Anyons and topological quantum computation

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Plan of the talk

- Motivation and background : What are anyons? Why is there so much interest in anyons these days?
- Kitaev model : Explicit model for Majorana modes, realization in semiconductor wires, current experimental scenario, extensions to other non-abelian anyons like parafermions
- Towards topological quantum computation : General model of anyons, fusion, braiding rules, unitary gates through combination of braiding matrices

Motivation and background

What are anyons?

- First point - All elementary particles are either fermions or bosons.

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\, \psi(\mathbf{r}_2, \mathbf{r}_1)$$

- But for emergent quasiparticles in condensed matter systems, this is not necessary. Simplest generalisation of bosons and fermions are abelian anyons that can have complex phases under exchange

- $$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{\pm i\theta}\psi(\mathbf{r}_2, \mathbf{r}_1)$$

Leinaas, Myrrheim, Wilczek

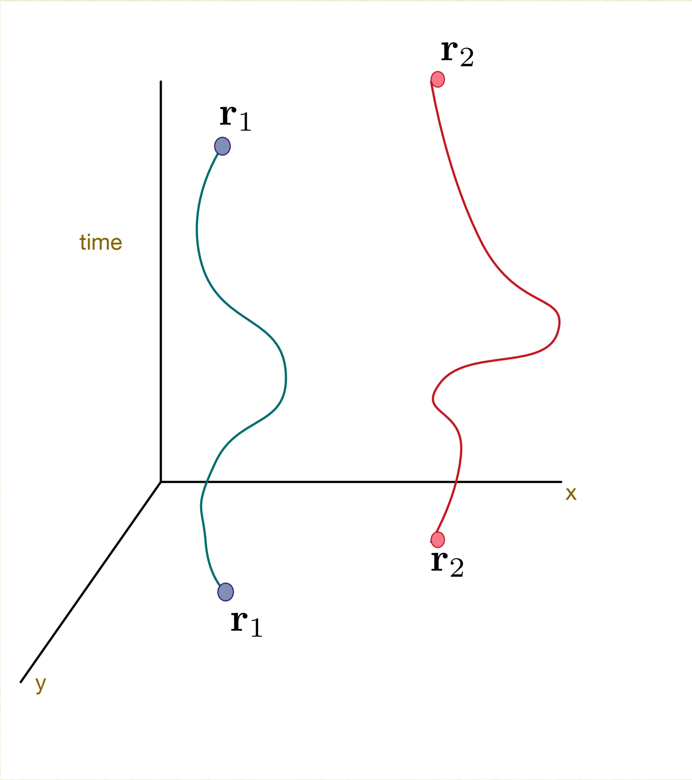
1977

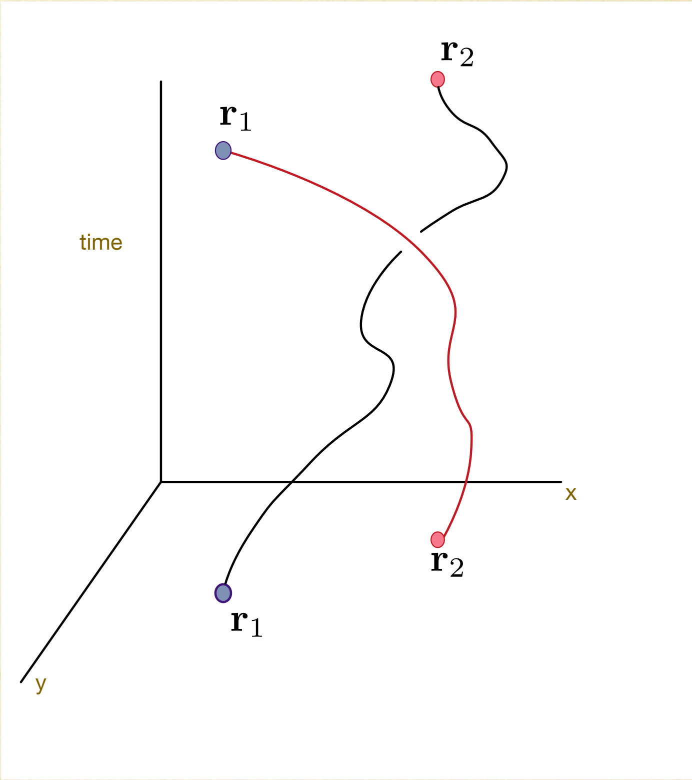
Why is this possible?

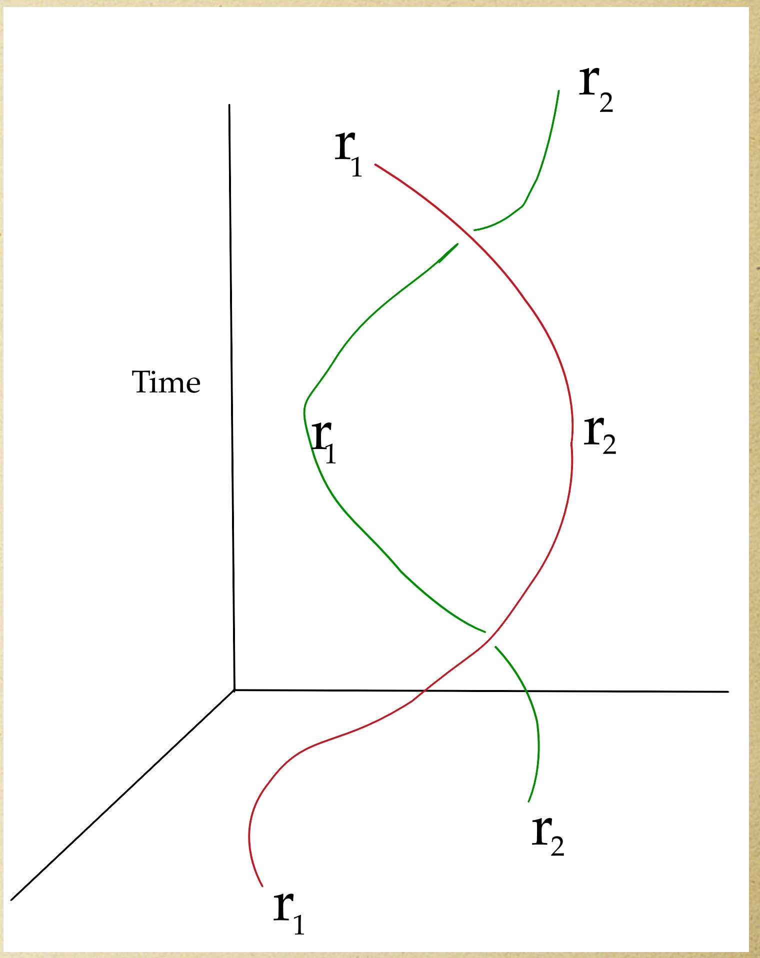
- If you think of exchange as an adiabatic process of taking particles to the other position, then double exchange means return to the original position or does it?

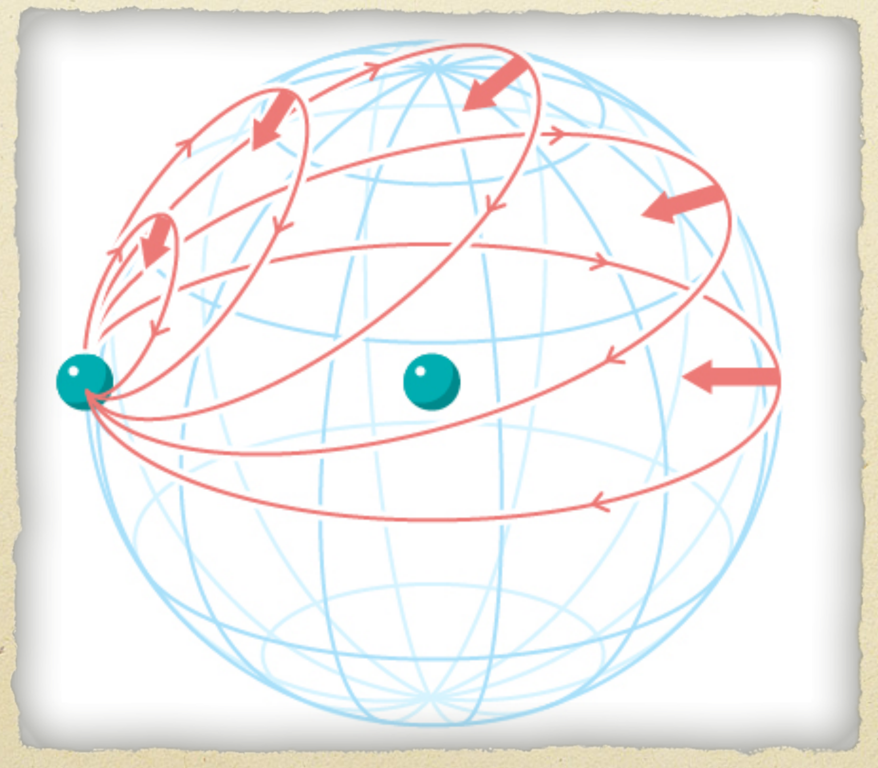
- In three dimensions, the loop formed by the relative coordinate for two exchanges can be shrunk to a point by taking it off the plane and it is as if nothing has happened

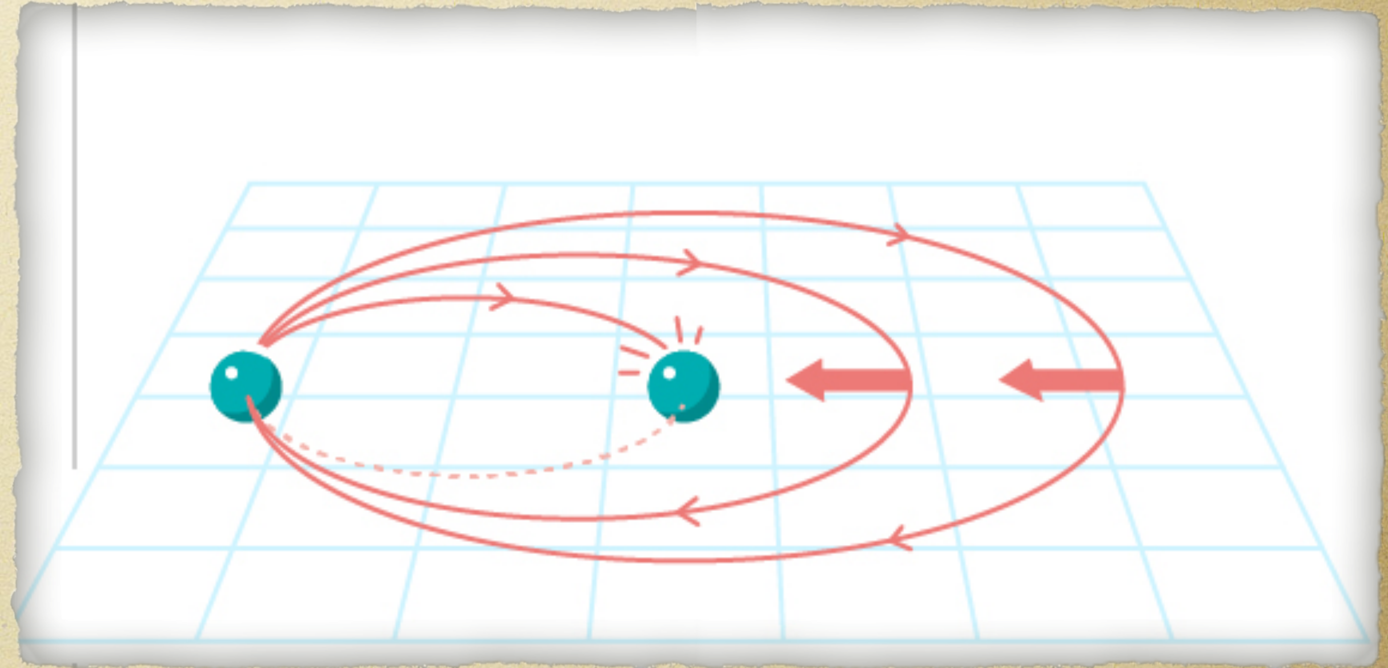
- But in two dimensions, this is not possible without the two particles crossing one another (relative coordinate going to zero)



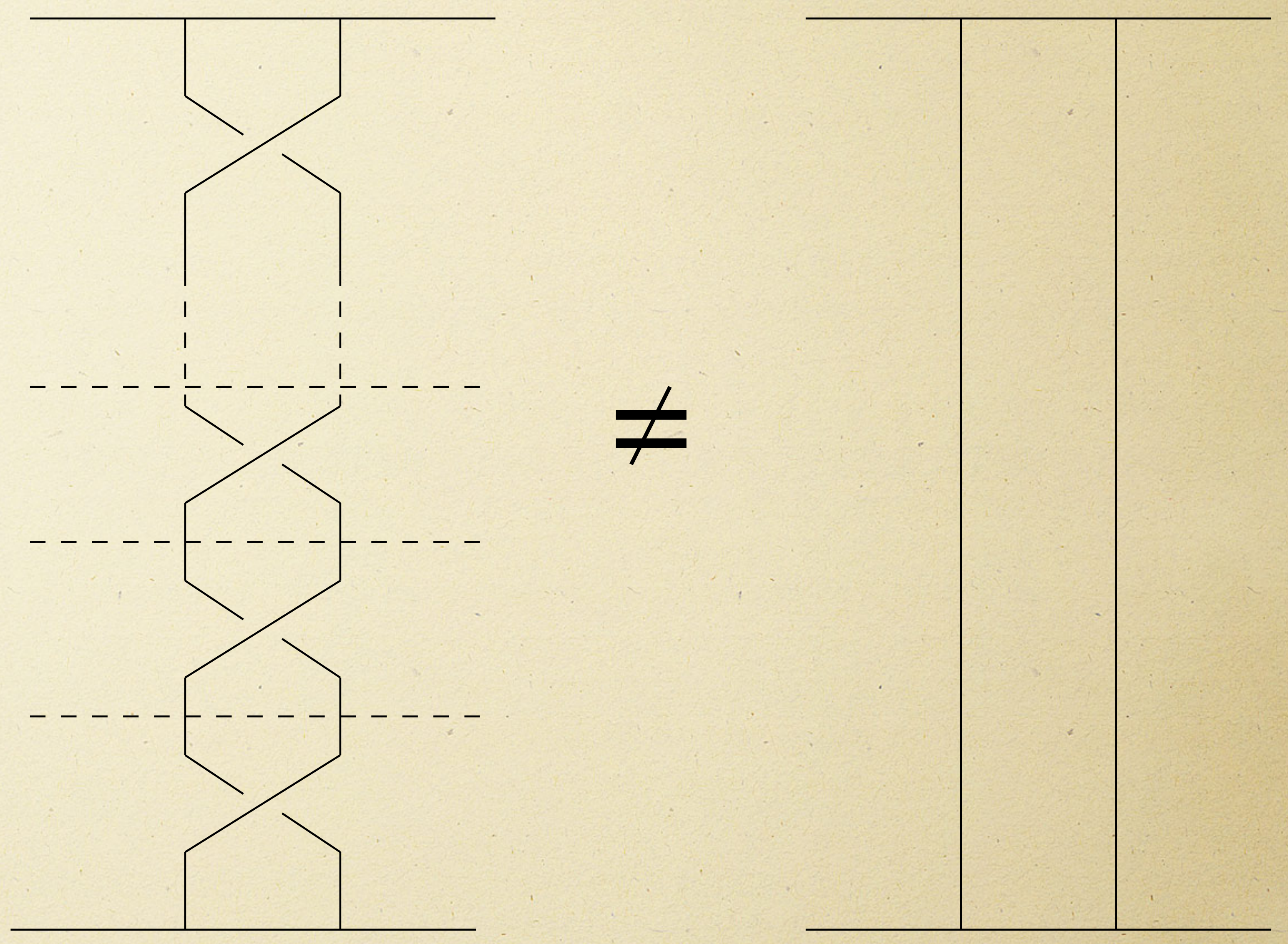






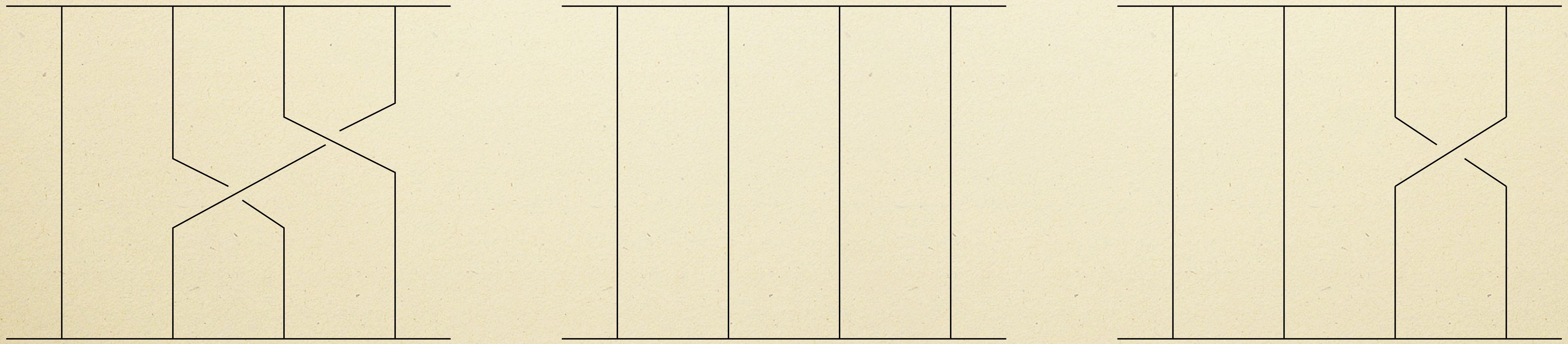


- Leads to `anyon' statistics in two dimensions
- Mathematically, distinction is that removal of origin (single point) in 2 dimensions makes the space multiply connected
- You can count the number of times you go around the origin
- So particles are classified in terms of the `braid group' (and not permutation group) in two dimensions



What is the braid group?

- Braid group is the group G of paths through configuration space
- Easiest way of seeing what the braid group is via pictures. Consider braid group with 4 particles - elements of B_4



(a) (b) (c)

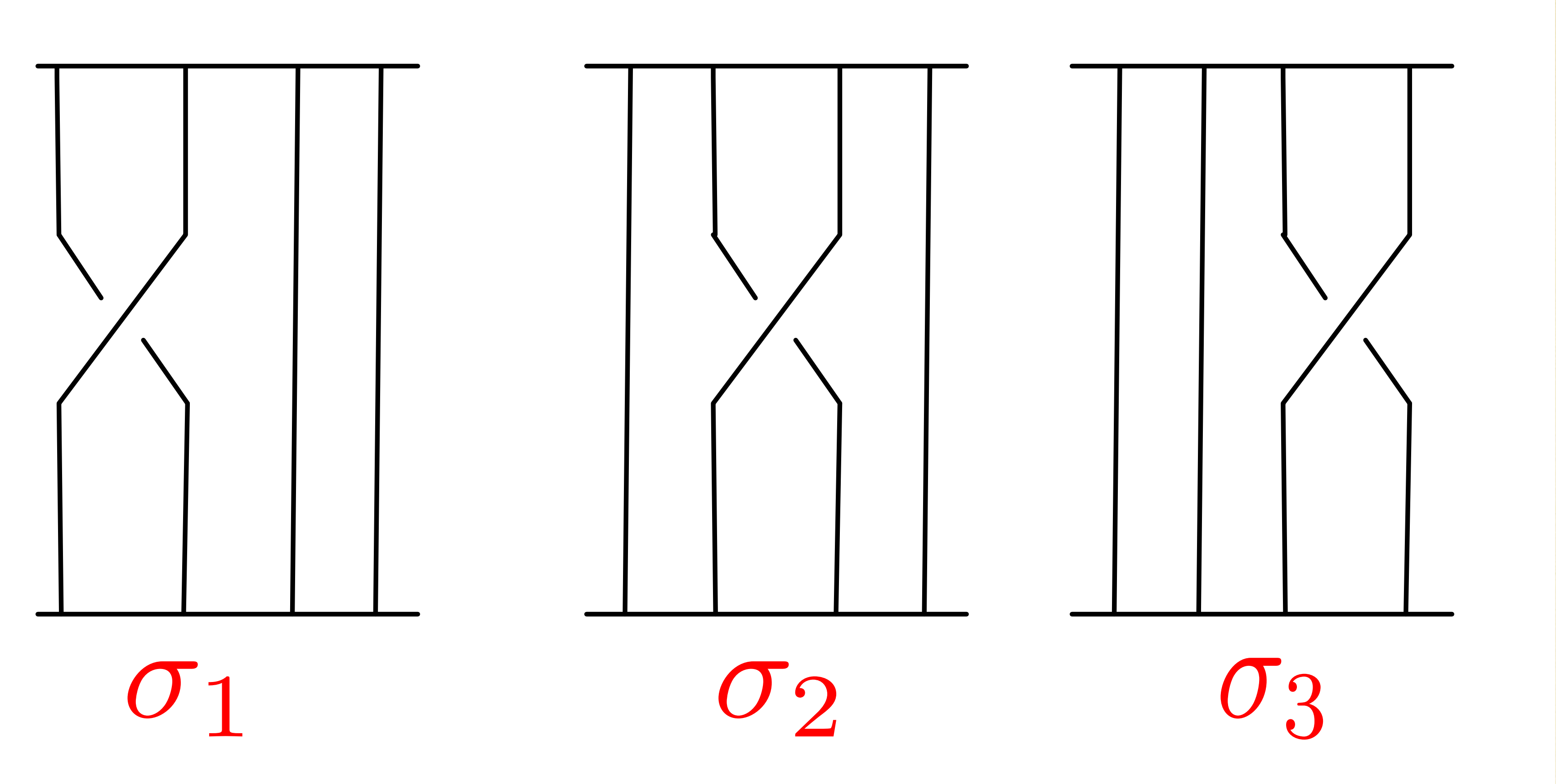


FIG. 7: The three generators of the braid group B_4

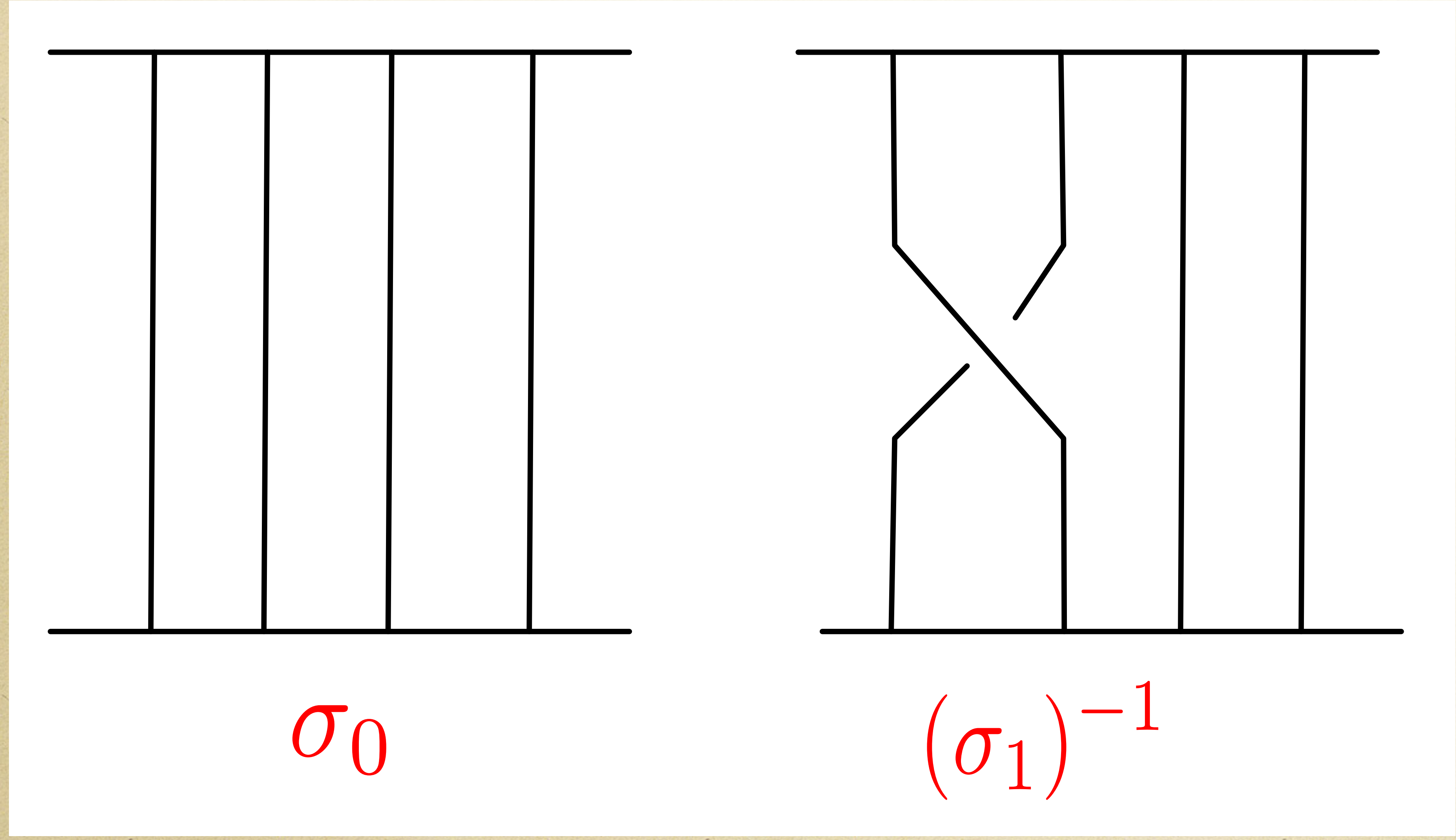


FIG. 8: The identity and the inverse of the generator σ_1.

- 3 generators of the braid group are σ_1, σ_2 and σ_3
- Identity is σ_0
- Inverse is $(\sigma_1)^{-1}$

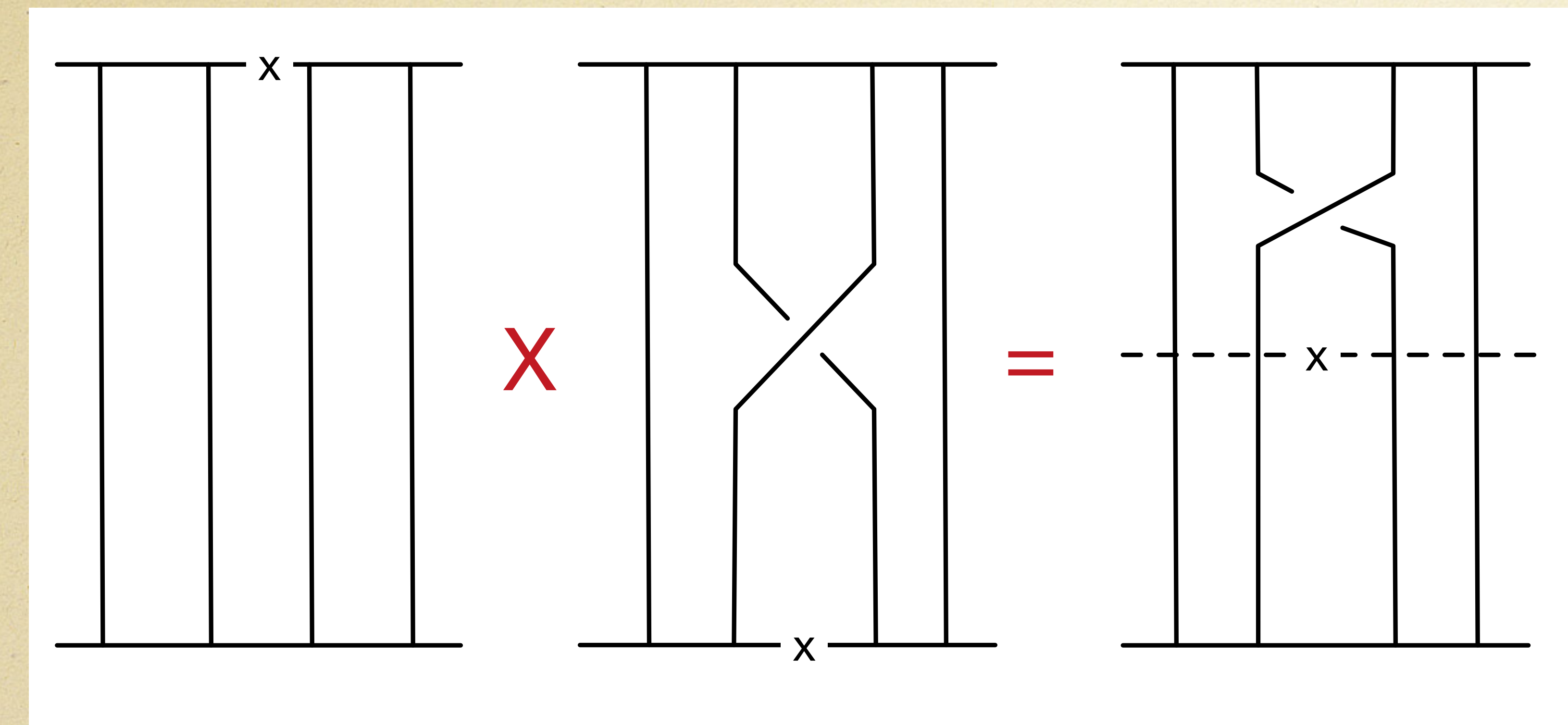
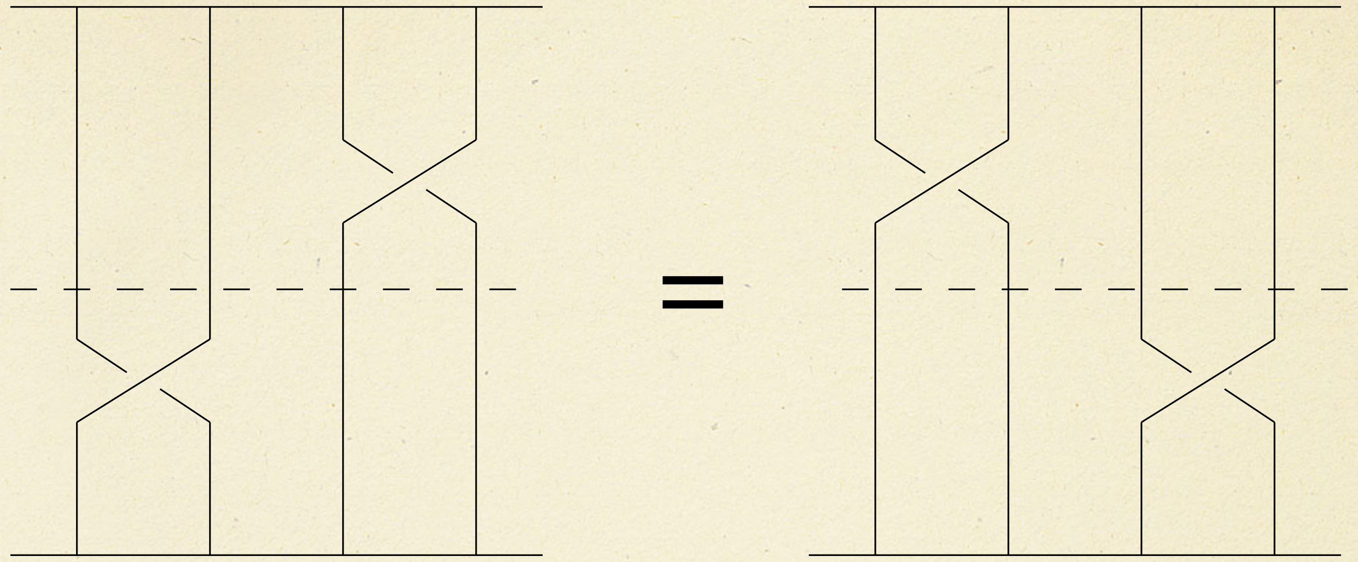
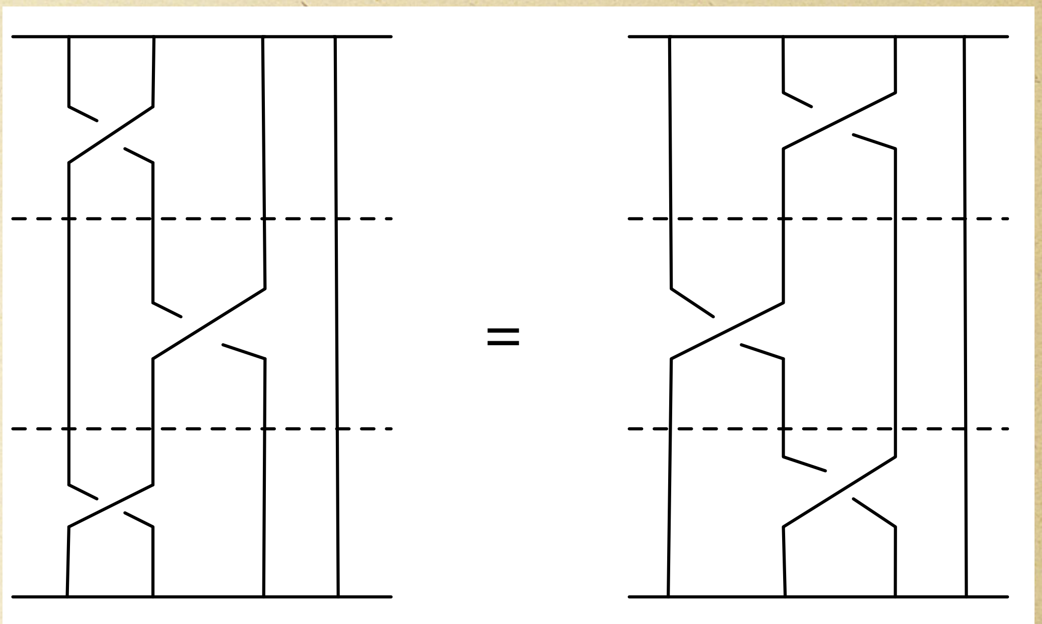


FIG. 9: Group multiplication.



$$\sigma_1\sigma_3 = \sigma_3\sigma_1$$



$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

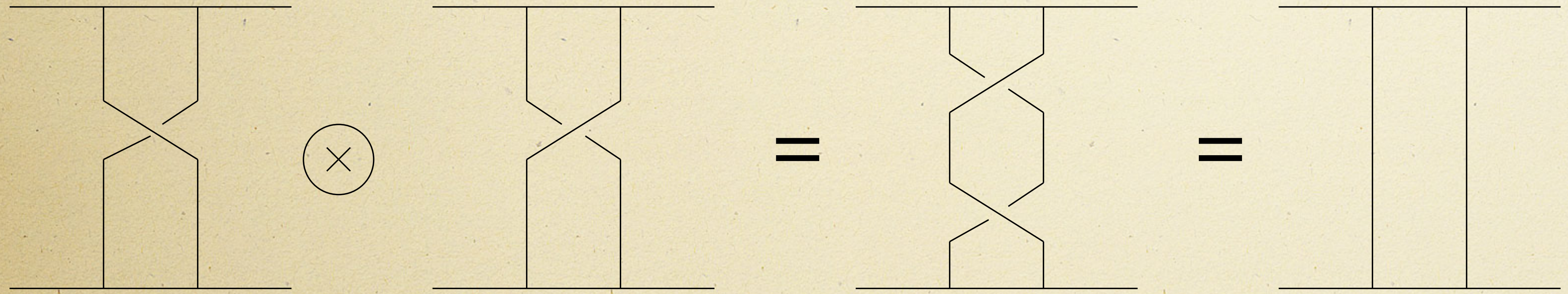


FIG. 10: Product of σ_1 and $(\sigma_1)^{-1}$ giving rise to identity.

- Group multiplication and Yang-Baxter relations

- The important point for bosons and even fermions, is that once you have exchanged the particles twice, then it can be completely forgotten that you exchanged at all
- For anyons, entire history is important - cannot be forgotten after any number of exchanges - reason for why even a system of free anyons is very hard to solve
- This phase leads to many new phases and phenomena, not surprising since even the minus/plus sign for fermions/bosons leads to phenomena like metals, insulators, for fermions and superfluidity and Bose condensation for bosons

Mutual statistics

- Can also drag distinguishable particles around each other in 2D. Will not come back to the same configuration (unlike in 3D)
- Here again, one can get a phase $S_{ab} = e^{i\theta_{ab}}$ (for taking particle `a' around particle `b')
- For identical anyons, exchange phase often written as $T_{ab} = \delta_{ab} e^{i\theta_a}$

Non-abelian anyons

- Anyons which acquire just a phase under exchange are abelian anyons - one dimensional representation of the braid group. To have non-abelian anyons, need higher dimensional representations of the braid group
- Can occur when there is a degenerate set of states with particles fixed at their positions
- Exchange of non-abelian anyons leads to rotation in the degenerate manifold

$$\psi_a(\mathbf{r}_1, \mathbf{r}_2) \to U_{ab}\psi_b(\mathbf{r}_2, \mathbf{r}_1)$$

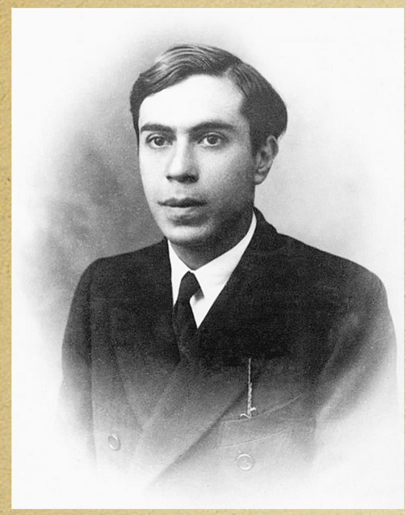
Frohlich, 1988

Why is there so much interest in anyons?
Topological quantum computation

- Quantum computer uses qubits instead of classical bits. The qubit represents the state of a wave function, - e.g, a spin 1/2 which can be in a state $|\uparrow>$ or $|\downarrow>$
- Unlike classical bits, qubits can also be in a state of superposition written as $\alpha|\uparrow> + \beta|\downarrow>$ or more commonly as $\alpha|0> + \beta|1>$
- For non-abelian anyons, ground state degeneracy means that multiple distinct states of the particles have same configuration of identical particles

- Prepare system in one ground state - exchange 2 quasi-particles - transformed by unitary transformation to another state in the ground state manifold
- Unitary transformations are the quantum gates which change the quantum states - building blocks for quantum circuits

Kitaev Model for Majorana modes - the simplest example of non-abelian anyons



Majorana modes - simplest non-abelian Anyon

- Majorana fermions initially introduced in high energy physics, as possible new elementary particles which can be their own anti-particles such as candidate neutrinos or possible particles such as supersymmetric partners of photons, neutral Higgs, etc
- Majorana modes as quasi-particles in condensed matter systems - called Majorana because they are self-conjugate
- These are not fermions, but instead behave as non-abelian anyons

Kitaev model for Majorana modes

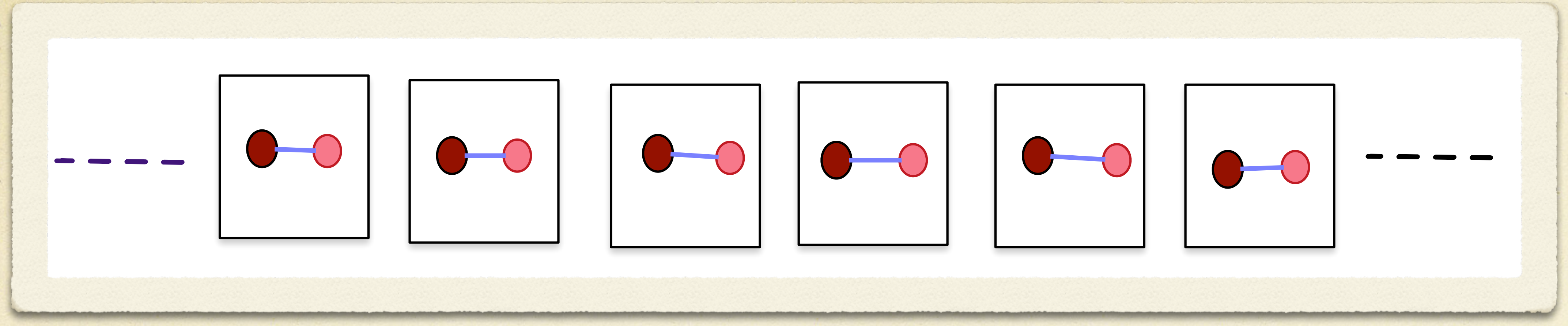
Kitaev 2001

- The `Ising' model for the Majorana modes is the Kitaev model which can be rewritten in terms of the Majorana modes

$$H = -\mu \sum_{x=1}^{N} c_x^\dagger c_x - \frac{1}{2} \sum_{x=1}^{N} (t c_x^\dagger c_{x+1} + \Delta c_x c_{x+1} + h.c.)$$

$$c_x = \frac{1}{2}(\gamma_{x,A} + i\gamma_{x,B}), \quad c_x^\dagger = \frac{1}{2}(\gamma_{x,A} - i\gamma_{x,B})$$

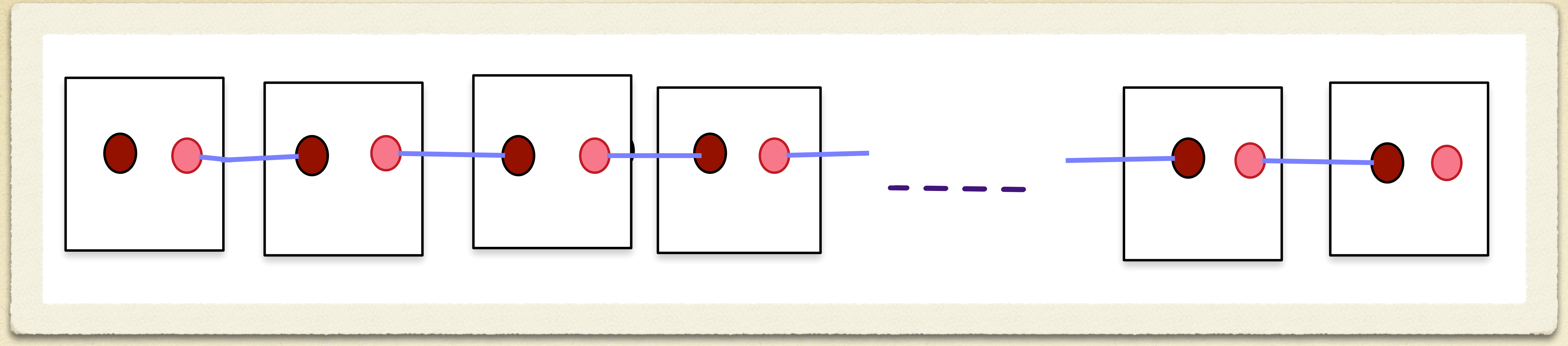
$$\{\gamma_a, \gamma_b\} = \delta_{ab}, \quad \gamma_a^2 = 1, \gamma_b^2 = 1 \qquad \gamma^\dagger = \gamma$$



- Idea is to think of fermions on a site being made of 2 `Majorana modes' and changing parameters so that the fermions get fractionalized and Majorana modes get separated

- When $\mu < 0$ and $t = \Delta = 0$, only bonds between Majoranas at same site

$$H = \frac{-\mu}{2} \sum_{x=1}^{N} (1 + i\gamma_{x,B}\gamma_{x,A})$$



- When $\mu = 0$ and $t = \Delta$, imagine new fermions made from Majoranas on nearest neighbour sites. Unpaired Majoranas at the two ends

$$H = -\frac{it}{2}\sum_{x=1}^{N-1} \gamma_{x,B}\gamma_{x+1,A}$$

- Note - highly correlated state in terms of original fermions, but simple in terms of new fermions defined on nearest neighbor sites

- Non-local fermion $d = \dfrac{\gamma_{A,1} + i\gamma_{B,N}}{2}$

- Energy is independent of whether this fermion state is occupied or not
- So non-unique ground state - rather, ground state is doubly degenerate - $|0>$ and $|1> = d^\dagger |0>$

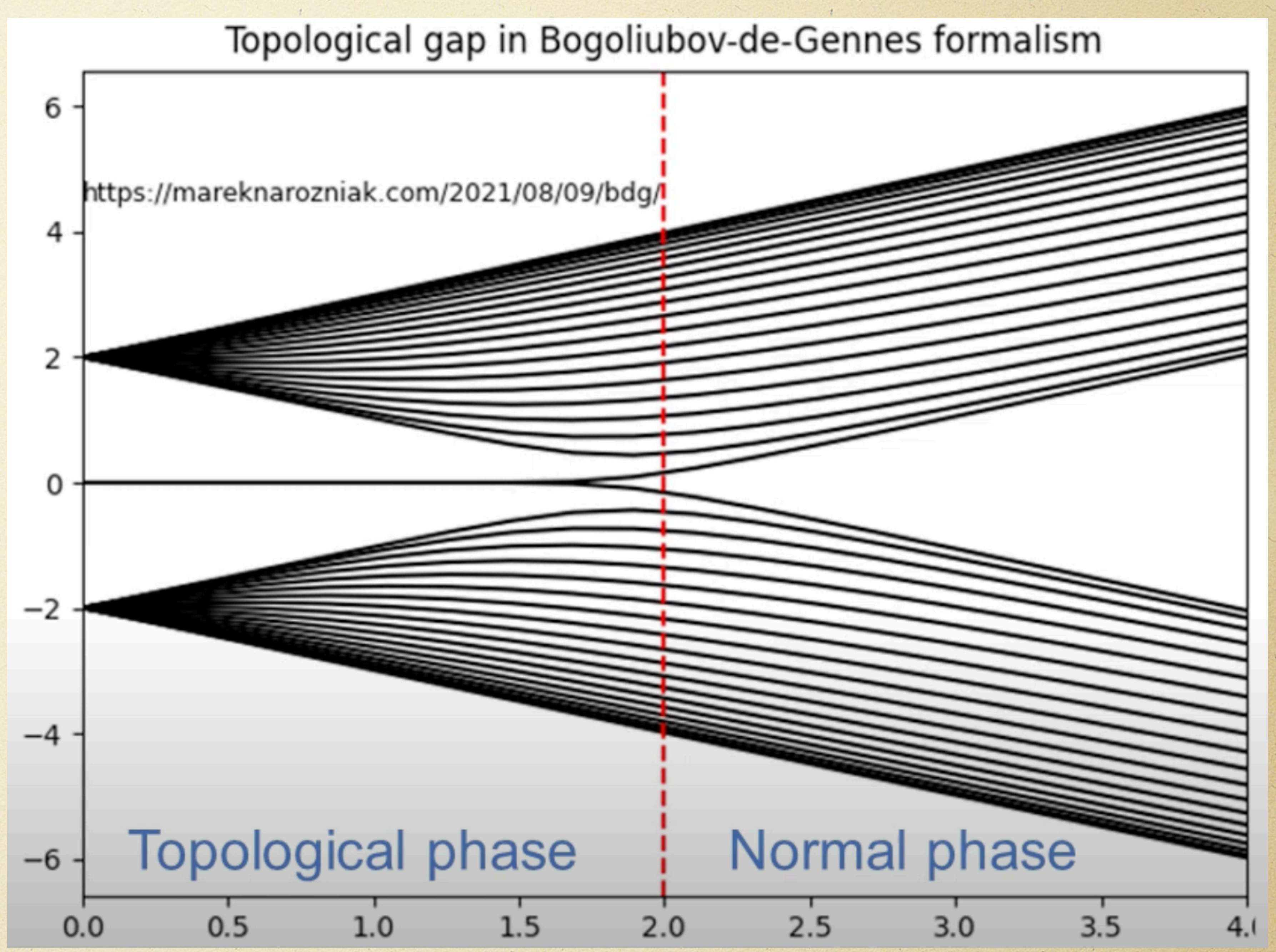
Main point

- Hamiltonian has no dependence on end Majoranas and is independent of whether or not the non-local fermion formed from them is occupied or not occupied - ground state is not unique, it is doubly degenerate

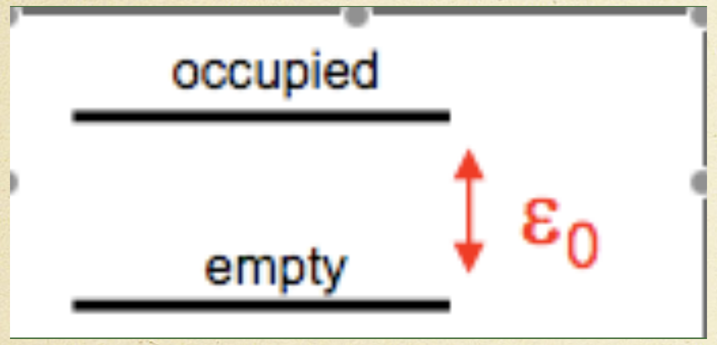
- This was Kitaev's magic trick. He found the model which could `fractionalise' the fermion and put different pieces of them at the two ends. So they behave as independent quasiparticles

- Range of parameters : $\mu > 2t$ for which system is topologically trivial (unique ground state) and $\mu < 2t$, for which system is topological with doubly degenerate ground state and end Majorana modes

- The model needs spinless fermions (or p-wave superconductors where the superconductor couples same spins)



Why are these end Majorana zero modes relevant for quantum computation?



$$d^\dagger d = 1$$

$$d^\dagger d = 0$$

$$\epsilon_0 = 0$$

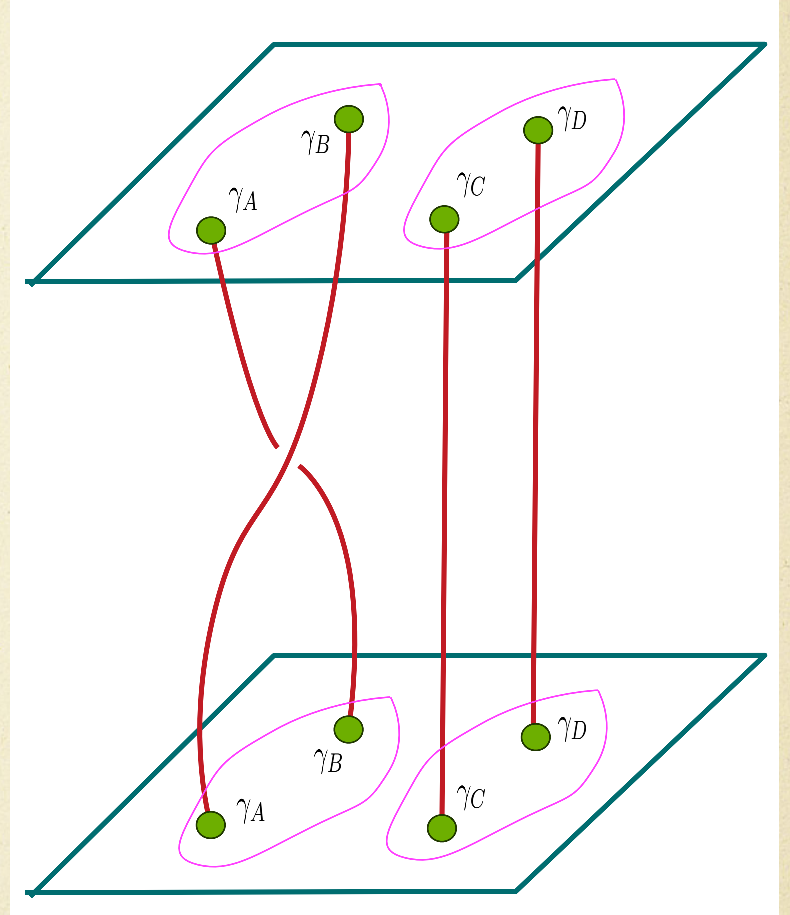
- The non-local fermion can either be occupied -forms the qubit. So the model has 2 zero energy bound states at the two ends which can be thought of as `fractional pieces' of an underlying non-local fermion
- Cannot be easily disturbed by local disorder because different pieces of fermion are at different locations - occupation of state can only change when all pieces are changed together - so state whose occupation or non-occupation forms qubit is highly stable

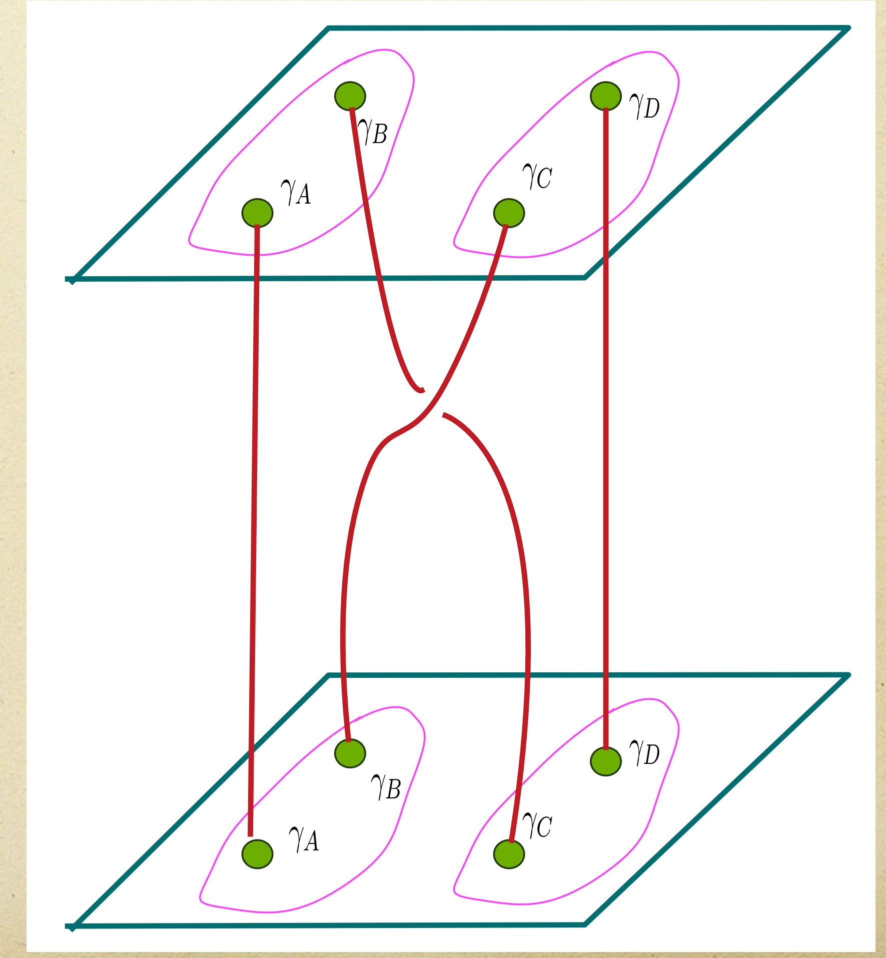
Fusion of the Majorana zero modes

- Now, let us assume we have isolated Majorana modes
- Can make `composite' anyons by `fusing' two anyons together into either the vacuum $|0>$ (no fermion) or $|1>$ (1 fermion)
- Since the fusion outcome is not unique, it is a non-abelian anyon

Braiding of the Majorana zero modes

- Majorana zero modes can be braided
- Top- Majoranas belonging to the same fermion have been braided
- Bottom - Majoranas belonging to two different fermions have been braided





- We can now explicitly find the operators that does the braiding
- Let us see what happens under exchange of 2 Majoranas
  $$\gamma_1 \to \gamma_1' = U_{12}^\dagger \gamma_1 U_{12} = e^{i\phi}\gamma_2$$
- U_{12} is the exchange operator and the phase can be chosen to be 1
- But then the phase of the other exchange $\gamma_2 \to \gamma_2' = U_{12}^\dagger \gamma_1 U_{12} = e^{i\phi}\gamma_1$ is determined.
- It is forced to be -1 because the product of $\gamma_1\gamma_2$ is forced to remain unchanged because of the ground state parity (value of $d^\dagger d$) is fixed

$$\gamma_1 = d + d^\dagger$$
$$\gamma_2 = (-i)(d - d^\dagger)$$
$$i\gamma_1\gamma_2 = (2d^\dagger d - 1)$$

- Only possible if $i\gamma_1\gamma_2 \to i\gamma_1'\gamma_2' = i\gamma_2(-\gamma_1) = i\gamma_1\gamma_2$
- So essentially, we have $\gamma_1 \to \gamma_2, \gamma_2 \to -\gamma_1$
- This is implemented by the exchange operator

$$U_{12} = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) = e^{\frac{\pi}{4}\gamma_1\gamma_2}$$

- In terms of fermion operators $U_{12} = F(d^\dagger d = n) = e^{i\frac{\pi}{4}(1-2n)}$
- So remains in the same state, since fermion parity cannot change

- To show how one moves in the degenerate ground state manifold, need to consider a minimum of four Majorana modes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and consider various exchanges

- We make 2 normal fermions as follows

$$d_1 = \frac{\gamma_1 + i\gamma_2}{2}, \quad d_1^\dagger = \frac{\gamma_1 - i\gamma_2}{2} \qquad d_2 = \frac{\gamma_3 + i\gamma_4}{2}, \quad d_2^\dagger = \frac{\gamma_3 - i\gamma_4}{2}$$

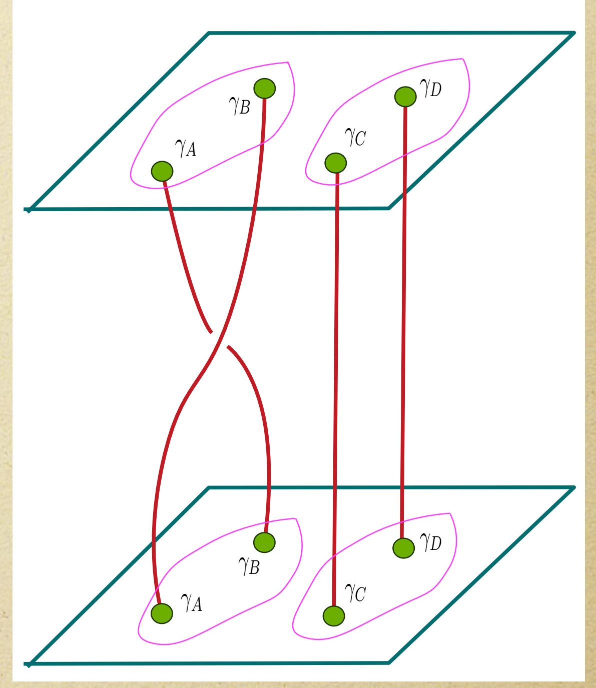
- States of the system given by $|n_1, n_2> = d_1^{\dagger n_1} d_2^{\dagger n_2} |0,0>$

- $|0,0> , |1,0> , |0,1> , |1,1>$

Ivanov,2000

- We can now explicitly find the operators that exchange the Majoranas

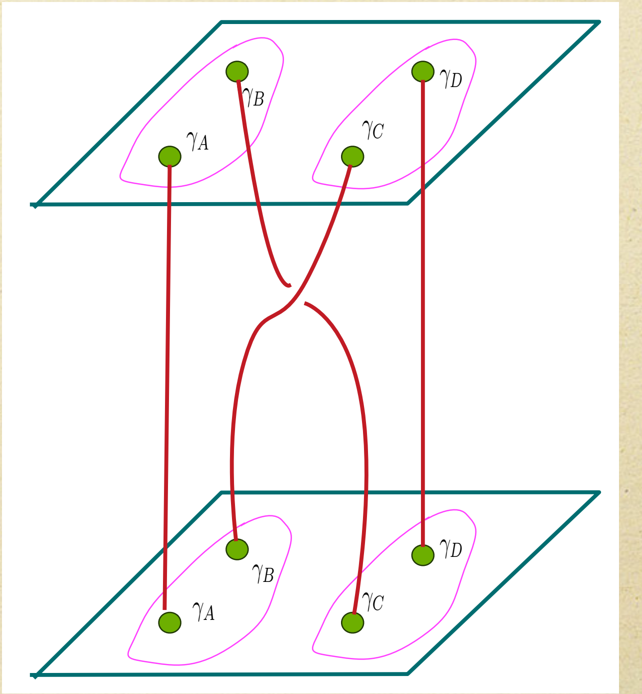
- Operator U_{12} exchanges Majoranas γ_1 and γ_2 keeping the others fixed and similarly, we have exchange operators U_{23} and U_{34}. Here it is only U_{23} which allows us to turn one pair of fermions into another



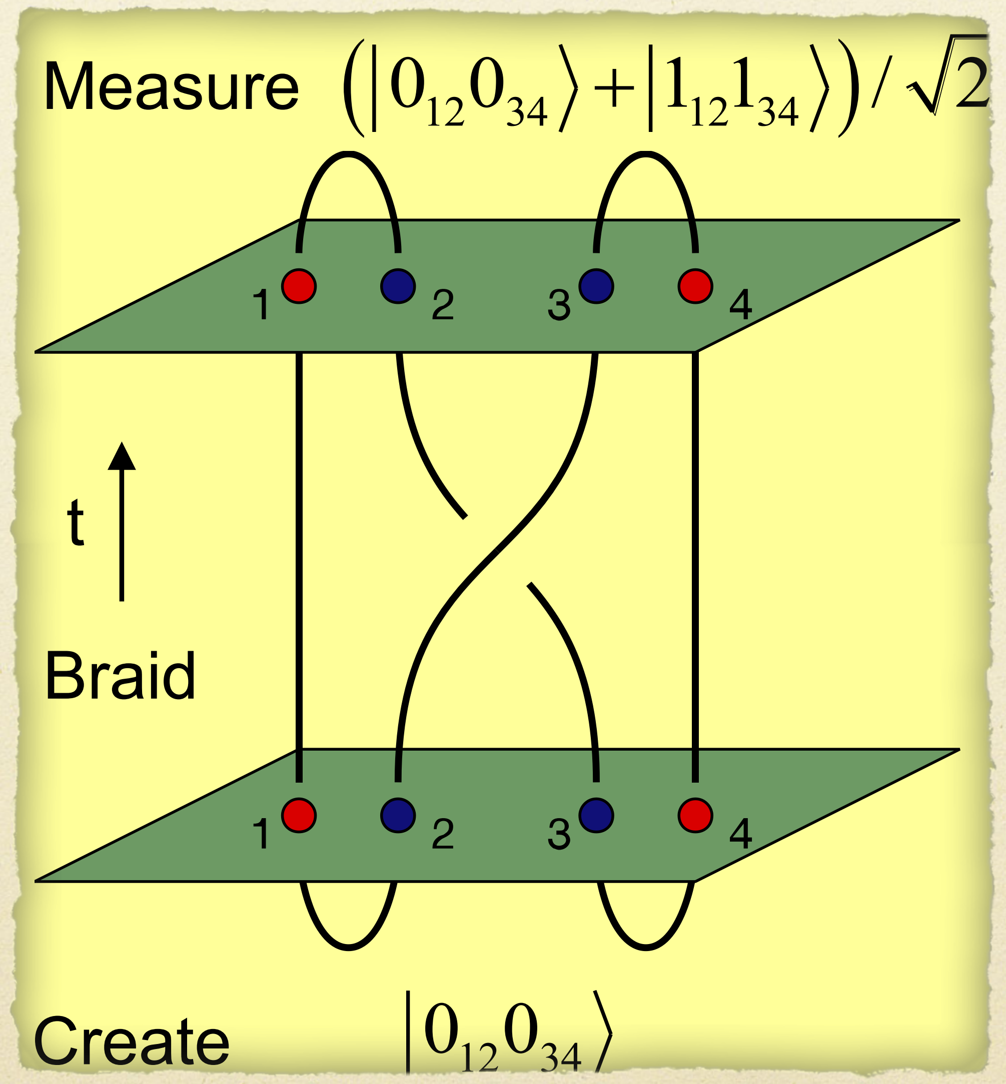
- $$U_{12}|n_1, n_2> = e^{i\frac{\pi}{4}(1-2n_1)}|n_1, n_2>$$

$$U_{23}|n_1, n_2> = \frac{1}{\sqrt{2}}[\,|n_1, n_2> + i(-1)^{n_1}|1-n_1, 1-n_2>\,]$$

$$U_{34}|n_1, n_2> = e^{i\frac{\pi}{4}(1-2n_2)}|n_1, n_2>$$



- Hence braiding of the Majoranas performs unitary operations - useful for quantum computation



Kitaev, Preskill, 2000

- So what is TOPOLOGICAL quantum computation?

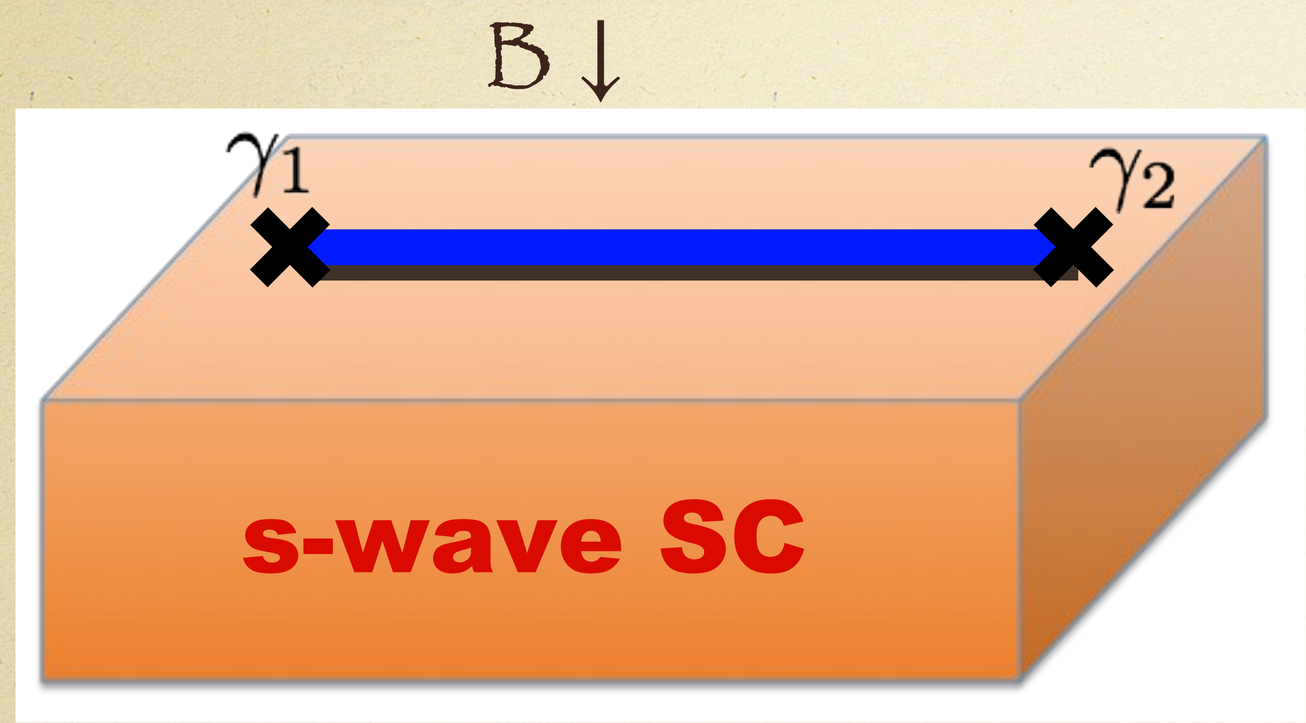
- Essentially using non-abelian anyons - qubits formed from the occupation/non-occupation of the non-local fermion and quantum gates are the unitary braiding matrices which can change the state of the qubit

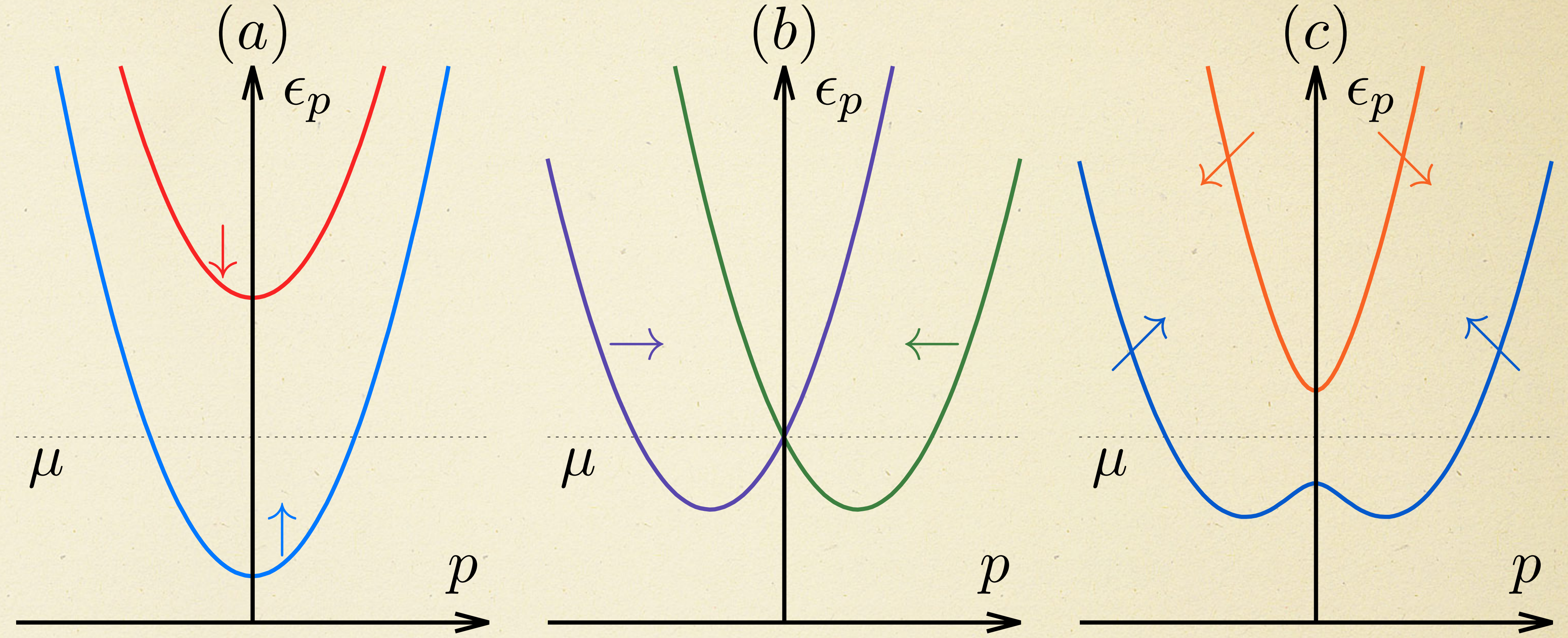
- Different pieces of fermion are at different locations - occupation of state cannot be easily changed by `local disorder'. So the qubit is topologically stable. (Stability proportional to length of wire). Braiding also cannot be changed by local deformation

- Hence, TOPOLOGICAL PROTECTION from decoherence, noise, errors - FAULT TOLERANT. Hence, expected to be easier for scalability as well,

Where to find or how to engineer Majorana modes?

Semiconductor nanowires



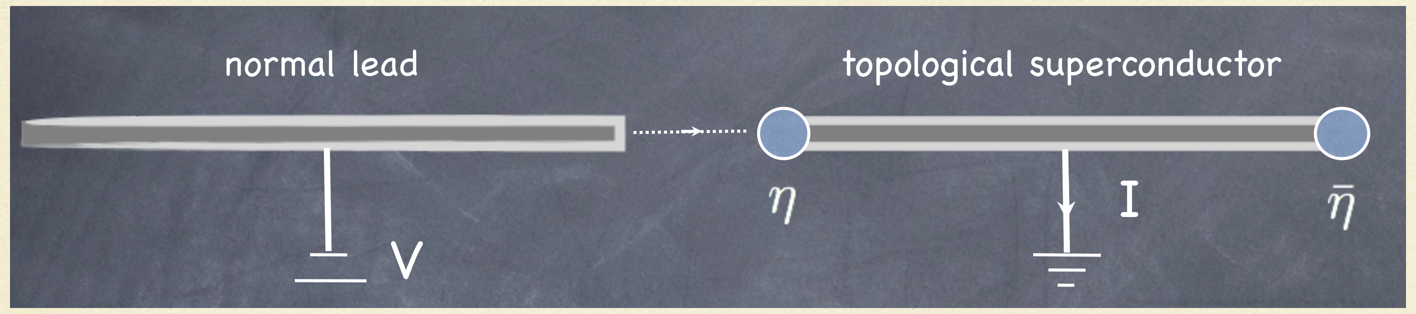


- Most promising platform - semiconductor wires (with strong spin-orbit coupling) on s-wave superconductors in an external magnetic field

$$H = \int dx \psi^{\dagger}[-\frac{\partial_x^2}{2m} - \mu - ui\hbar\partial_x\sigma_y - \frac{g\mu_b B}{2}\sigma_z]\psi + \Delta[\psi_{\uparrow}\psi_{\downarrow} + h.c.]$$

- Engineered to mimic the Kitaev model, so expect to have a Majorana bound state at the edge of the topological superconductor

Theoretical proposal



- Tunneling from a normal lead into the end of the wire should give signature of the Majorana mode which should be a state at zero energy
- Expected zero bias conductance $$[\frac{\delta I}{\delta V}]_{V=0} = \frac{2e^2}{h}$$

Lutchyn,Sau and Das Sarma, 2010

Oreg,Refeal and von Oppen, 2010

Quest for Majoranas

- Many experiments have tried to look for the Majorana modes

Zero-bias peaks and splitting in an Al-InAs nanowire topological superconductor as a signature of Majorana fermions

Anindya Das†, Yuval Ronen†, Yonatan Most, Yuval Oreg, Moty Heiblum* and Hadas Shtrikman

Signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices

V. Mourik,[1*] K. Zuo,[1*] S. M. Frolov,[1] S. R. Plissard,[2] E. P. A. M. Bakkers,[1,2] L. P. Kouwenhoven[1]†

Spin-resolved Andreev levels and parity crossings in hybrid superconductor–semiconductor nanostructures

Eduardo J. H. Lee[1], Xiaocheng Jiang[2], Manuel Houzet[1], Ramón Aguado[3], Charles M. Lieber[2] and Silvano De Franceschi[1]*

Parity lifetime of bound states in a proximitized semiconductor nanowire

A. P. Higginbotham[1,2†], S. M. Albrecht[1†], G. Kiršanskas[1], W. Chang[1,2], F. Kuemmeth[1], P. Krogstrup[1], T. S. Jespersen[1], J. Nygård[1,3], K. Flensberg[1] and C. M. Marcus[1]*

Anomalous Zero-Bias Conductance Peak in a Nb–InSb Nanowire–Nb Hybrid Device

M. T. Deng,† C. L. Yu,† G. Y. Huang,† M. Larsson,† P. Caroff,‡ and H. Q. Xu†,§,*

Consensus?

Quantized Majorana conductance

Hao Zhang[1]*, Chun-Xiao Liu[2]*, Sasa Gazibegovic[3]*, Di Xu[1], John A. Logan[4], Guanzhong Wang[1], Nick van Loo[1], Jouri D. S. Bommer[1], Michiel W. A. de Moor[1], Diana Car[3], Roy L. M. Op het Veld[3], Petrus J. van Veldhoven[3], Sebastian Koelling[3], Marcel A. Verheijen[3,5], Mihir Pendharkar[6], Daniel J. Pennachio[4], Borzoyeh Shojaei[4,7], Joon Sue Lee[7], Chris J. Palmstrøm[4,6,7], Erik P. A. M. Bakkers[3], S. Das Sarma[2] & Leo P. Kouwenhoven[1,8]

Majorana zero-modes—a type of localized quasiparticle—hold great promise for topological quantum computing[1]. Tunnelling spectroscopy in electrical transport is the primary tool for identifying the presence of Majorana zero-modes, for instance as a zero-bias peak in differential conductance[2]. The height of the Majorana zero-bias peak is predicted to be quantized at the universal conductance value of $2e^2/h$ at zero temperature[3] (where e is the charge of an electron and h is the Planck constant), as a direct consequence of the famous Majorana symmetry in which a particle is its own antiparticle. The Majorana symmetry protects the quantization against disorder, interactions and variations in the tunnel coupling[3–5]. Previous experiments[6], however, have mostly shown zero-bias peaks much smaller than $2e^2/h$, with a recent observation[7] of a peak height close to $2e^2/h$. Here we report a quantized conductance plateau at $2e^2/h$ in the zero-bias conductance measured in indium antimonide semiconductor nanowires covered with an aluminium superconducting shell. The height of our zero-bias peak remains constant despite changing parameters such as the magnetic field and tunnel coupling, indicating that it is a quantized conductance plateau. We distinguish this quantized Majorana peak from possible non-Majorana origins by investigating its robustness to electric and magnetic fields as well as its temperature dependence. The observation of a quantized conductance plateau strongly supports the existence of Majorana zero-modes in the system, consequently paving the way for future braiding experiments that could lead to topological quantum computing.

A semiconductor nanowire coupled to a superconductor can be tuned into a topological superconductor with two Majorana zero-modes localized at the wire ends[1,8,9]. Tunnelling into a Majorana mode will show a zero-energy state in the tunnelling density-of-states, that is, a zero-bias peak (ZBP) in the differential conductance (dI/dV)[2,6]. This tunnelling process is an 'Andreev reflection', in which an incoming electron is reflected as a hole. Particle–hole symmetry dictates that the zero-energy tunnelling amplitudes of electrons and holes are equal, resulting in a perfect resonant transmission with a ZBP height quantized at $2e^2/h$ (refs 3, 4, 10), irrespective of the precise tunnelling strength[3–5]. The Majorana nature of this perfect Andreev reflection is a direct result of the well-known Majorana symmetry property 'particle equals antiparticle'[1].

This robust conductance quantization has not yet been observed[6,7,13,14]. Instead, most of the ZBPs have a height considerably less than $2e^2/h$. This discrepancy was first explained by thermal averaging[15–18], but that explanation does not hold when the peak width exceeds the thermal broadening (about $3.5k_BT$)[13,14]. In that case, other averaging mechanisms, such as dissipation[19], have been invoked. The main source of dissipation is a finite quasiparticle density-of-states within the superconducting gap, often referred to as a 'soft gap'. Substantial advances have been achieved in 'hardening' the gap by improving the quality of materials, eliminating disorder and interface roughness[20,21], and better control during device processing[22,23], all guided by a more detailed theoretical understanding[24]. We have recently solved all these dissipation and disorder issues[21], and here we report the resulting improvements in electrical transport leading to the elusive quantization of the Majorana ZBP.

Figure 1a shows a micrograph of a fabricated device and schematics of the measurement set-up. An InSb nanowire (grey) is partially covered (two out of six facets) by a thin superconducting aluminium shell (green)[21]. The 'tunnel gates' (coral red) are used to induce a tunnel barrier in the non-covered segment between the left electrical contact (yellow) and the Al shell. The right contact is used to drain the current to ground. The chemical potential in the segment covered with Al can be tuned by applying voltages to the two long 'super-gates' (purple).

Transport spectroscopy is shown in Fig. 1b, which displays dI/dV as a function of voltage bias V and magnetic field B (aligned with the nanowire axis), while fixed voltages are applied to the tunnel- and super-gates. As B increases, two levels detach from the gap edge (at about 0.2 meV), merge at zero bias and form a robust ZBP. This is consistent with the Majorana theory: a ZBP is formed after the Zeeman energy closes the trivial superconducting gap and re-opens a topological gap[8,9]. The gap re-opening is not visible in a measurement of the local density-of-states because the tunnel coupling to these bulk states is small[25]. Moreover, the finite length (about 1.2 μm) of the proximitized segment (that is, the part that is superconducting because of the proximity effect from the superconducting Al coating) results in discrete energy states, turning the trivial-to-topological phase transition into a smooth crossover[26]. Figure 1c shows two line-cuts from Fig. 1b extracted at 0 T and 0.88 T. Importantly, the height of the ZBP reaches the quantized value of $2e^2/h$. The line-cut at zero bias in the lower panel of Fig. 1b shows that the ZBP height remains close to $2e^2/h$ over a sizable range in B field (0.75–0.92 T). Beyond this range, the height drops, most probably because of a closure of the superconducting gap in the bulk Al shell.

We note that the sub-gap conductance at $B = 0$ (black curve, left panel, Fig. 1c) is not completely suppressed down to zero, reminiscent of a soft gap. In this case, however, this finite sub-gap conductance does not reflect any finite sub-gap density-of-states in the proximitized wire. It arises from Andreev reflection (that is, transport by dissipationless Cooper pairs) due to a high tunnelling transmission, which is evident from the above-gap conductance (dI/dV for $V > 0.2$ mV) being larger than e^2/h. As this softness does not result from dissipation, the Majorana peak height should still reach the quantized value[27]. In Extended Data Fig. 1, we show that this device tuned into a low-transmission regime,

Chiral Majorana fermion modes in a quantum anomalous Hall insulator–superconductor structure

Qing Lin He,[1*†] Lei Pan,[1†] Alexander L. Stern,[3] Edward C. Burks,[4] Xiaoyu Che,[1] Gen Yin,[1] Jing Wang,[5,6] Biao Lian,[6] Quan Zhou,[6] Eun Sang Choi,[7] Koichi Murata,[1] Xufeng Kou,[1,8*] Zhijie Chen,[4] Tianxiao Nie,[1] Qiming Shao,[1] Yabin Fan,[1] Shou-Cheng Zhang,[6*] Kai Liu,[4] Jing Xia,[3] Kang L. Wang[1,2*]

Editorial Expression of Concern

On 21 July 2017, *Science* published the Report "Chiral Majorana fermion modes in a quantum anomalous Hall insulator—superconductor structure" by Q. L. He *et al.* (*1*). Since that time, raw data files were offered by the authors in response to queries from readers who had failed to reproduce the findings. Those data files did not clarify the underlying issues, and now their provenance has come into question. While the authors' institutions investigate further, we are alerting readers to these concerns.

H. Holden Thorp
Editor-in-Chief

Have Majoranas been finally seen? 2207/02472, PRB 107,245423 (2024)

C. Nayak et al, 2024

- Title of the paper : InAs-Al Hybrid devices passing the topological gap protocol - Microsoft team - over 127 authors with 35 graduate students
- Combination of local and non-local transport measurements which implies high probability of having actually detected a topological phase hosting Majorana modes
- Seen in roughly 1/2 of 25 samples studied. Passes the cut proposed by Pan and Das Sarma, 2020, to prove the existence of the Majorana

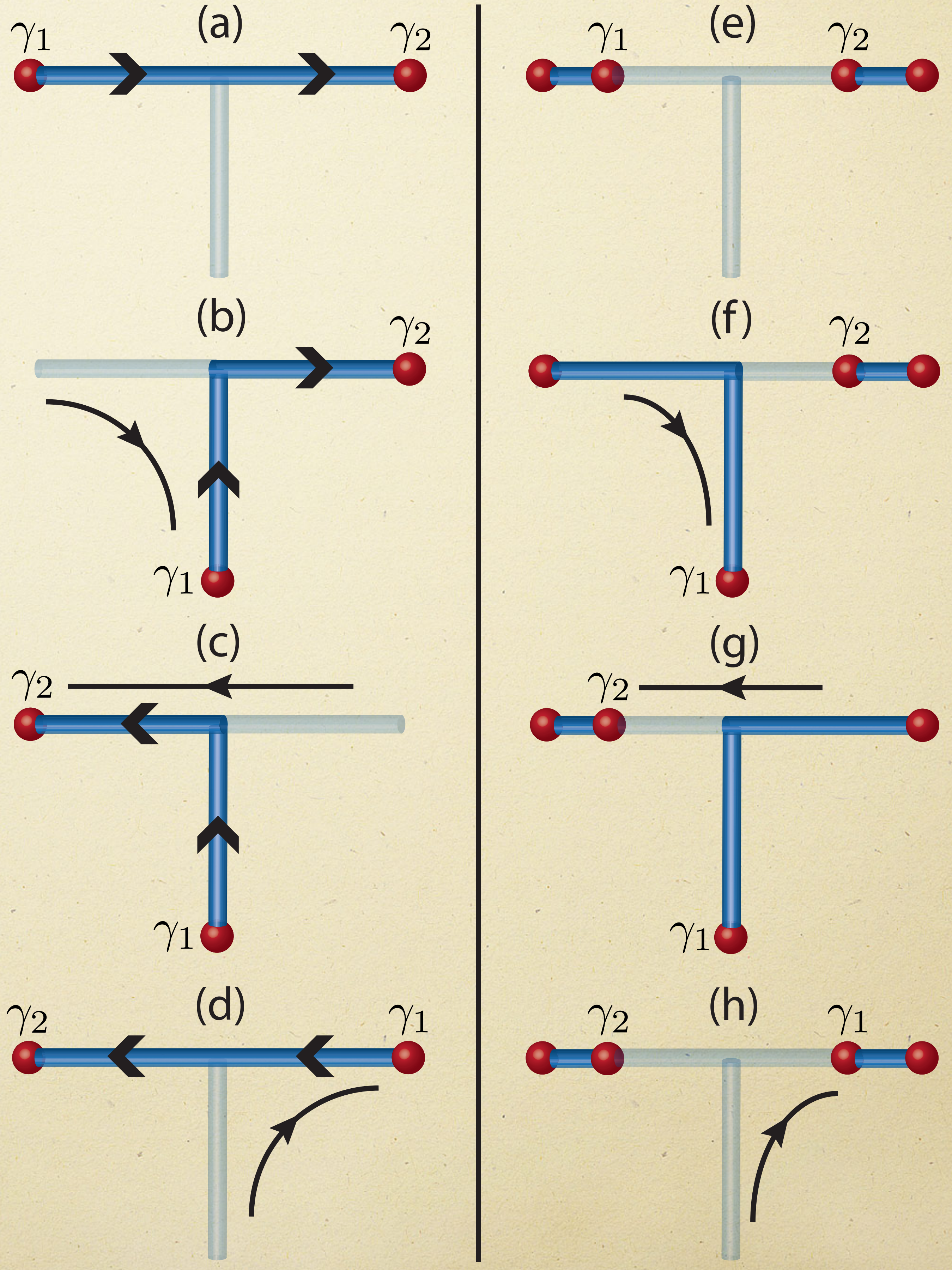
- “Our main result is that several devices, fabricated according to the design's engineering specifications, have passed the topological gap protocol defined in Pikulin et al. [arXiv:2103.12217]. This protocol is a stringent test composed of a sequence of three-terminal local and non-local transport measurements performed while varying the magnetic field, semiconductor electron density, and junction transparencies. Passing the protocol indicates a high probability of detection of a topological phase hosting Majorana zero modes as determined by large-scale disorder simulations. Our experimental results are consistent with a quantum phase transition into a topological superconducting phase that extends over several hundred milli-Tesla in magnetic field and several millivolts in gate voltage, corresponding to approximately one hundred micro-electron-volts in Zeeman energy and chemical potential in the semiconducting wire. These regions feature a closing and re-opening of the bulk gap, with simultaneous zero-bias conductance peaks at both ends of the devices that withstand changes in the junction transparencies. The extracted maximum topological gaps in our devices are 20-60 μeV. This demonstration is a prerequisite for experiments involving fusion and braiding of Majorana zero modes”

I believe Majoranas have been seen
But jury is still out - no consensus yet!

Need to braid these 1 D Majorana end modes

- Can be done, in principle using T-junctions

Alicea *et al*, Nature Phys.2011



More exotic non-abelian anyons ? Parafermions?
Fibonacci anyons?

What are parafermions and why study them?

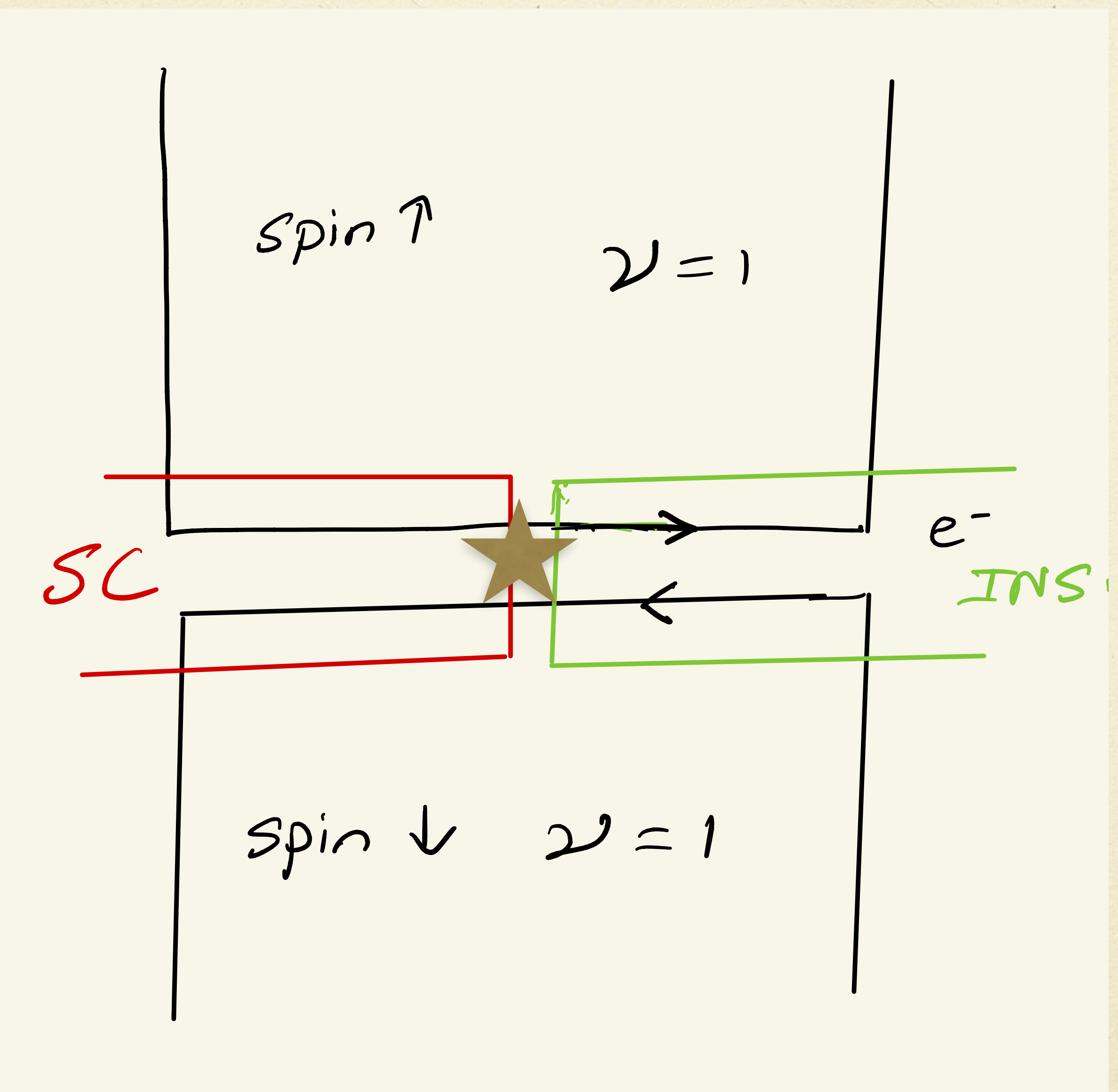
Fradkin, Kadanoff,1980
P . Fendley, 2012

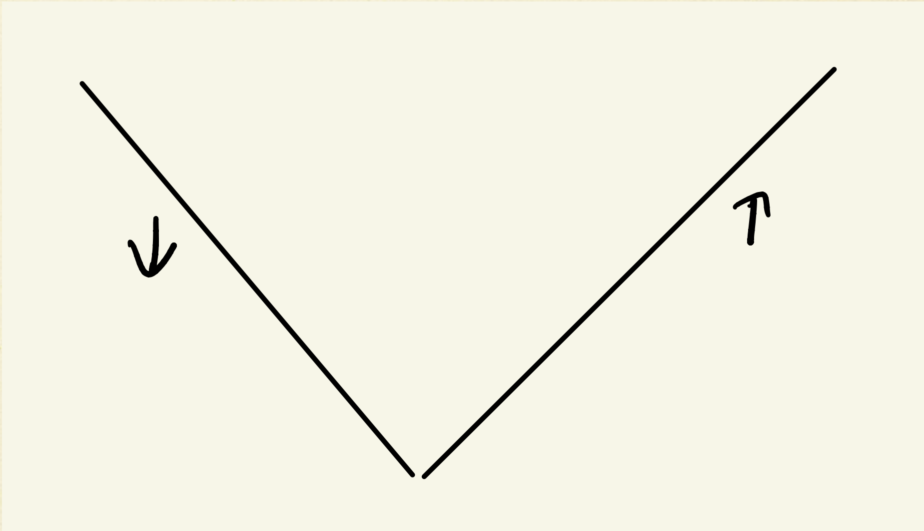
- Parafermions are more exotic quasiparticles than Majoranas, and cannot be realized in a free fermion model - building blocks themselves need to be fractionally charged
- Motivation - braiding of Majoranas cannot lead to universal quantum computation - it does not allow for all possible unitary operations. Parafermions allows for more gates (With Fibonacci anyons, can engineer universal topological quantum computer)
- Can be constructed using quantum Hall platforms (still in the range of theoretical proposals)

Quantum Hall systems can be used to engineer Majorana and parafermion modes

Majorana modes engineered at edges

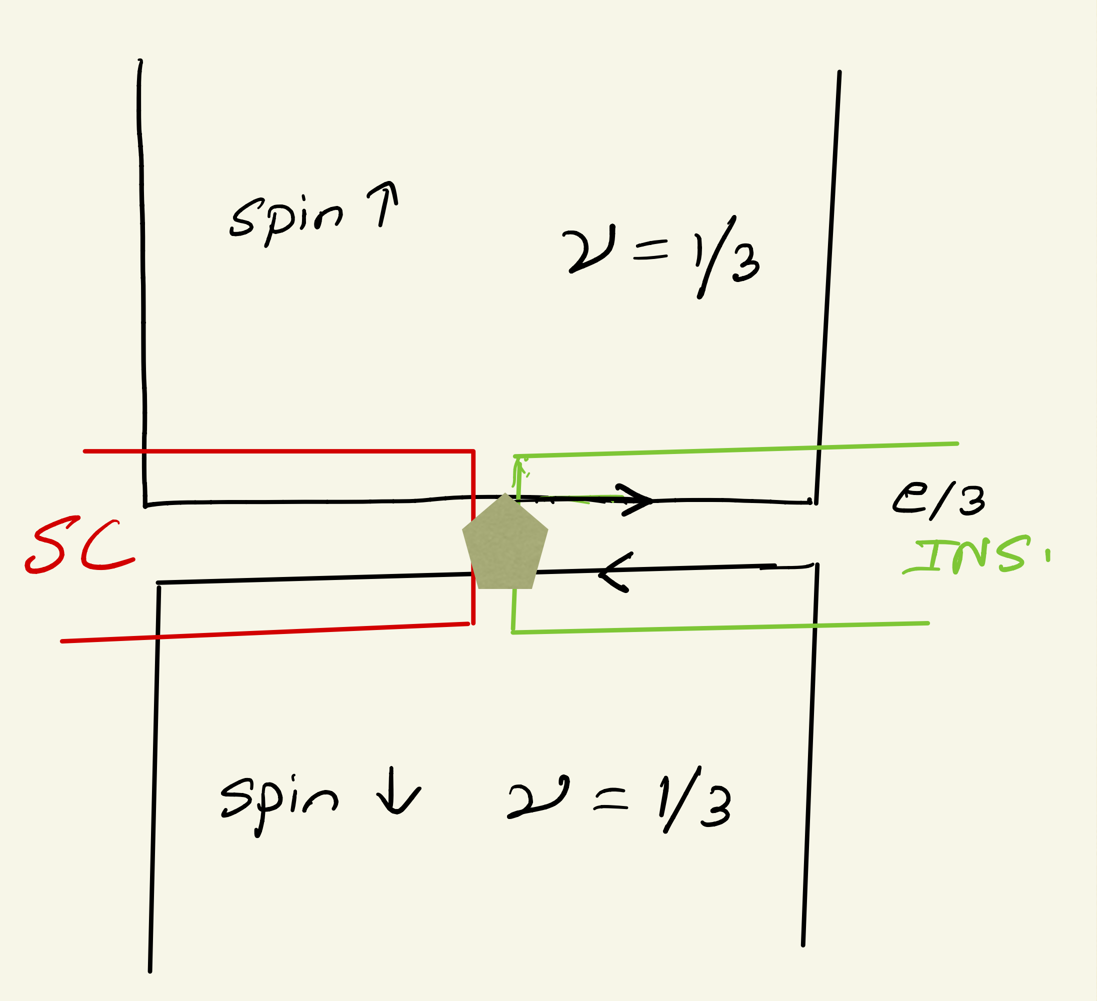
Ma and Zyuzin, EPL21,(1993)
F. Amet *et al*, Science352,(2016)





- Essential idea is that gapless edge states (of opposite chiralities) mimics a wire and gapping out by superconductor and ferromagnet essentially mimics the Kitaev model. So Majorana modes created at the interface between two different gappings

Parafermions at FQHE edges



- Parafermions are similar to Majoranas but need FQHE. Building blocks are themselves fractionally charged quasi-particles - eg, e/3 quasi-holes at edge of the fractional quantum Hall state with conductance $G = e^2/3h$
- Backscattering (Andreev) at SC and (normal at) Insulator traps quasi-particle - quasi hole bound states- parafermions

Clarke, Alicea and Shtengel,2012

Lindner, Berg, Refeal and Stern,2012

Our recent work

- What happens to Majoranas/parafermions in quantum Hall systems when there is edge reconstruction? Kishore Iyer, Amulya Ratnakar, Sourin Das and S.R., PRB, 110, L161302 (2024).

- Spontaneous fractional Josephson Current from parafermions - Kishore Iyer, Amulya Ratnakar, Aabir Mukhopadhaya, S.R and Sourin Das, PRB 107, L121408 (2023)

General model of anyons and how they can be used for topological quantum computation

General model for non-abelian anyons

A . Kitaev,2001 Preskill,2004

Das Sarma, Freedman, Nayak,2005

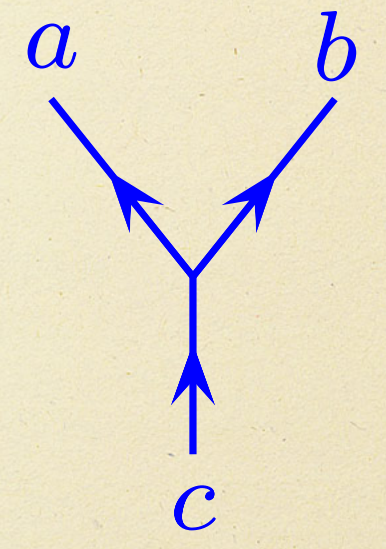
- 2D system with particles (anyons) labelled: $a, b, c \ldots\ldots$
- Vacuum or 0 anyon, anti-anyons: $\bar{a}, \bar{b}, \bar{c} \ldots\ldots$
- Rules for fusion (and splitting) : $a \times b = \sum_c N_{ab}^c c$, N_{ab}^c non-negative integers (but usually, $N_{ab}^c = 0{,}1$)

Nayak, Simon, Stern, Freedman, Das Sarma, RMP,2008

Topological quantum, Steve Simon,2023

More on fusion

- For abelian anyons, only one way to fuse and for non-abelian anyons, multiple ways to fuse
- Fusion rules specify allowed ways two anyons can combine. State of 2 anyons will be in any one or superposition of allowed ways
- Fusion is commutative $a \times b = b \times a$



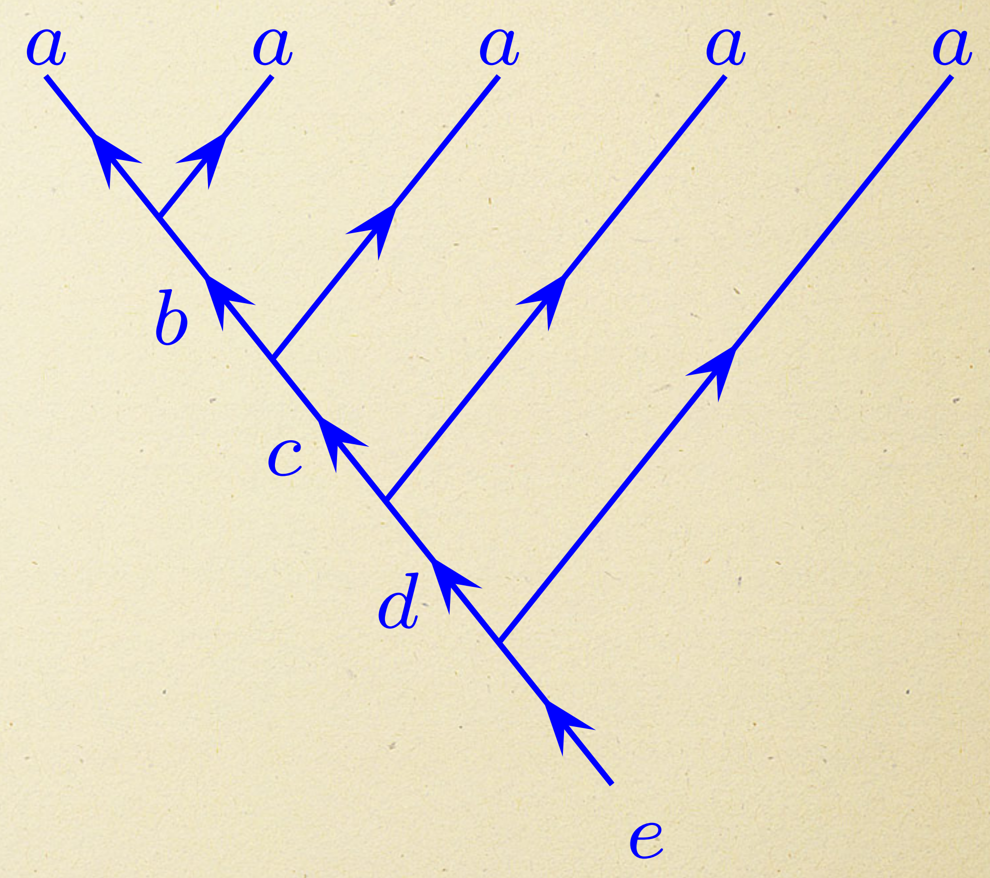
if $N_{ab}^{c} \neq 0$

- Can fuse multiple anyons -e.g n=5 anyons $e=\sum_{b,c,d} N_{aa}^{b} N_{ba}^{c} N_{ca}^{d} N_{da}^{e}$

- Define quantum dimension of anyon as $=d_a=$ largest eigenvalue of N_a- can be fractional

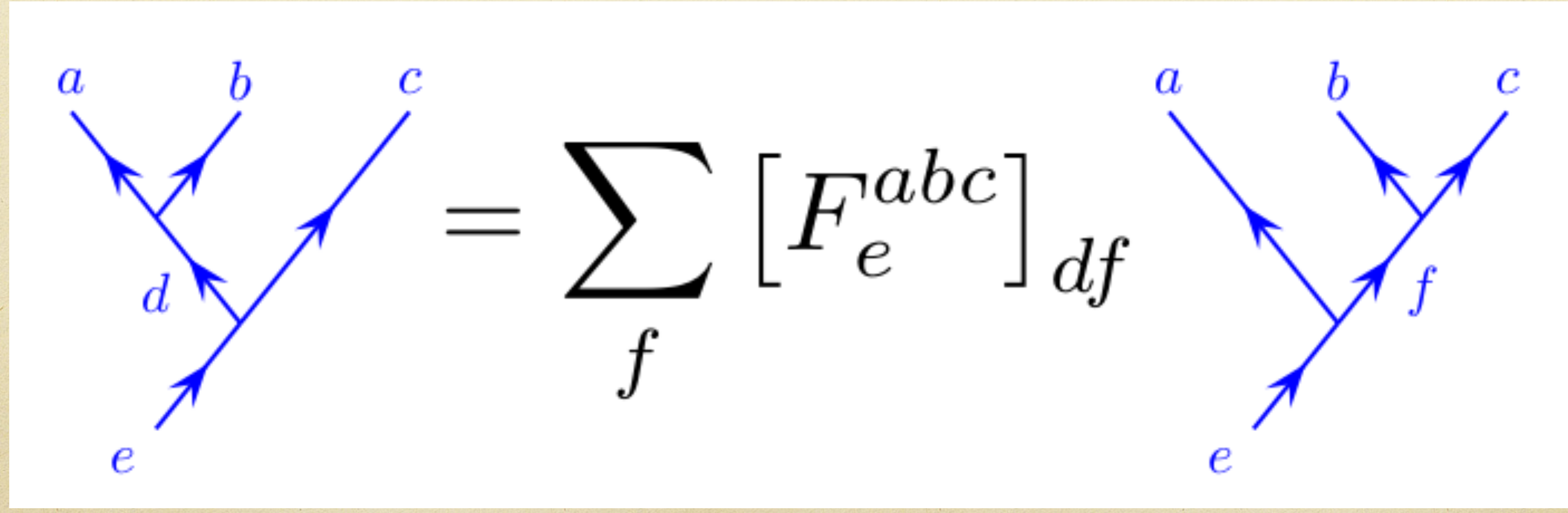
- Dimension of Hilbert space d_a^n

- Example : Quantum dimension of Majorana $=\sqrt{2}$ (4 Majoranas = 2 fermions needed to have 4 dimensional Hilbert space)

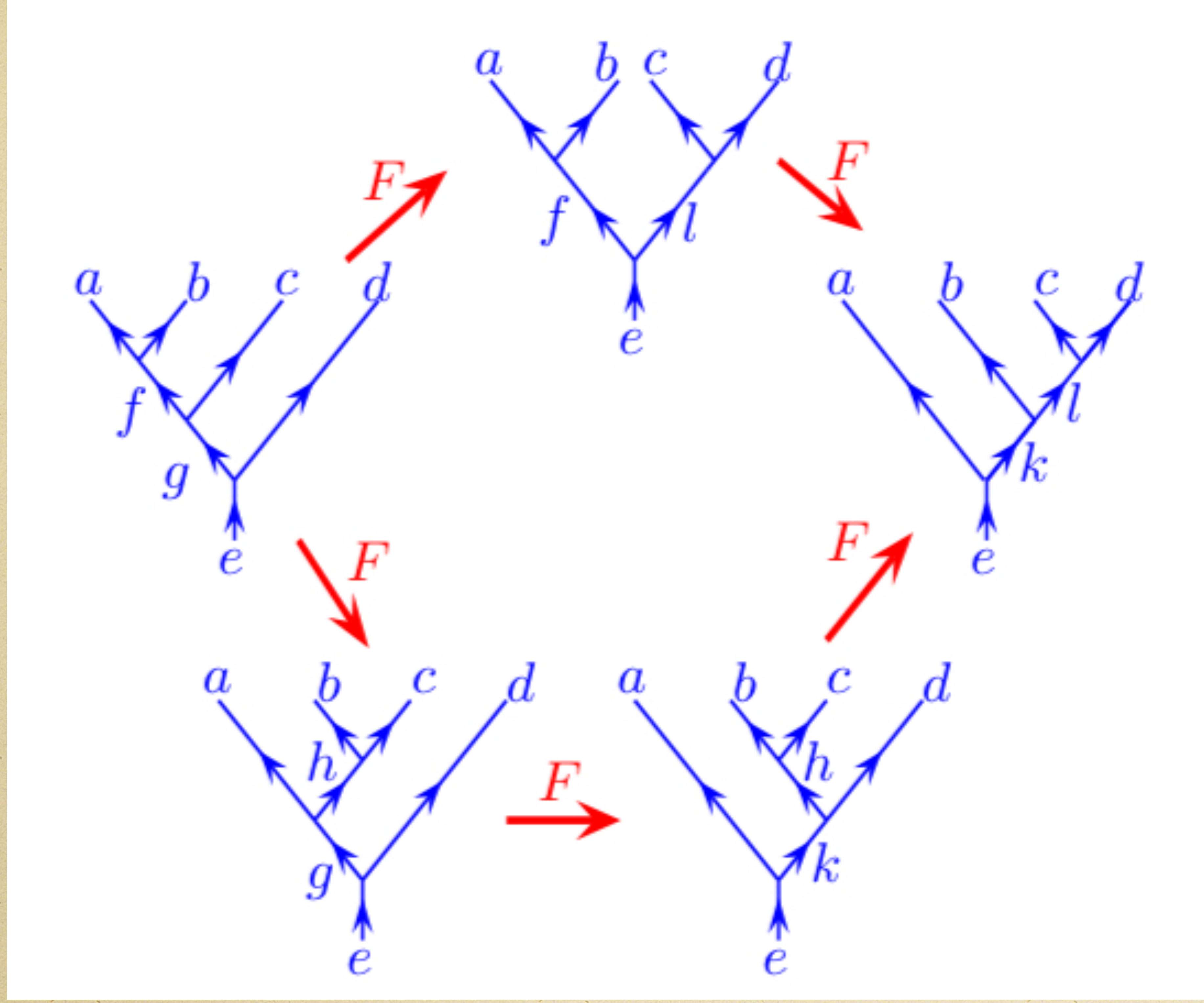


- Fusion rules are associative.

- F matrices : When we have more than 2 anyons, we can fuse in different orders. Two choices related by a unitary matrix called F matrix (basis change). Here F^{abc}_e is a unitary matrix

- 

- To make the F matrices consistent, we need to look at 4 anyons - pentagon equation

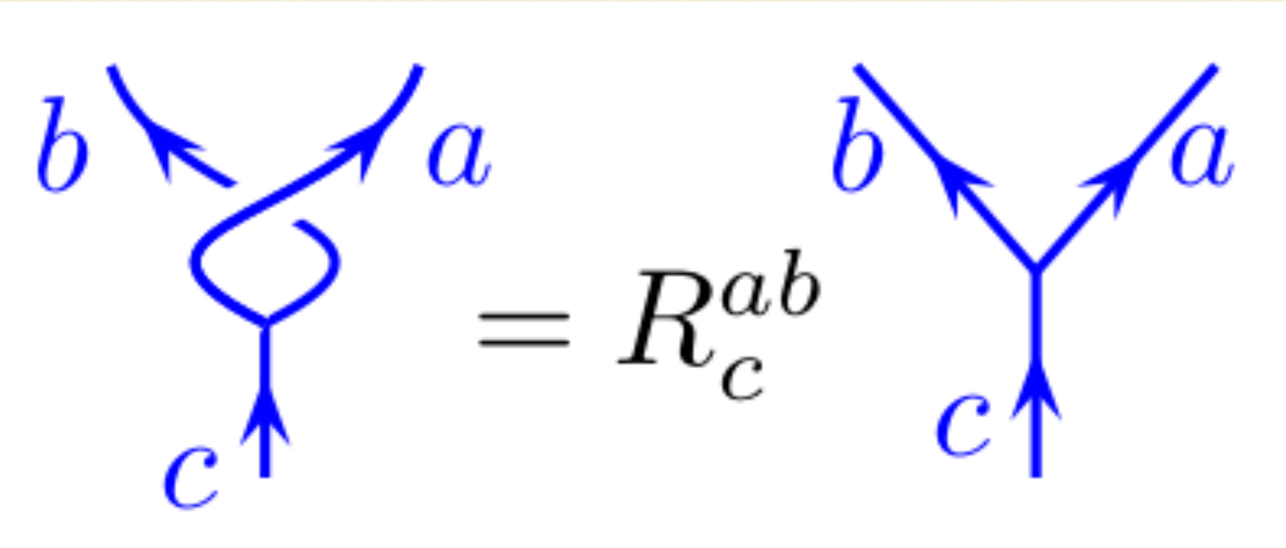


$$[F_e^{fcd}]_{gl}[F_e^{abl}]_{fk} = \sum_h [F_g^{abc}]_{fh}[F_e^{ahd}]_{gk}[F_k^{bcd}]_{hl}$$

- Don't need any further consistency conditions if we add more anyons

Braiding

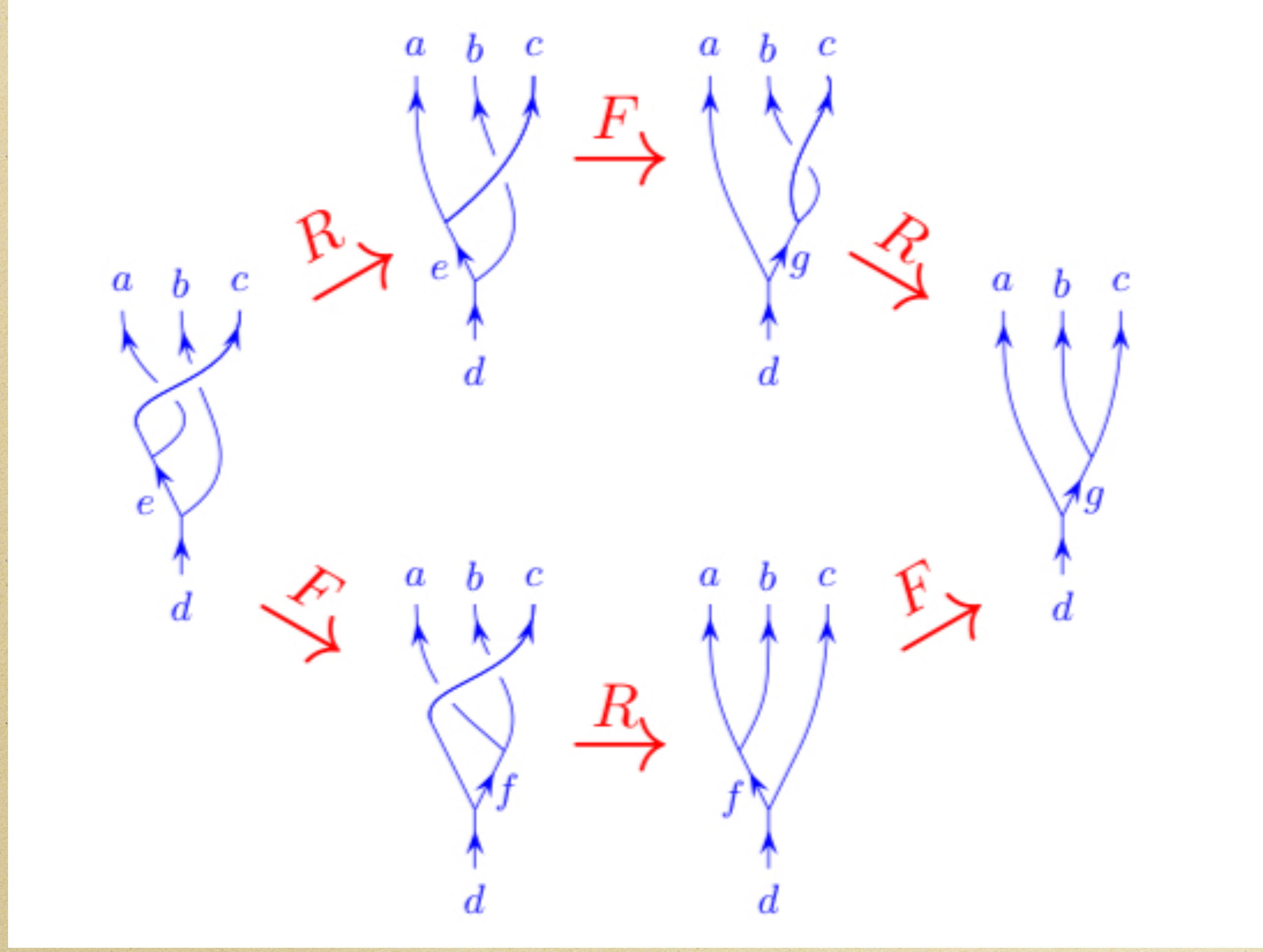
- Rules for braiding : what happens when 2 neighboring anyons are exchanged



- Also, when we have more than 2, we need to swap 2 at a time. Turns out exchange statistics depends on other anyons in the system and is a unitary matrix and not just a phase

R matrices

- Consistency conditions : The hexagon rules

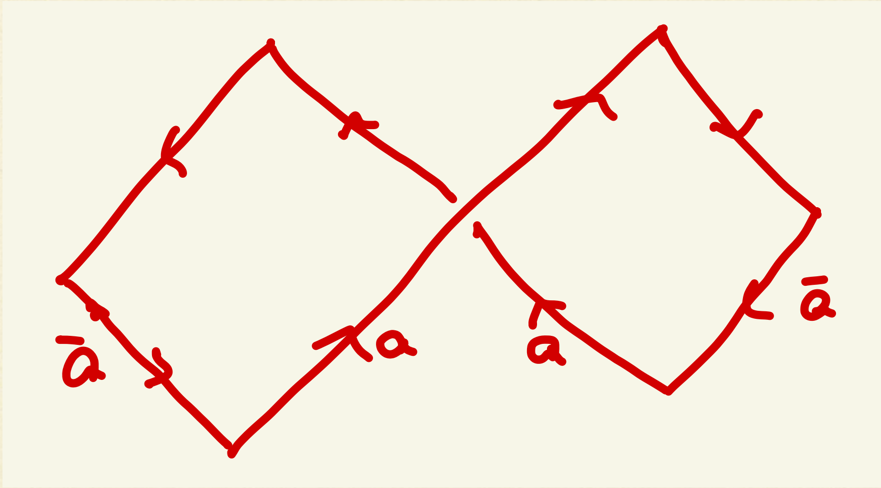


$$R^{ca}_{e}[F^{acb}_{d}]_{eg}R^{cb}_{g} \quad = \quad \sum_{f}[F^{cab}_{d}]_{ef}R^{cf}_{d}[F^{abc}_{d}]_{fg}$$

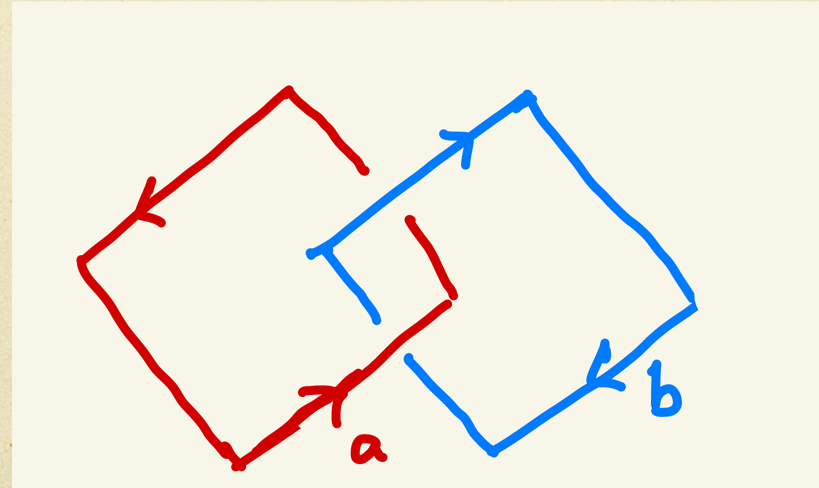
Solution of the pentagon and hexagon rules

- Complicated set of equations in terms of the F and R matrices
- No general solution. TQFT's not fully classified
- But for any given anyon model, it is possible to solve the consistency conditions for the F and R matrices and compute what are called the topological invariants, the S and the T matrices, which define the topological phase

- We had defined exchange phase as $T_{ab} = \delta_{ab} e^{i\theta_a}$. Turns out that $\theta_a = \dfrac{1}{d_a} \sum_c R_c^{aa} d_c$



- Can also compute the S-matrix $S_{ab} = \dfrac{1}{D} \sum_c N_{ab}^c R_c^{ab} R_c^{ba} d_c$, $D = \sum_a d_a^2$



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- To recap : Essentially started with an anyon model : $a, b, c, \ldots$
- Defined fusion rules, associativity and the F matrices
- Defined braiding and the R matrices
- For consistency, need pentagon and hexagon rules. This defines the model.
- By solving these consistency conditions, we get the topological spin θ_a or T_{ab} and the mutual statistics S_{ab}
- We can now go onto understand how it can be used for quantum computation

- Many other connections that we haven't explored
- Relation to anyons as punctures on Riemann surfaces
- Verlinde relation relating S-matrix (obtained from braiding) to N_{ab}^{c} (related to fusion). Turns out that fusion rules can be derived from the S-matrix.
- Relation to Hopf links, knot theory, Chern-Simons theory, K-theory, unitary modular tensor category theory, ……

Examples

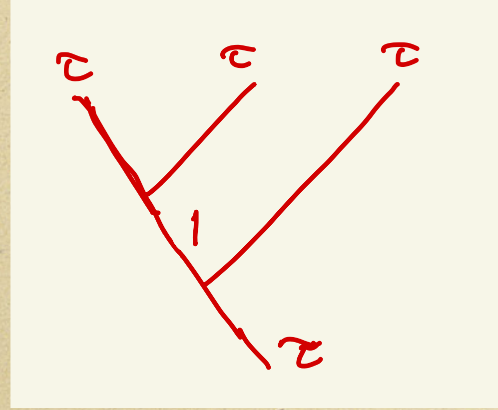
- Toric code model : Excitations with mutual abelian anyon statistics
- Majorana model - with Majorana modes $\gamma \times \gamma = 0 + f$. So model with 3 anyons, 0, γ and f
- Fibonacci model, with just 0 and τ
- Z_3 parafermion models, and more generally Z_N parafermion models

Quantum computing with anyons

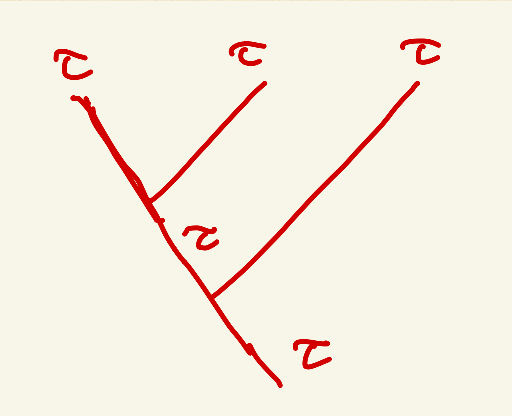
- Easiest to do it for a specific example : the Fibonacci model

$$\tau \times \tau = 1 + \tau,\ 1 \times 1 = 1,\ 1 \times \tau = \tau \times 1 = \tau$$

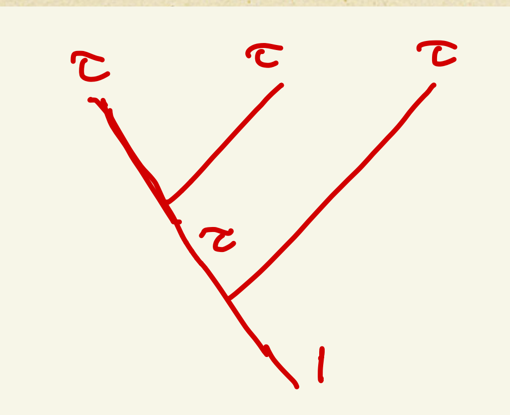
- 3 states possible when 3 anyons are fused - forms our qubit



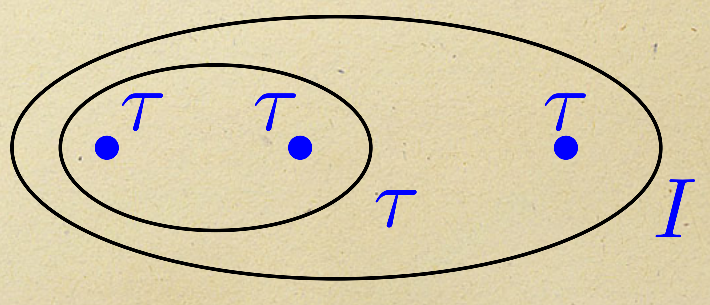
$$= |0> = ((\tau, \tau)_1, \tau)_\tau$$



$$= |1> = ((\tau, \tau)_\tau, \tau)_\tau$$



$$= |N> = ((\tau, \tau)_\tau, \tau)_1 =$$



F and R matrices

- We can now solve the pentagon equation to get the F matrices and the hexagon equation to get the R matrices
- Only non-trivial F matrix turns out to be

$$F^{\tau\tau\tau}_{\tau} = F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \qquad \phi^{-1} = \frac{(\sqrt{5}-1)}{2}$$

- We also find that $R^{\tau\tau}_{\tau} = e^{+3i\pi/5}$, $R^{\tau\tau}_{1} = e^{-4i\pi/5}$

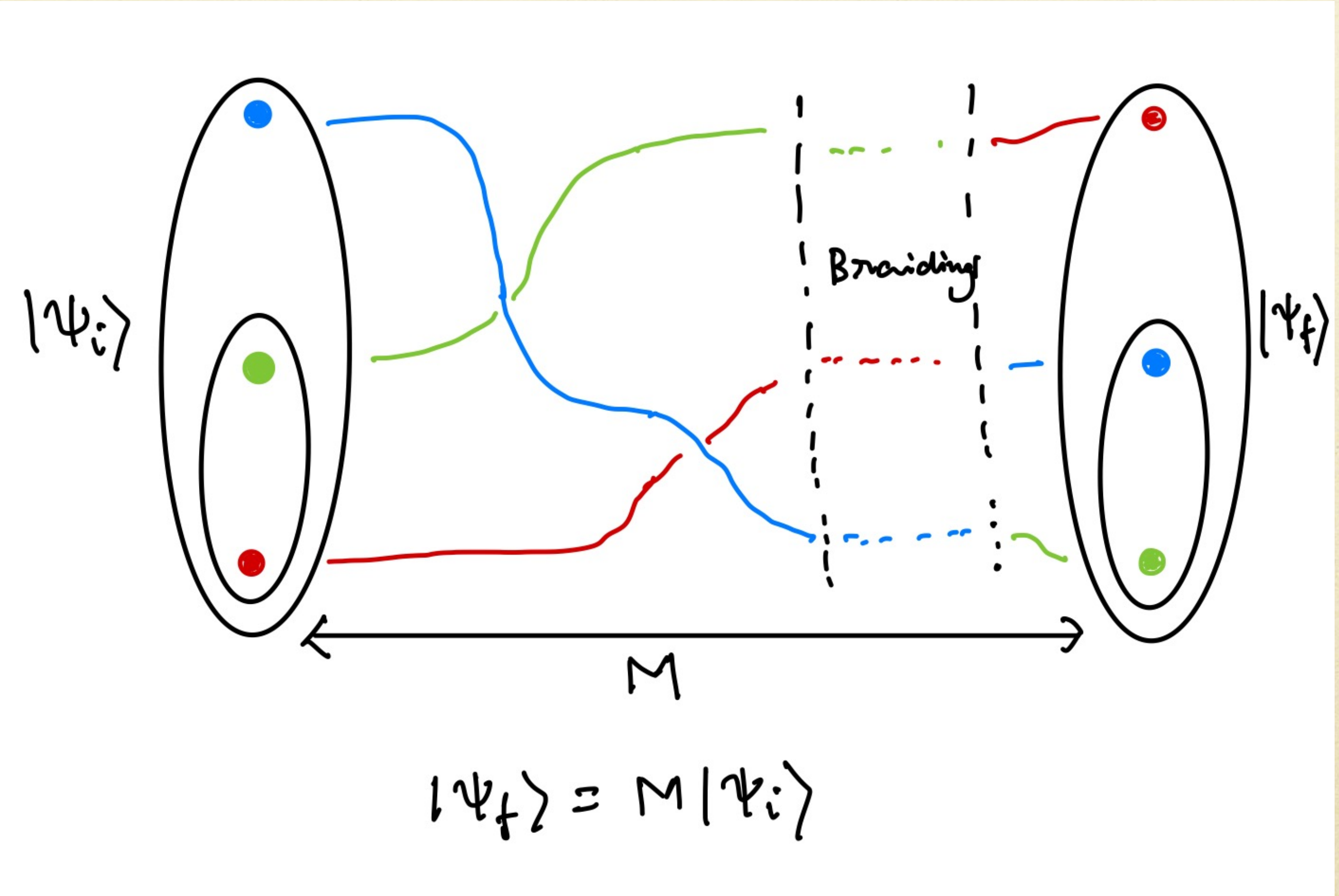
- We now can implement unitary operations on our qubit by braiding
- Braiding first 2 τ's can be implemented by the R matrix since they are in the same fusion channel - $B_{12_0} = e^{-4i\pi/5}$, $B_{12_1} = e^{+3i\pi/5}$, $B_{12_N} = e^{+3i\pi/5}$
- But to braid 2nd τ with 3rd τ or 1st τ with 3rd τ, need to first change basis by applying F matrix and then braid using R matrix and then bring back to old basis by using $F^{-1}(=F)$.
- Essentially $B_{23} = F_\tau B_{12} F_\tau$ and using braid group properties $B_{13} = B_{12} B_{23} B_{13}^{-1}$
-

- So in the basis $(|N>, |0>, |1>)$

$$B_{12} = \begin{pmatrix} e^{3\pi i/5} & & \\ & e^{-4\pi i/5} & \\ & & e^{3\pi i/5} \end{pmatrix}$$

$$B_{23} = \begin{pmatrix} e^{3\pi i/5} & & \\ & \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ & \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

- Any arbitrary braiding operator is a combination of B_{12} and B_{23} operators and we can reach any 2×2 unitary matrix using their combinations
- Topologically protected from noise and decoherence
- Hence, fault tolerant quantum computation



Preskill's bet

- In March 2020, made a bet to be settled on March 1, 2030 at midnight
- Will someone have finally made a topological quantum computer ?
  Preskill : yes   Dowling : no.   Stake : Pizza and beer

Anyon,anyon

John Preskill,2005

Anyon, anyon, where do you roam?
Braid for a while before you go home.

Though you're condemned just to slide on a table,
A life in 2D also means that you're able
To be of a type neither Fermi nor Bose
And to know left from right --- that's a kick, I suppose.

You and your buddy were made in a pair
Then wandered around, braiding here, braiding there.
You'll fuse back together when braiding is through
Well bid you adieu as you vanish from view.

No one can say, not at this early juncture
If someday we'll store quantum data in punctures
With quantum states hidden where no one can see,
Protected from damage through topology.

Anyon, anyon, where do you roam?
Braid for a while before you go home.

- Hope I have convinced you that non-abelian anyons are interesting from many points of view, theoretical, experimental and for applications
- Thank you all for coming and listening to me!