Supersymmetric index of black holes

Ashoke Sen

ICTS, Bengaluru, India

Collaborators: P.V. Athira, A.H. Anupam, Chandramouli Chowdhury, Subramanya Hegde, P. Shanmugapriya, Amitabh Virmani

Acornhoek, December 2024

Black holes

A black hole in a two derivative theory in asymptotically flat D dimensional space-time can carry

– U(1) charges Q_k

– angular momentum J_i in Cartan subalgebra of SO(D-1)

– mass M

The Bekenstein-Hawking entropy takes the form

$$S_{bh} = \frac{A}{4G_N} = f(Q, M, J)$$

Q, J have multiple components in general

A two derivative action scales as λ^{D-2} under

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu}, \qquad B_{\mu\nu} \to \lambda^2 B_{\mu\nu}, \qquad A_\mu \to \lambda A_\mu, \qquad \phi \to \phi$$

$\Rightarrow$ scaling property of the entropy in D dimensions

$$f(\lambda^{D-3} Q, \lambda^{D-3} M, \lambda^{D-2} J) = \lambda^{D-2} f(Q, M, J)$$

To take macroscopic limit, we take

$$M \sim \lambda^{D-3}, \qquad Q \sim \lambda^{D-3}, \qquad J \sim \lambda^{D-2}$$

and take λ large

Then

$$S_{bh} \sim \lambda^{D-2}$$

Goal: Compare the entropy with log(degeneracy) from some microscopic counting of degeneracy of states.

Chemical potentials

$$\beta = \frac{\partial \mathbf{S_{bh}}}{\partial \mathbf{M}}, \qquad \mu = \frac{\mathbf{1}}{\beta}\frac{\partial \mathbf{S_{bh}}}{\partial \mathbf{Q}}, \qquad \omega = \frac{\mathbf{1}}{\beta}\frac{\partial \mathbf{S_{bh}}}{\partial \mathbf{J}}$$

Scaling from $\mathbf{S_{bh}} \sim \lambda^{\mathbf{D-2}}, \quad \mathbf{M}, \mathbf{Q} \sim \lambda^{\mathbf{D-3}}, \quad \mathbf{J} \sim \lambda^{\mathbf{D-2}}$

$$\beta \sim \lambda, \qquad \mu \sim \mathbf{1}, \qquad \omega \sim \lambda^{-\mathbf{1}}$$

Wald's formula gives correction to the entropy from classical higher derivative terms and the scaling properties are modified.

Loop corrections give additional terms of order $\ln \lambda$ which also violate the scaling properties

While entropy=log(degeneracy) has been tested in many examples in string theory, there remained one open issue

Strominger, Vafa; $\cdots$

The computation of degeneracy is done by representing the black hole states as states of some brane system carrying the same charge as the black hole

The dynamics of the branes is understood at weak string coupling and that is where the counting is done

The black hole description is valid only when gravitational coupling is strong enough so that the horizon size exceeds the Compton radius.

How can we compare the two expressions?

Answer: Try to compare supersymmetric index of BPS black holes which does not change as we vary the coupling constant.

Supersymmetric Index

Consider a supersymmetric theory in asymptotically flat D space-time dimensions

$$I \equiv Tr_{Q,J',k=0}\left[e^{-\beta H} e^{2\pi i J_0} (2J_0)^n\right]$$

J_0 some particular Cartan generator of SO(D-1), k: momentum

J' represents Cartan generators other than J_0

The trace is taken over states at fixed Q, J' and k=0

The trace gets contribution from only supersymmetric states that break 2n (or less) J'-invariant supersymmetries

Bachas, Kiritsis hep-th/9611205; Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline hep-th/9708062

Dabholkar, Gomes, Murthy, A.S., arXiv:1009.3226

n=0 $\Rightarrow$ Witten index

– gets contribution from only vacuum states that preserve all supersymmetries

$$\mathrm{I} = \mathrm{Tr}_{Q,J',k=0}\left[e^{-\beta H} e^{2\pi i J_0} (2J_0)^n\right] \equiv e^{S_{BPS} - \beta\, M_{BPS}}$$

$e^{S_{BPS}}$ is called the index and receives contribution from supersymmetric (black hole) microstates

1. In D=4 the rotation group is SU(2)

J_0 is the third generator of the rotation group, J' trivial

2. In D=5 the rotation group is $SO(4) = SU(2)_L \times SU(2)_R$

We can take $J_0 = J_{3R}$, $J' = J_{3L}$

S_{BPS} is a function of J_{3L} and electric charges

On the microscopic side we can easily compute the index but on the black hole side we calculate the degeneracy via entropy

Assumption: Index and degeneracy are equal at generic coupling

Can we do better by computing the index on the black hole side?

Iliesiu, Kologlu, Turiaci; Cabo-Bizet, Cassani, Martelli, Murthy

Gravitational path integral for the index

Perform a path integral over all fields, weighted by $\exp[-\text{action}] \times (2J_0)^n$, with the following asymptotic conditions

1. Euclidean time τ and the azimuthal angles ϕ periodically identified as

$$(\tau, \phi) \equiv (\tau + \beta, \phi - i\omega\beta)$$

2. The time components of the gauge fields take asymptotic values

$$A_\tau = -i\,\mu$$

3. Choose $\beta\omega_0 = -2\pi i$, ω_0: conjugate to J_0

Computes

Gibbons, Hawking

$$Z = \mathrm{Tr}\left[e^{-\beta H - \beta\mu.Q - \beta\omega'.J' + 2\pi i J_0} (2J_0)^n\right]$$

We shall analyze the formula from the statistical side and the gravitational side

Statistical side

$$Z = \mathrm{Tr}\left[e^{-\beta H-\beta\mu . Q-\beta\omega' . J'+2\pi i J_0}(2J_0)^n\right]$$

Use

$$\mathrm{Tr}_{Q,J',k=0}\left[e^{-\beta H}e^{2\pi i J_0}(2J_0)^n\right] = e^{S_{BPS}-\beta M_{BPS}}$$

$$\Downarrow$$

$$Z = \int d^{n_V}Q\, d^{n'_c}J'\, d^{n_T}k\, e^{\left[S_{BPS}-\beta M_{BPS}-\beta k^2/2M_{BPS}-\beta\omega' . J'-\beta\mu . Q\right]}$$

k: momenta invariant under $\omega' . J'$

$\Rightarrow$ an n_T dimensional space of momenta to integrate over

n'_c: number of Cartan generators J'

n_V: number of U(1) gauge fields

$$Z = \int d^{n_V}Q\, d^{n_c'}J'\, d^{n_T}k\, e^{\left[S_{BPS} - \beta M_{BPS} - \beta k^2/2M_{BPS} - \beta\omega'.J' - \beta\mu.Q\right]}$$

The contribution to the integral is dominated by the saddle point where the integrand has a maximum

Gaussian integral around the saddle point produces correction $\propto \ln\lambda$

e.g. k integration gives $\sim (M_{BPS}/\beta)^{n_T/2} \sim e^{\frac{n_T}{2}(D-4)\ln\lambda}$

Q, J′ integrals give $\left(\det\frac{\partial^2 S_{BPS}}{\partial Q^2}\right)^{-1/2}\left(\det\frac{\partial^2 S_{BPS}}{\partial J'^2}\right)^{-1/2} \sim \lambda^{\frac{n_V(D-4)+n_c'(D-2)}{2}}$

Net result:

$$\ln Z = S_{BPS} - \beta M_{BPS} - \beta\omega'.J' - \beta\mu.Q - C_E\ln\lambda + \cdots$$

with J′, Q evaluated at the saddle, and

$$C_E = -\frac{1}{2}\left[(n_V + n_T)(D-4) + n_c'(D-2)\right]$$

Gravitational side

Gibbons, Hawking

$$\mathbf{Z} = \mathbf{Tr}\left[\mathbf{e}^{-\beta \mathbf{H} - \beta\mu.\mathbf{Q} - \beta\omega'.\mathbf{J}' + 2\pi i \mathbf{J}_0}(2\mathbf{J}_0)^{\mathbf{n}}\right]$$

Dominant contribution to Z comes from a supersymmetric Euclidean black hole solution with parameters (β, ω, μ)

$$\ln Z = S_{rot} - \beta M_{rot} - \beta\omega'.J' + 2\pi i J_0 - \beta\mu.Q + C \ln\lambda + \cdots$$

$(2J_0)^n$ is needed to saturate the integral over fermion zero modes associated with broken supersymmetry

S_{rot}: Classical entropy (including higher derivative terms)

$C \ln\lambda$ term is one loop quantum correction to the path integral

$\cdots$: higher order corrections

Combined result

Gravity:

$$\ln Z = S_{rot} - \beta M_{rot} - \beta\omega'.J' + 2\pi i J_0 - \beta\mu.Q + C\ln\lambda$$

Statistical mechanics

$$\ln Z = S_{BPS} - \beta M_{BPS} - \beta\omega'.J' - \beta\mu.Q - C_E\ln\lambda$$

$$\Rightarrow S_{BPS} = S_{rot} + 2\pi i J_0 + \beta(M_{BPS} - M_{rot}) + (C + C_E)\ln\lambda$$

Non-trivial identities:

$$M_{BPS} = M_{rot}, \qquad S_{BPS} = S_{rot} + 2\pi i J_0 + (C + C_E)\ln\lambda$$

$$\mathbf{M_{BPS} = M_{rot}}, \qquad \mathbf{S_{BPS} = S_{rot} + 2\pi i J_0 + (C + C_E) \ln \lambda}$$

– can be tested in various ways

rhs has to be computed using rotating, supersymmetric black holes with $\beta\omega_0 = -2\pi i$

If the microscopic result for the index is known then the lhs can be computed as log of the index

If the microscopic result is not known, we can still compare the rhs with the entropy of an extremal, zero temperature black hole with near horizon AdS_2 geometry

– computes degeneracy instead of index

– tests the index = degeneracy hypothesis for black holes

1. Classical 2-derivative theories:

$$M_{BPS} = M_{rot}, \qquad S_{BPS} = S_{rot} + 2\pi i J_0$$

This has been verified in various theories

– Minimal N=2 supergravity in D=4 — Iliesiu, Kologlu, Turiaci

– N=2 supergravity in D=5 — Anupam, Chowdhury, A.S.

– General N=2 supergravity in D=4 — Boruch, Iliesiu, Murthy,Turiaci

We compare the rhs with extremal black hole entropy with $\beta = \infty$, $J_0 = 0$, providing a test for degeneracy=index

In special cases, degeneracy was shown to be equal to the microscopic index earlier.

2. Classical theory with higher derivative terms

Prediction:

$$\mathbf{S_{BPS} = S_{rot} + 2\pi i J_0}$$

– now rhs needs to be computed using Wald entropy

We compare this with the Wald entropy of extremal black hole

This has been checked in N=2 supergravity in D=4 with a class of higher derivative terms Hegde, A.S., Shanmugapriya, Virmani

Wald entropy of extremal black holes is known from earlier work on attractor mechanism Lopes Cardoso, de Wit, Mohaupt

3. Logarithmic corrections

Anupam, Athira, Chowdhury, A.S.

$$\mathbf{S_{BPS} = S_{rot} + 2\pi i J_0 + (C + C_E)\ln\lambda}$$

$$\mathbf{C_E = -\frac{1}{2}\left[(n_v + n_T)(D-4) + n_c'(D-2)\right]}$$

We need to compute the $\mathbf{C \ln\lambda}$ term from gravitational path integral

Power counting $\Rightarrow$ such contributions come from one loop contribution of massless fields

Contribution comes from two sources

1. Non-zero eigenvalues of the kinetic operator

– can be evaluated using the heat kernel method

2. Zero eigenvalues of the kinetic operator

– can be evaluated by

(a) changing variables that relates the zero modes to broken symmetry transformation parameters

(b) then integrating over the symmetry transformation parameters

Final result: Logarithmic correction $(\mathbf{C} + \mathbf{C_E}) \ln \lambda$ to the index gives the same result as $\cdots$

$\cdots$ logarithmic correction to the degeneracy, computed from the near horizon geometry of an extremal, non-rotating black hole

$\cdots$ even though the intermediate steps are quite different

In particular, in the second approach there is no contribution $\mathbf{C_E} \ln\lambda$ due to change of ensemble

The entire contribution comes from the gravitational path integral

The black hole index also agrees with the microscopic results when they are known

e.g. in theories with 16, 32 supersymmetries in D=4, 5

4. Small black holes

Consider heterotic string theory compactified on a 10-D dimensional torus

The spectrum of the string contains states carrying electric charges under various U(1) gauge fields in the theory

Denote by Q the charge vector

A subset of these states are invariant under half of the supersymmetries of the theory

Dabholkar. Harvey

$\mathbf{S_{BPS}} =$ log(index) of these states grows as $2\sqrt{2}\pi\sqrt{Q^2}+\cdots$

Q^2: some quadratic combination of charges

Question: Can they be described as black holes carrying the same charge vector Q?

$$S_{BPS} = 2\sqrt{2}\pi\sqrt{Q^2} + \cdots$$

Can we compare this with BPS black hole entropy carrying the same charge?

– long history spanning 1995-2004 A.S.; Peet; Dabholkar; $\cdots$

This will not be reviewed today

Instead we shall change the question:

Can S_{BPS} be compared with

$$S_{rot} + 2\pi i J_0 \quad \text{at } \omega_0 = -2\pi i/\beta$$

Chowdhury, A.S., Shanmugapriya, Virmani; Chen, Murthy, Turiaci

J_0: Angular momentum in one plane

Recall the scaling property:

$$M \sim \lambda^{D-3}, \qquad Q \sim \lambda^{D-3}, \qquad J_0 \sim \lambda^{D-2}$$

$$S_{rot} \sim \lambda^{D-2}$$

Microscopic result:

$$S_{BPS} = 2\sqrt{2}\pi\sqrt{Q^2} \sim \lambda^{D-3}$$

Macroscopic result:

$$S_{rot}, \; 2\pi i J_0 \sim \lambda^{D-2} >> S_{BPS}$$

– apparent contradiction!

Calculate and see what we get for $S_{rot} + 2\pi i J_0$

$$\text{Action} = C_D \int d^D x \sqrt{-\det G}\, e^{-\Phi} \Big[R_G + G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} G^{\mu\nu}\, \mathrm{Tr}(\partial_\mu M L \partial_\nu M L)$$
$$- \frac{1}{12} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} - G^{\mu\mu'} G^{\nu\nu'} F^{(j)}_{\mu\nu} (LML)_{jk} F^{(k)}_{\mu'\nu'} \Big],$$

$$F^{(j)}_{\mu\nu} = \partial_\mu A^{(j)}_\nu - \partial_\nu A^{(j)}_\mu ,$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2A^{(j)}_\mu L_{jk} F^{(k)}_{\nu\rho} + \partial_\nu B_{\rho\mu} + 2A^{(j)}_\nu L_{jk} F^{(k)}_{\rho\mu} + \partial_\rho B_{\mu\nu} + 2A^{(j)}_\rho L_{jk} F^{(k)}_{\mu\nu},$$

M: a $(36-2D)\times(36-2D)$ matrix valued scalar subject to $MLM^T = L$, $L = \mathrm{diag}(1^{26-D}, -1^{10-D})$

C_D: a constant = $1/16\pi G_N$

Black hole solution: **Horowitz, A.S.**

$$\begin{aligned} ds^2 &= (\rho^2 - b^2\cos^2\theta)\Big\{ \Delta^{-1}(\rho^2 - b^2\cos^2\theta) d\tau^2 + (\rho^2 - b^2)^{-1} d\rho^2 + d\theta^2 \\ &\quad + \Delta^{-1}\sin^2\theta[\Delta - b^2\sin^2\theta(\rho^2 - b^2\cos^2\theta + 2m_0\rho^{5-D}\cosh\alpha)]\, d\phi^2 \\ &\quad + 2\Delta^{-1} m_0 \rho^{5-D} b \sin^2\theta d\tau d\phi + \rho^2\cos^2\theta(\rho^2 - b^2\cos^2\theta)^{-1} d\Omega^{D-4} \Big\}, \end{aligned}$$

$$\Delta \equiv (\rho^2 - b^2\cos^2\theta)^2 + 2m_0\rho^{5-D}\cosh\alpha\,(\rho^2 - b^2\cos^2\theta) + m_0^2 \rho^{10-2D}$$

$$m_0, b, \alpha : \text{parameters}, \qquad d\Omega^{D-4} : \text{metric on unit (D-4)-sphere}$$

$$A^{(i)}_\mu = \cdots, \quad B_{\mu\nu} = \cdots, \quad e^{-\Phi} = \cdots, \quad M = \cdots$$

Horizon at $\rho = b$

$\sqrt{Q^2} = \sqrt{2}\,(D-3)\, C_D\, g_s^{-2}\, m_0\, v_{D-2}$,		v_k : volume of unit k-sphere
$S_{rot} = \frac{m_0 b\, v_{D-2}}{4G_N}$,	$J_0 = i\,\frac{v_{D-2}}{8\pi G_N} m_0 b$,	$S_{rot} + 2\pi i J_0 = 0$

– survives inconsistency

$$\mathbf{S_{rot}} + 2\pi \mathbf{iJ_0} = 0, \qquad \mathbf{S_{BPS}} \sim \lambda^{\mathbf{D-3}}$$

To proceed further we need to recall that string theory has higher derivative corrections to the low energy effective action

Next order terms contain two extra derivatives

– need to contract with g$^{\mu\nu}$

Since g$_{\mu\nu} \sim \lambda^2$, the next order correction to S$_{rot}$ will be suppressed by an additional power of λ^{-2}

Since at the leading order S$_{rot}$, J$_0 \sim \lambda^{D-2}$, after correction,

$$\mathbf{S_{rot}} + 2\pi \mathbf{iJ_0} \sim \lambda^{\mathbf{D-4}} << \mathbf{S_{BPS}} \sim \lambda^{\mathbf{D-3}}$$

Higher derivative corrections cannot cure this problem if the geometry is smooth.

Closer inspection reveals that the metric is singular over a subspace of the horizon

For D=4 this subspace is north and south poles

Study the solution near this subspace in the large λ limit

Result: The solution takes a universal form, independent of the parameters of the black hole and D, except for

– some overall constant shift in the dilaton

– a factor in the metric which is an almost flat D-4 dimensional sphere

$\Rightarrow$ the stringy corrections are also universal

A scaling argument gives the following form of $S_{rot} + 2\pi i J_0$

$C_D \times g_s^{-2} b^{4-D} m_0 \times b^{D-4} v_{D-4} \times K$, K : Unknown numerical constant

$$\sqrt{Q^2} = \sqrt{2}\,(D-3)\,C_D\,g_s^{-2}\,m_0\,v_{D-2}$$

$$v_k = \text{Volume of unit k-sphere} = 2\pi^{(k+1)/2}/\Gamma[(k+1)/2]$$

This gives

$$S_{rot} + 2\pi i J_0 = \frac{K}{(D-3)\sqrt{2}}\,\frac{1}{\pi}\,\frac{\Gamma((D-1)/2)}{\Gamma((D-3)/2)}\,\sqrt{Q^2} = \frac{K}{2\pi\sqrt{2}}\,\sqrt{Q^2}$$

Compare with

$$S_{BPS} = 2\sqrt{2}\pi\sqrt{Q^2}$$

Note: Both formulæ are independent of D and asymptotic values of Φ and M, and scale with Q in the same way

– would agree if $K = 8\pi^2$.

To find K, we need to determine how higher derivative corrections modify the near horizon geometry and S_{rot}

Universal part of the solution near the singularity

$$ds^2 \simeq dR^2 + 4R^2 d\Theta^2 + 4R^2 \sin^2\Theta d\phi^2 + 4R^2\cos^2\Theta d\tau^2$$

$$0 \le \Theta \le \frac{\pi}{2}, \qquad \phi \equiv \phi + 2\pi, \qquad \tau \equiv \tau + 2\pi ,$$

$$M \simeq I_{36-2D} ,$$

$$A^{(j)}_{(E)\tau} \simeq 0,$$

$$A^{(j)}_{(E)\phi} \simeq 0 , \qquad \text{for } j \neq 36-2D, \qquad (E) : \text{Euclidean} ,$$

$$A^{(36-2D)}_{(E)\tau} \simeq \sqrt{2}R\cos^2\Theta,$$

$$A^{(36-2D)}_{(E)\phi} \simeq \sqrt{2}R\sin^2\Theta ,$$

$$H_{(E)} \simeq 8\,R^2\,\sin\Theta\,\cos\Theta\, d\Theta\wedge d\phi\wedge d\tau ,$$

$$\exp[-\Phi] \simeq \frac{1}{2R} .$$

The solution is universal but singular at $R = 0$

$\Rightarrow$ higher derivative corrections should be universal and produce a universal number K for the entropy

Summary

1. Gravitational path integral can be used to directly compute supersymmetric index of BPS states

2. In cases that have been studied, the index agrees with the entropy of the supersymmetric extremal black holes, leading to the confirmation of index=degeneracy for black holes

3. This formalism produces correctly

– Bekenstein-Hawking entropy of two derivative theory

– Wald correction to the entropy

– logarithmic correction to the entropy

– small black hole entropy, up to an overall undetermined constant

Lesson for microstates / fuzzballs

We have seen that the index of a supersymmetric black hole grows in the same way as its entropy, at least to the leading and first subleading orders

$\Rightarrow$ any proposal for supersymmetric black hole microstates must produce non-vanishing index.

Otherwise there remains the possibility that these are artifacts of working at special points in the moduli space and will disappear once we go to a generic point.