QNEC bounds on quenches in critical many-body systems

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Based on

1. Quantum thermodynamics of holographic quenches and bounds on the growth of entanglement from the QNEC, Tanay Kibe, Pratik Roy and Ayan Mukhopadhyay, Phys.Rev.Lett. 128 (2022) 19, 191602 • arXiv: 2109.09914 [hep-th]
2. Erasure tolerant quantum memory and the quantum null energy condition in holographic systems, Avik Banerjee, Tanay Kibe, Nehal Mittal, Ayan Mukhopadhyay, Pratik Roy, Phys. Rev. Lett. 129 (2022), 191601 • arXiv: 2202.00022 [hep-th]
3. Generalized Clausius inequalities and entanglement production in holographic two-dimensional CFTs, Tanay Kibe, Ayan Mukhopadhyay and Pratik Roy • arXiv: 2412.xxxxx
4. Work in progress with Pratik Roy

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Introduction and motivation

Goal

Unraveling (quantum thermodynamic) bounds on critical many-body systems using the quantum null energy condition

Quantum Thermodynamics

This is an emerging field with widespread applications.

- The one shot work cost of creating a state and the extractable work from a state is bounded by the hypothesis testing relative entropy [Yunger Halpern and Renes, 2016])
- Quantum entanglement can lead to anomalous heat flows [Bera et al., 2017]
- Few results for quantum many-body systems

Irreversible entropy production

- $\Delta S = \Delta S_{\mathrm{rev}} + \Delta S_{\mathrm{irr}}$
- The classical Clausius inequality bounds the irreversible entropy production as $\Delta S_{\mathrm{irr}} \geq 0$

Quantum irreversible entropy production has two contributions

$$\Delta S_{\mathrm{irr}} = S(\rho_E || \rho_E^{(0)}) + I_{\rho_{SE}}(S:E),$$

where

$$I_{\rho_{SE}}(S:E) = S(\rho_{SE} || \rho_S \otimes \rho_E)$$

1. $S(\rho_E || \rho_E^{(0)}) \rightarrow$ the loss of information contained purely in the environment
2. $I_{\rho_{SE}}(S:E) \rightarrow$ loss of information in system-environment correlations

For system-environment couplings with a global fixed point ρ_S^*

$$U\left(\rho_S^* \otimes \rho_E^{(0)}\right) U^\dagger = \rho_S^* \otimes \rho_E^{(0)}$$

$$\Delta S_{\mathrm{irr}} = S(\rho_S^{(0)} || \rho_S^*) - S(\rho_S || \rho_S^*).$$

$\rho_S^{(0)}$- initial system state
ρ_S^*- equilibrium state (fixed point of quantum channel)
ΔS_{irr} is manifestly positive due to monotonicity of relative entropy

Quantum bounds on irreversible entropy production

- Quantum thermodynamics $\rightarrow$ lower bound on ΔS_{irr} in terms of the Bures distance between the out-of-equilibrium state and the final equilibrium state [Deffner and Lutz, 2010]
- Also an upper bound related to the Bremermann-Bekenstein bound [Bekenstein, 1981] on the maximum rate of information transfer with a given amount of energy
- Can we find similar bounds for critical many-body systems described by conformal field theory?

QNEC: the key tool

Setup: Quenches in a 1+1 d CFT

Which quenches are physically allowed?

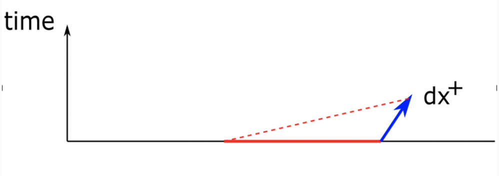
Quantum Null Energy Condition (QNEC)

$$\mathcal{Q}_\pm := 2\pi\langle t_{\pm\pm}\rangle - \left(\partial_\pm^2 S - \frac{6}{c}\left(\partial_\pm S\right)^2\right) \geq 0,$$

Any quench that violates QNEC is not physical

$$\mathcal{Q}_\pm := 2\pi\langle t_{\pm\pm}\rangle - \left(\partial_\pm^2 S - \frac{6}{c}\left(\partial_\pm S\right)^2\right)$$

Derivatives are with respect to left and right null deformations of the end point of the interval



QNEC can also be written as [Leichenauer et al., 2018]

$$\frac{\delta^2}{\delta x^\pm} S_{\rm rel}(\rho_R|\sigma_R) \geq 0$$

QNEC has been proven for

- Free QFTs [Bousso et al., 2016, Malik and Lopez-Mobilia, 2020]
- Holographic QFTs assuming entanglement wedge nesting [Koeller and Leichenauer, 2016]
- Two-dimensional CFTs assuming the state is cyclic [Balakrishnan et al., 2019]
- General Poincaré-invariant QFTs for states with finite averaged null energy and relative entropy with respect to the vacuum [Ceyhan and Faulkner, 2020]

Holographic global quench

Holographic model for quenches

- Instantaneous transition between momentum carrying thermal states at time $u = 0$
- Holographic dual is two BTZ geometries glued across a null shock

The energy momentum tensor of the CFT is

$$\langle t_{\pm\pm}\rangle = \frac{c}{12\pi}(\Theta(-u)L^i_\pm(x^\pm) + \Theta(u)L^f_\pm(x^\pm)), \quad \langle t_{+-}\rangle = 0,$$

Constant $L_\pm = \mu_\pm^2$ correspond to a BTZ black brane

Bulk spacetime

$$\mathrm{d}s^2 = \frac{-2du\,dz + (-1 + 2m(u,y)z^2)du^2 + 2j(u,y)z^2du\,dy + dy^2}{z^2},$$

with

$$m(u) = \theta(-u)({\mu_+^i}^2 + {\mu_-^i}^2) + \theta(u)({\mu_+^f}^2 + {\mu_-^f}^2),$$
$$j(u) = \theta(-u)({\mu_+^i}^2 - {\mu_-^i}^2) + \theta(u)({\mu_+^f}^2 - {\mu_-^f}^2)$$

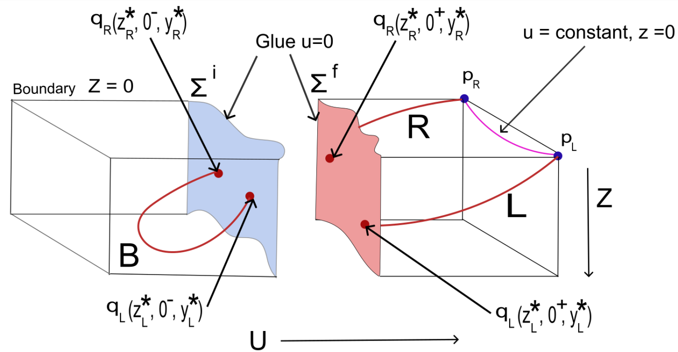
Einstein equations are satisfied with a bulk stress tensor with non-zero components:

$$T_{uu} = q(u)z + p(u)j(u)z^3, \qquad T_{uy} = p(u)z,$$

where

$$8\pi G q(u) = \delta(u)({\mu^f_+}^2 - {\mu^i_+}^2 + {\mu^f_-}^2 - {\mu^i_-}^2),$$
$$8\pi G p(u) = \delta(u)({\mu^f_+}^2 - {\mu^i_+}^2 - {\mu^f_-}^2 + {\mu^i_-}^2).$$

Cut and glue method to compute entropy



- Map the pre and post quench geometries to Poincaré AdS via two separate diffeomorphisms (uniformization maps)
- the quench surface (u=0) maps to two separate surfaces which are glued by identifying coordinates
- Compute geodesic lengths to obtain entanglement entropy [Ryu and Takayanagi, 2006, Hubeny et al., 2007]

- Intersection points are solved using extremization conditions for the geodesic at the shock
- These are algebraic equations for $q_{L,R}$
- Lots of technical subtleties

Post-quench entanglement entropy growth

1. **Early time quadratic growth**: entropy grows as u^2 for small times

$$\Delta S = \frac{c}{6}\left({\mu^f_+}^2 + {\mu^f_-}^2 - {\mu^i_+}^2 - {\mu^i_-}^2\right) u^2$$

2. **Intermediate time linear growth**: for
$\ell \to \infty, \quad u \to \infty, \quad 0 < \frac{u}{\ell} \leq \frac{1}{2}$

$$\Delta S = \frac{c}{6}\left(2(s^f - s^i)u\right)$$

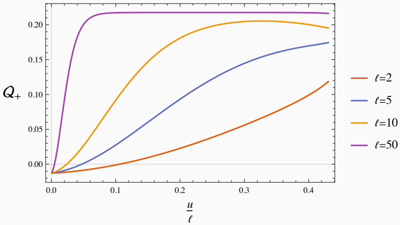
3. **Approach to equilibrium**: For times $u \approx \frac{\ell}{2}$ the entropy behaves as

$$S_f - S \sim \left(\frac{\ell}{2} - u\right)^{3/2}$$

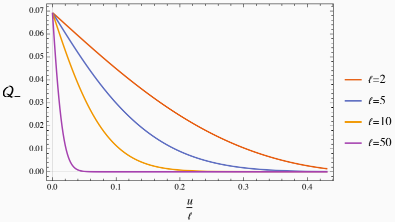
Generalize earlier results from [Liu and Suh, 2014, Hubeny et al., 2013] for non-rotating to non-rotating BTZ quenches.

Generalized Clausius inequality

Keep p_R fixed at $(z = \epsilon, u, \ell)$ and evaluate $\mathcal{Q}_{\pm}$ by deforming $p_L = (\epsilon, u, 0)$

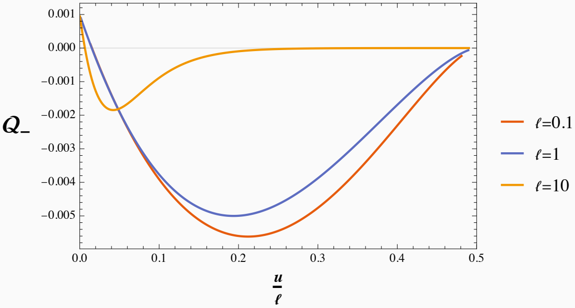


(a)



(b)

- It is enough to check $\mathcal{Q}_+$ at $u \to 0$
- $\mathcal{Q}_-$ should be checked at all times

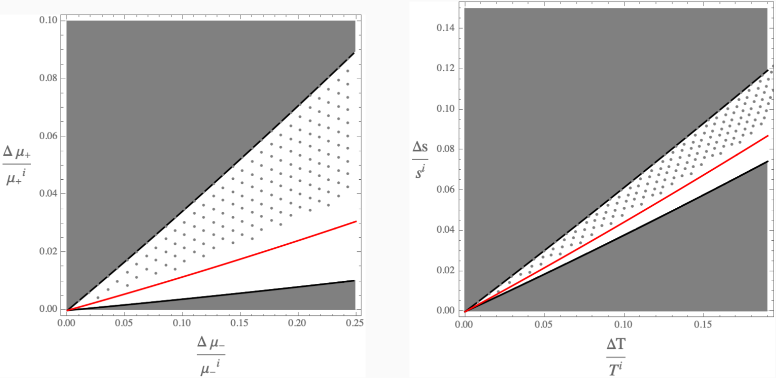


Early time

Analytics possible when $\mu^f_\pm u \ll 1$

$$\mathcal{Q}_+ = \frac{c}{6}\frac{1}{4}\left(3{\mu^f_+}^2 - {\mu^f_-}^2 - 3{\mu^i_+}^2 + {\mu^i_-}^2\right),$$
$$\mathcal{Q}_- = \frac{c}{6}\frac{1}{4}\left(3{\mu^f_-}^2 - {\mu^f_+}^2 - 3{\mu^i_-}^2 + {\mu^i_+}^2\right).$$

The white region in the figure below is allowed by QNEC.



$$T^{i,f} = \frac{2}{\pi} \frac{\mu_+^{i,f} \mu_-^{i,f}}{\mu_+^{i,f} + \mu_-^{i,f}}, \quad s^{i,f} = \frac{c}{6} \left(\mu_+^{i,f} + \mu_-^{i,f} \right)$$

Bounds are stronger than the classical Clausius inequality (NEC)

Why is QNEC violated?

Holographic proof of QNEC [Koeller and Leichenauer, 2016]:

- Assumes the NEC is satisfied in the bulk (true for us)
- Bulk is a *smooth* classical geometry which is the solution of a two-derivative gravitational theory

In our case

- Likely that the bulk null shock cannot be realized as a limit of a smooth solution of Einstein's gravity minimally coupled to matter fields.
- $\Delta J = 0$ is always allowed.
- Consistent with [Bhattacharyya and Minwalla, 2009] where the Vaidya spacetime is realized using a massless scalar field

A question

- These holographic global quenches are different from the more standard Cardy-Calabrese CFT quenches
- Do we find similar QNEC bounds in those setups?

Global and local quenches in CFTs

Global quench [Calabrese and Cardy, 2005]

- Prepare system in the translation invariant eigenstate $|\psi_0\rangle$ of H_0
- Regulate: $e^{-\epsilon H}|\psi_0\rangle$
- Quench: evolve with critical CFT Hamiltonian H: $e^{-itH-\epsilon H}|\psi_0\rangle$

For an interval A

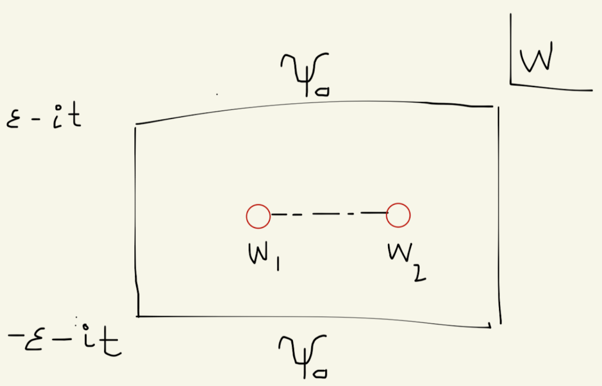
$$S_A^{(n)} = \mathrm{Tr}(\rho_A^n) = c_n \langle \Phi_n(z_1)\Phi_{-n}(z_2)\rangle_{UHP}$$

$$= c_n \left(\frac{|z_1 - \bar{z}_2||z_2 - \bar{z}_1|}{|z_1 - z_2||\bar{z}_1 - \bar{z}_2||z_1 - \bar{z}_1||z_2 - \bar{z}_2|} \right)^{2n\Delta_n} \mathcal{F}_n(\eta)$$

$\Phi_{n,-n}$ are twist fields with $\Delta_n = \frac{c}{24}(1 - \frac{1}{n^2})$
$\mathcal{F}_n(\eta) \approx 1$ when $\eta \approx 0, 1$ [Calabrese and Cardy, 2007b]

$$w = \frac{2\epsilon}{\pi} \log z,$$

maps the UHP to the strip with width 2ϵ.



The entanglement entropy is:

$$S_A = -\partial_n S_A^{(n)}\Big|_{n=1}$$

The stress tensor can be computed using the Schwarzian and is

$$\langle T(w)\rangle = \langle \bar{T}(\bar{w})\rangle = \frac{c\pi}{192\epsilon^2}$$

The averaged null energy diverges

$$ANE = \int dw \, \langle T(w)\rangle \to \infty$$

QNEC has been proven only for states with finite ANE and relative entropy with respect to the vacuum

We choose an interval from $x_1 = 0$ to $x_2 = \ell$ and evaluate QNEC by deforming the first point

QNEC is satisfied for this global quench, unlike the holographic case

$$\mathcal{Q}_+ = \frac{c\pi^2 \,\text{sech}\left(\frac{\pi(\ell-2t)}{4\epsilon}\right)^2}{48\epsilon^2} \geq 0,$$

$$\mathcal{Q}_- = \frac{c\pi^2 \,\text{sech}\left(\frac{\pi(\ell+2t)}{4\epsilon}\right)^2}{48\epsilon^2} \geq 0$$

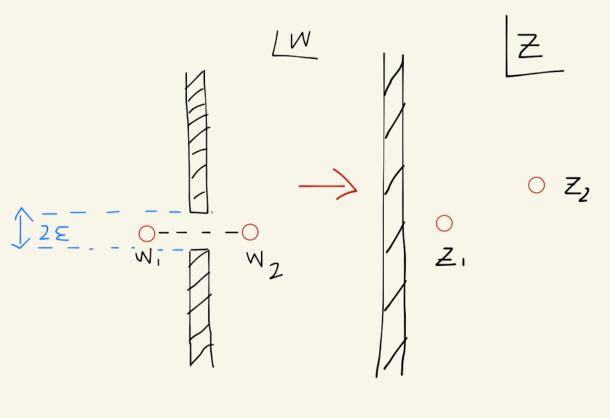
Local joining quench [Calabrese and Cardy, 2007a]

- Cut a CFT into two half lines A and $\bar{A}$
- Prepare a state that is the vacuum on each of the half lines:

$$|0\rangle_A \otimes |0\rangle_{\bar{A}}$$

- Join the two halves at $t = 0$ and evolve with CFT Hamiltonian of the full line H

$$z(w) = \frac{w}{\epsilon} + \sqrt{\frac{w^2}{\epsilon^2} + 1}$$



The entanglement entropy can be computed as a correlation function as before

The stress tensor is

$$\langle T(w) \rangle = \frac{c}{12\pi} \frac{\epsilon^2}{(w^2 + \epsilon^2)^2}$$

$$ANE = \int dw \, \langle T(w) \rangle = \frac{3\pi}{2\epsilon} \to \infty$$

At least one QNEC is violated for any choice of an equal time interval

For $x_1^{\pm}, x_2^{\pm} > 0$ and $x_2 > x_1$, in the $t, x_1, x_2 \gg \epsilon$ limit

$$\mathcal{Q}_+(1) = \frac{c}{6}\frac{(x_1 - t)(x_2 - t)(x_1^2 - 5x_1x_2 + x_2^2 - 3(x_1 + x_2)t - 3t^2)}{2(x_1 - x_2)^2(x_1 + t)(x_2 + t)\epsilon^2} < 0$$

- General proof of QNEC [Ceyhan and Faulkner, 2020] assumes: finite ANE and finite relative entropy with respect to the vacuum
- This state has finite relative entropy since it is in the identity sector (fusion of two identities can only produce identity and descendants) [Stéphan and Dubail, 2011]
- The local joining quench is a counter example to the QNEC when the ANE is not finite

Floquet CFT on a circle [Jiang and Mezei, 2024]

- 1+1-d CFT on a circle of circumference 2π
- For time T_0 evolve with

$$H_0 = \int_0^{2\pi} dx T_{00}(x) = L_0 + \bar{L}_0 - \frac{c}{12}$$

- Then for time T_1 evolve with

$$H_1 = \int_0^{2\pi} dx \sin^2\left(\frac{x}{2}\right) T_{00}(x) = L_0 - \frac{L_1 + L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1 + \bar{L}_{-1}}{2} - \frac{c}{12}$$

- Time reversal symmetric case: start with the vacuum at the midpoint of H_0 evolution
- Time evolution in Heisenberg picture$\rightarrow$ time dependent $\mathrm{SL}(2, \mathbb{R})$ transformation

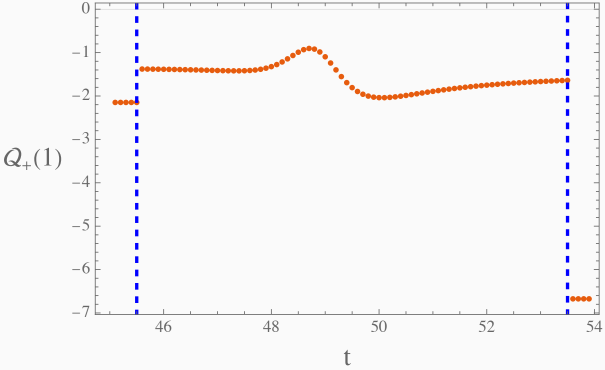
- Entanglement entropy computed as a two point function using the time dependent $\mathrm{SL}(2, \mathbb{R})$
- Energy momentum tensor calculated by mapping to the UHP vacuum and using the Schwarzian

System has three phases depending on $T_{0,1}$ [Wen and Wu, 2018]:

1. Heating phase (entropy grows linearly in time)
2. Non-heating phase (entropy oscillates in time)
3. Phase transition (entropy is logarithmic in time)

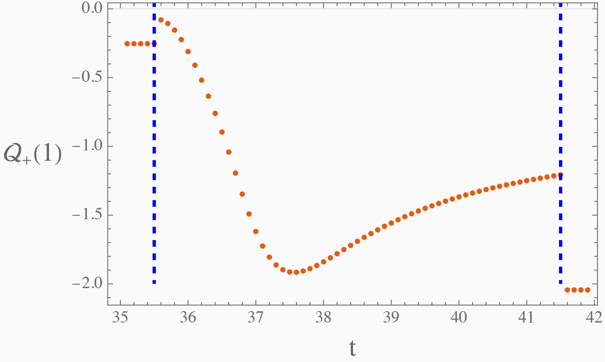
Heating phase QNEC

With $x_1 = 0, x_2 = 2$ and $T_0 = 1$, $T_1 = 8$



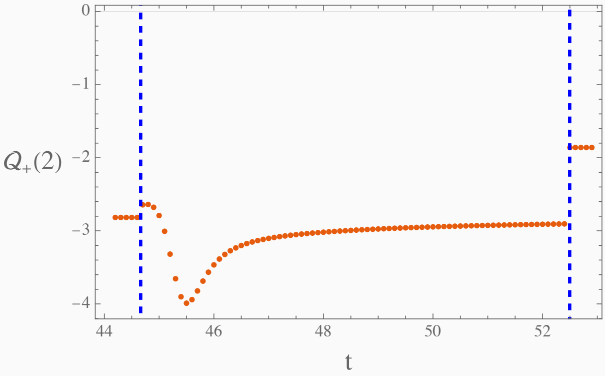
Non-Heating phase QNEC

With $x_1 = 0, x_2 = 2$ and $T_0 = 1$, $T_1 = 6$



QNEC at Phase transition

With $x_1 = 0, x_2 = 2$ and $T_0 = 1$, $T_1 = 7.8326$



- ANE has to be evaluated numerically by integrating upto some cutoff null coordinate $x_c^{\pm}$
- Find that ANE $\to \infty$ as $x_c^{\pm} \to \infty$
- Another counter example to QNEC when the ANE is not finite

Outroduction

Conclusion

- QNEC in holographic quenches $\rightarrow$ bounds on entropy production
- Likely due to bulk not being a limit of a solution to Einstein gravity coupled to matter
- NEC need not imply entanglement wedge nesting in discontinuous spacetimes
- CFT QNECs $\rightarrow$ counter examples to possible generalizations of Faulker and Ceyhan's proof

Outlook

- Can we place bounds on slower holographic quenches?
- [Almheiri et al., 2019] setup is an interesting model for quenches
- possible relaxations of the assumptions in Faulkner and Ceyhan's proof
- Implications of QNEC for spin chains via Temperley-Lieb algebra
- Bounds from Rènyi QNEC [Moosa et al., 2021]

Thank you.

Almheiri, A., Engelhardt, N., Marolf, D., and Maxfield, H. (2019).
The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole.
JHEP, 12:063.

Balakrishnan, S., Faulkner, T., Khandker, Z. U., and Wang, H. (2019).
A General Proof of the Quantum Null Energy Condition.
JHEP, 09:020.

Bekenstein, J. D. (1981).
Energy cost of information transfer.
Phys. Rev. Lett., 46:623–626.

Bera, M. N., Riera, A., Lewenstein, M., and Winter, A. (2017).
Generalized laws of thermodynamics in the presence of correlations.
Nature Communications, 8(1).

Bhattacharyya, S. and Minwalla, S. (2009).

Weak Field Black Hole Formation in Asymptotically AdS Spacetimes.
JHEP, 09:034.

Bousso, R., Fisher, Z., Koeller, J., Leichenauer, S., and Wall, A. C. (2016).
Proof of the quantum null energy condition.
Physical Review D, 93(2).

Calabrese, P. and Cardy, J. (2007a).
Entanglement and correlation functions following a local quench: a conformal field theory approach.
J. Stat. Mech., 0710(10):P10004.

Calabrese, P. and Cardy, J. (2007b).
Quantum Quenches in Extended Systems.
J. Stat. Mech., 0706:P06008.

Calabrese, P. and Cardy, J. L. (2005).
Evolution of entanglement entropy in one-dimensional systems.

J. Stat. Mech., 0504:P04010.

Ceyhan, F. and Faulkner, T. (2020).
Recovering the QNEC from the ANEC.
Commun. Math. Phys., 377(2):999–1045.

Deffner, S. and Lutz, E. (2010).
Generalized clausius inequality for nonequilibrium quantum processes.
Phys. Rev. Lett., 105:170402.

Hubeny, V. E., Rangamani, M., and Takayanagi, T. (2007).
A covariant holographic entanglement entropy proposal.
Journal of High Energy Physics, 2007(07):062–062.

Hubeny, V. E., Rangamani, M., and Tonni, E. (2013).
Thermalization of Causal Holographic Information.
JHEP, 05:136.

Jiang, H. and Mezei, M. (2024).
New horizons for inhomogeneous quenches and Floquet CFT.

Koeller, J. and Leichenauer, S. (2016).
Holographic proof of the quantum null energy condition.
Physical Review D, 94(2).

Leichenauer, S., Levine, A., and Shahbazi-Moghaddam, A. (2018).
Energy density from second shape variations of the von Neumann entropy.
Phys. Rev. D, 98(8):086013.

Liu, H. and Suh, S. J. (2014).
Entanglement Tsunami: Universal Scaling in Holographic Thermalization.
Phys. Rev. Lett., 112:011601.

Malik, T. A. and Lopez-Mobilia, R. (2020).
Proof of the quantum null energy condition for free fermionic field theories.
Phys. Rev. D, 101(6):066028.

Moosa, M., Rath, P., and Su, V. P. (2021).

A Rényi quantum null energy condition: proof for free field theories.
JHEP, 01:064.

Ryu, S. and Takayanagi, T. (2006).
Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence.
Phys. Rev. Lett., 96:181602.

Stéphan, J.-M. and Dubail, J. (2011).
Local quantum quenches in critical one-dimensional systems: entanglement, the Loschmidt echo, and light-cone effects.
Journal of Statistical Mechanics: Theory and Experiment, 2011(8):08019.

Wen, X. and Wu, J.-Q. (2018).
Floquet conformal field theory.

Yunger Halpern, N. and Renes, J. M. (2016).
Beyond heat baths: Generalized resource theories for small-scale thermodynamics.

Phys. Rev. E, 93:022126.