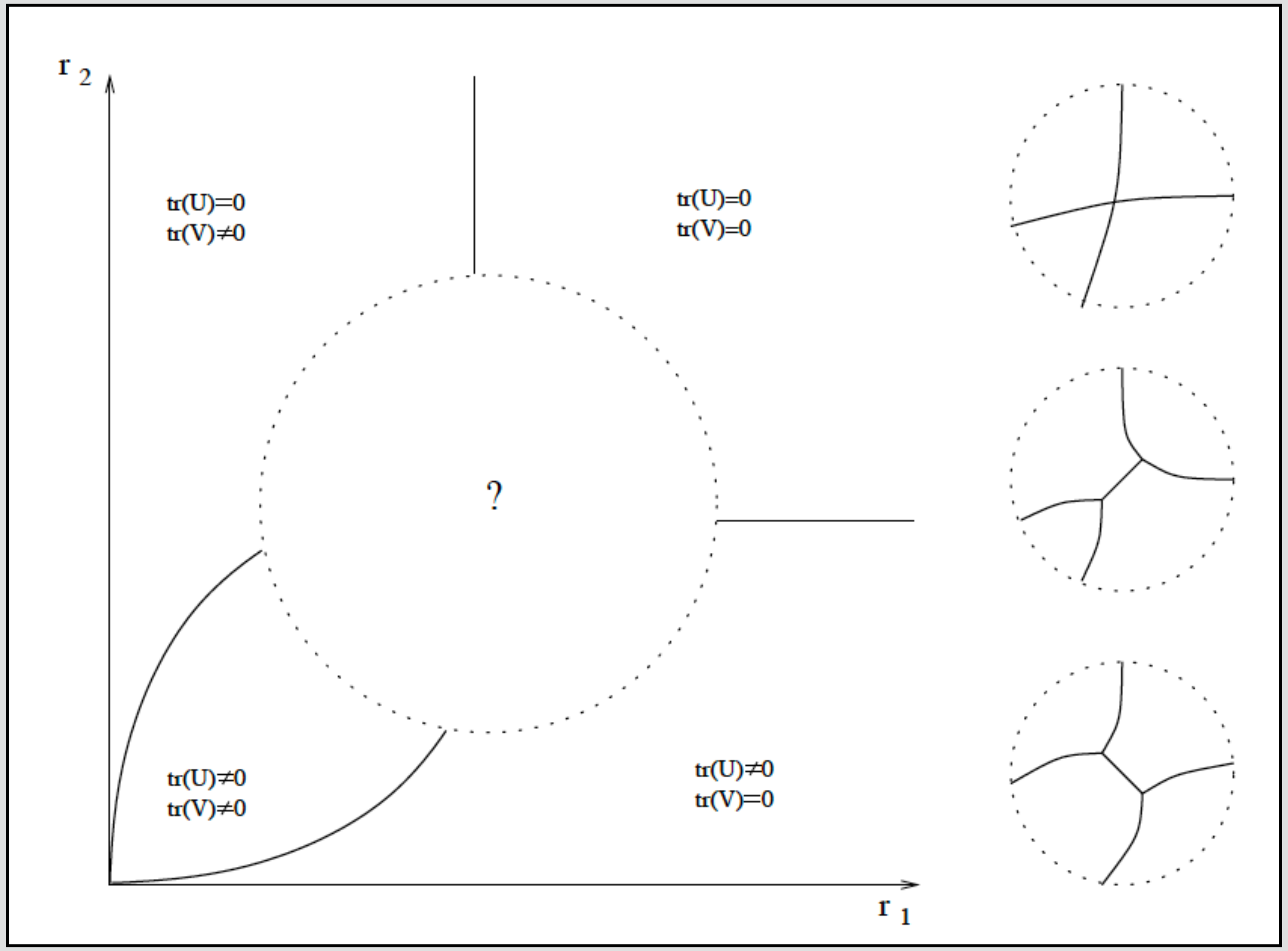
Rules out possibility of **topology changing transition**





Future Directions

Complete this phase diagram



Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk, Wiseman
JHEP **01** (2006) 140

Future Directions

Study the "**extent of scalars**" $\mathrm{Tr}\, X_i^2$

Related to **bound states of D branes**

Another question:

How does the phase diagram look in $2d\ \mathcal{N} = (8{,}8)$ SYM?

END