4d Modularity

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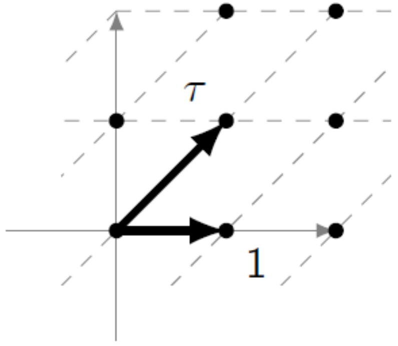
Sam van Leuven

WITS UNIVERSITY

Review: modularity in 2d CFT

- In 2d CFT: local operator on $\mathbb{R}^2$ $\Leftrightarrow$ state on S^1
- $Z(\beta,\theta) = \mathrm{tr}_{\mathcal{H}(S^1)}\, e^{-\beta\left(\Delta-\frac{c}{12}\right)+i\theta P}$ $\Leftrightarrow$ path integral on $T^2_\tau \cong S^1 \times S^1_\beta$

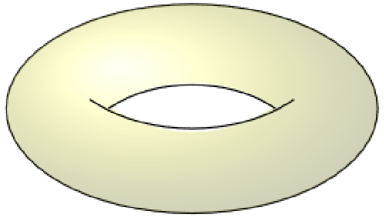


$$\tau = \frac{\theta + i\beta}{2\pi}$$

- $SL(2,\mathbb{Z})$ invariance:

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = Z(\tau,\bar{\tau})$$

$$ad-bc=1, \qquad a,b,c,d \in \mathbb{Z}$$



Application

- S-transformation relates low $(\tau \to i\infty)$ and high $(\tau \to 0)$ temperatures:

$$Z(\tau,\bar{\tau}) = Z\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right)$$

$$S^1 \leftrightarrow S^1_\beta$$

$\Rightarrow$ universal density of states at high energy controlled by central charge Cardy 86

$$S(E) = 2\pi\sqrt{\frac{cE}{3}}, \qquad E \gg c$$

Many other applications

In 2d CFT:

- Classification
- Fusion rules
- Bootstrap
- Heavy OPE coefficients
- …

Cappelli et al 87, Verlinde 88
Hellerman 09, Chang-Lin 16,
Kraus-Maloney 16,
Collier-Maloney et al 19

In string theory:

- Worldsheet string theory
- Black hole entropy
- Quantum gravitational path integral
- Holographic CFTs

Strominger-Vafa 96
Dijkgraaf-Verlinde^2 96
Dijkgraaf-Maldacena et al 00
Dabholkar-Murthy-Gomes 11/14
Hartman-Keller-Stoica 14
Benjamin-Cheng et al 15

In mathematics…

Higher dimensions

- Is $\mathrm{CFT}_{d>2}$ data universal at high energy as well?
- Does modularity, as a tool, generalize?
- Local operators on $\mathbb{R}^d \Leftrightarrow$ states on S^{d-1}.
- For $d > 2$, no symmetry between S^{d-1} and S^1_β.
- Various proposed tools:
  - Modularity for free (conformally coupled) fields
  - AdS/CFT
  - Change geometric background
  - Thermal effective action

Cardy 91, Kutasov-Larsen 01
Lei-SvL 24

Verlinde 00, Gibbons-Perry-Pope 04/05
Shaghoulian 15

Belin-de Boer et al 16,
Shaghoulian 15/16
Hofman-Vitouladitis 24

Bhattacharyya-Lahiri et al 07
Benjamin-Lee et al 23
Allameh-Shaghoulian 24

Exact results in SCFT

- Access to exact expressions for supersymmetric $Z[\mathcal{M}_d]$

- Unexpected signs of modularity in:
  - 4d $\mathcal{N}=2$ Schur index: $\mathcal{M}_4 = S^3 \times S^1$
  - 4d/6d susy Cardy formula: $\mathcal{M}_d = S^{d-1} \times S^1$
  - 3d $\mathcal{N}=2$ half-index: $\mathcal{M}_3 = D^2 \times S^1$
  - 4d/6d $\mathcal{N}=1$ superconformal index (SCI): $\mathcal{M}_d = S^{d-1} \times S^1$

Razamat 12, Beem et al 13
Dedushenko-Fluder 19
Pan-Peelaers 21
Beem-Razamat-Singh 21
Di Pietro-Komargodski 14
Cheng-Chun-Ferrari et al 18
Gadde 20, Jejjala-Lei-Li-SvL 22

- Interesting applications in AdS_{d+1}/CFT_d

Choi-Kim et al 18, Benini-Milan 18

- Physics/geometry of $SL(2,\mathbb{Z})$ action?

Motivation

- Bottom-up argument for modularity of 4d $\mathcal{N} = 1$ superconformal index

$\mathcal{M}_4 = S^3 \times S^1$

- Playground: free chiral multiplet

- Generalizations: non-trivial SCFTs and other dimensions

Superconformal index

- For 4d $\mathcal{N} = 1$ SCFTs:

Romelsberger 05, Kinney et al 05

$$\mathcal{I}(z_i;\sigma,\tau) = \mathrm{tr}_{\mathcal{H}(S^3)} e^{-\beta\delta}(-1)^F\, p^{J_1-\frac{r}{2}}\, q^{J_2-\frac{r}{2}}\, y_i^{Q_i}$$

$$\delta = \{\mathcal{Q},\mathcal{Q}^\dagger\} = \Delta - J_1 - J_2 + \frac{3}{2}r$$

$$p = e^{2\pi i\sigma},$$
$$q = e^{2\pi i\tau},$$
$$y_i = e^{2\pi i z_i}$$

- Only (quarter-)BPS states: $\mathcal{Q}|\psi\rangle = 0$ with $\delta = 0$

- Index is protected quantity; exactly calculable for many non-trivial SCFTs

Chiral multiplet on S^3

- $\mathcal{N} = 1$ chiral multiplet: $\{\phi, \psi_\alpha\} \cup \{\bar{\phi}, \bar{\psi}_{\dot{\alpha}}\}$

- Consider $\mathcal{O}(x)$ which obey $\delta = \Delta - J_1 - J_2 + \frac{3}{2} r = 0$

- "Single-letter" susy Hilbert space: $\mathcal{H}'_{\text{susy}}(S^3) = \left\{\partial^m_{+\dot{+}} \partial^n_{+\dot{-}} (\psi_+, \bar{\phi}) |0\rangle\right\}$

- Full $\mathcal{H}_{\text{susy}}(S^3)$ constructed from all "words"

Chiral multiplet index

- Chiral multiplet:

$$\mathcal{I}_R(z;\sigma,\tau) = \mathrm{Pexp}\left[\frac{(pq)^{\frac{R}{2}}y-(pq)^{1-\frac{R}{2}}y^{-1}}{(1-p)(1-q)}\right] = \Gamma\left(z+\frac{R}{2}(\sigma+\tau);\sigma,\tau\right)$$

Dolan-Osborn 08

$$\{\partial_{+\dot{+}}^{m}\partial_{+\dot{-}}^{n}(\bar{\phi},\psi_+)|0\rangle\}$$

- Elliptic Gamma function:

Felder-Varchenko 99

$$\Gamma(z;\sigma,\tau) = \prod_{m,n=0}^{\infty}\frac{1-y^{-1}p^{m+1}q^{n+1}}{1-yp^mq^n}$$

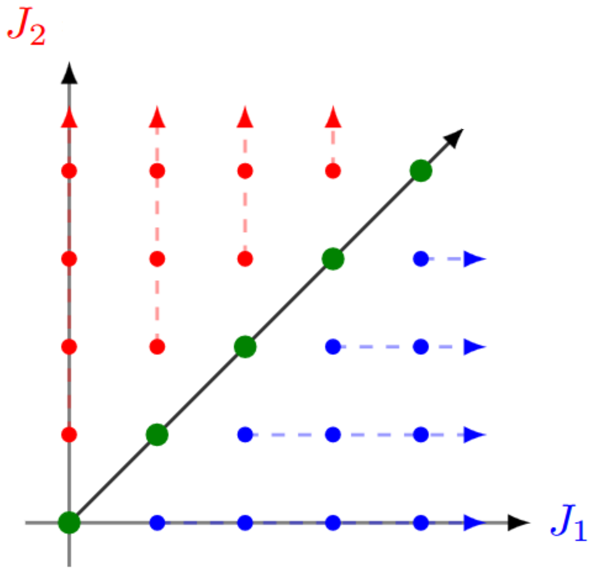
Hilbert space factorization

- Split: $\mathcal{H}'_{\text{susy}}(S^3) = \{\partial^m_{+\dot{+}}\partial^n_{+\dot{-}}(\bar{\phi},\psi_+)|0\rangle\} = \{|\psi\rangle \mid J_1 \leq J_2\} \oplus \{|\psi\rangle \mid J_1 > J_2\}$

Razamat-Willett 13

- Then: $\mathcal{H}_{\text{susy}}(S^3) = \mathcal{H}^{j_2\leq 0}_{\text{susy}}(S^3) \otimes \mathcal{H}^{j_2\geq 0}_{\text{susy}}(S^3)$

  with $j_2 \equiv \frac{1}{2}(J_1 - J_2)$



Factorization index

- On factorized Hilbert space:

$$\mathcal{I}_R(z;\sigma,\tau) = \frac{1}{\Gamma(-z+(1-R/2)(\sigma+\tau);\sigma+\tau,\tau)} \times \Gamma(z+R/2(\tau+\sigma);\tau+\sigma,\sigma)$$

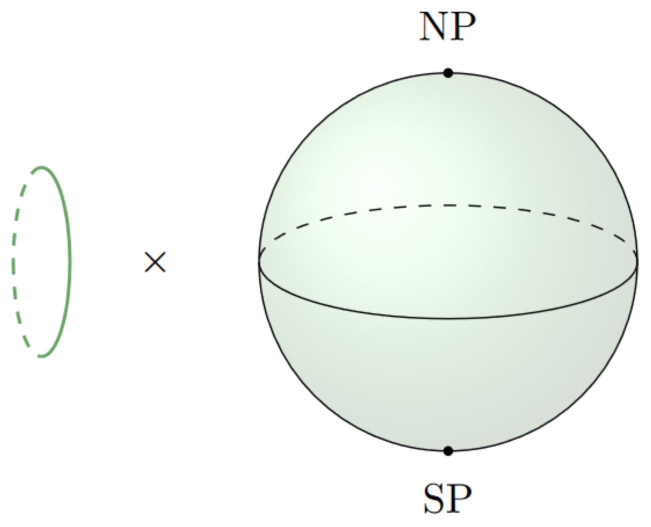
  - cf. $\mathcal{I}_R(z;\sigma,\tau) = \Gamma\left(z+\frac{R}{2}(\sigma+\tau);\sigma,\tau\right)$

  - Consistent with factorization properties of $\Gamma(z;\sigma,\tau)$

Felder-Varchenko 99

Claim 1

- This factorization can be interpreted geometrically
- To describe this, first view S^3 as Hopf fibration: $S^1_H \hookrightarrow S^3 \rightarrow S^2$
- S^3 restricted to a hemisphere of S^2 trivializes fibration: $HS^2 \times S^1_H$



Claim 1 continued

- At the level of the Hilbert space:

$$\mathcal{H}^{j_2\le 0}_{\text{susy}}(S^3)\otimes\mathcal{H}^{j_2\ge 0}_{\text{susy}}(S^3)=\mathcal{H}^{+}_{\text{susy}}(HS^2_S\times S^1_H)\otimes\widetilde{\mathcal{H}}^{+}_{\text{susy}}(HS^2_N\times S^1_H)$$

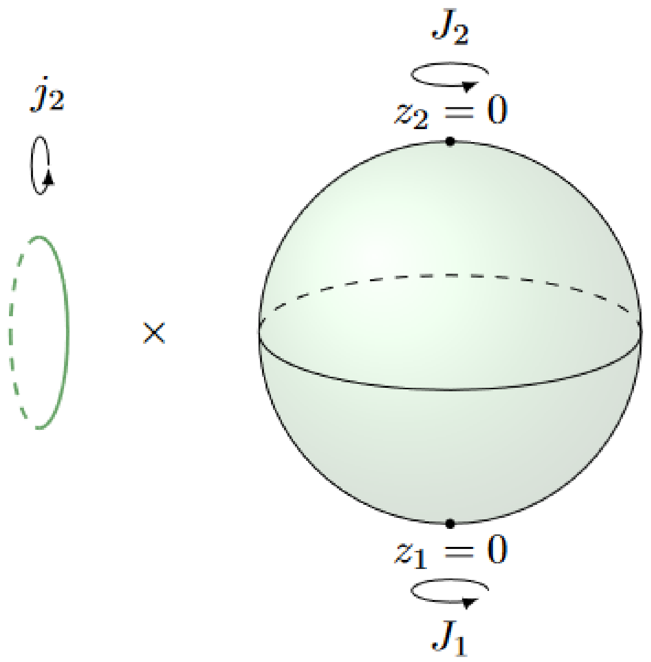
- Factors correspond to “hemi-sphere indices”:

$S^3=\{|z_1|^2+|z_2|^2=1\}$

$$\mathcal{I}_R(z;\sigma,\tau)=\mathcal{B}^+_R(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}^+_R(z;\tau+\sigma,\sigma)$$

where

- $\mathcal{B}^+_R(z;\sigma+\tau,\tau)=\mathrm{tr}_{\mathcal{H}^+_{\text{susy}}(HS^2_S\times S^1_H)}(-1)^F\,(pq)^{J_1-\frac{r}{2}}\,q^{-2j_2}\,y^Q$
- $\widetilde{\mathcal{B}}^+_R(z;\tau+\sigma,\sigma)=\mathrm{tr}_{\widetilde{\mathcal{H}}^+_{\text{susy}}(HS^2_N\times S^1_H)}(-1)^F\,(pq)^{J_2-\frac{r}{2}}\,p^{2j_2}\,y^Q$



Claim 2

- Trivial fibration allows KK reduction:

$$\mathcal{H}_{\text{susy}}(S^3) = \mathcal{H}^+_{\text{susy}}(HS^2_N \times S^1_H) \otimes \widetilde{\mathcal{H}}^+_{\text{susy}}(HS^2_S \times S^1_H) = \bigotimes_{m=0}^{\infty} \mathcal{H}^{(m)}(S^1_H) \otimes \widetilde{\mathcal{H}}^{(m)}(S^1_H)$$

- 4d index as infinite product of 2d indices

$$\mathcal{Z}[S^3 \times S^1_{(\sigma,\tau)}] \sim \prod_m \mathcal{B}^{(m)}[T^2_\tau] \times \widetilde{\mathcal{B}}^{(m)}[T^2_\sigma]$$

$T^2_{\tau/\sigma} \cong S^1_H \times S^1_\beta$
at northern/southern
hemisphere

inherits $SL(2,\mathbb{Z})$ covariance of the latter!

- Bottom-up derivation of “modular factorization”

Jejjala-Lei-Li-SvL 22

Remaining plan

1. Quantization on $HS^2 \times S^1$ and "hemi-sphere index"

2. 4d Modularity

Hemisphere reduction

- Quantize 4d chiral on $HS^2 \times S^1$ (in non-trivial R-symmetry bg)

Closset-Shamir 13
Longhi-Nieri-Pittelli 19

- Supersymmetric sector → short multiplets of 2d $\mathcal{N} = (0,2)$ susy:

$$\{\phi, \psi_\alpha\} \cup \{\bar{\phi}, \bar{\psi}_{\dot{\alpha}}\} \to \begin{cases} \text{D:} & \left\{\chi_+^{(m)}\right\} \cup \left\{\bar{\chi}_+^{(-m)}\right\} \\ \text{R:} & \left\{\rho^{(-m)}, \chi_-^{(-m)}\right\} \cup \left\{\bar{\rho}^{(m)}, \bar{\chi}_-^{(m)}\right\} \end{cases}$$

Fermi multiplets:
left-moving complex fermion

2d chiral multiplets:
complex boson and
right-moving complex fermion

  - m labels J_3 eigenvalue, generator of rotations on HS^2
  - b.c. admit infinite number of (zero-)modes: $m \geq 0$

Dedushenko 23

Hemisphere index

- Schematically: $\mathcal{H}^a_{\mathrm{BPS}}(HS^2 \times S^1) = \prod_{m\geq 0} \mathcal{H}_a^{(m)}(S^1)$ with $a = \mathrm{D}, \mathrm{R}$

- Define an index:

S^2 isometry and R-charge

flavor symmetry

$$\mathcal{B}^a(z;\sigma,\tau) = \mathrm{tr}_{\mathcal{H}^a(HS^2\times S^1)}(-1)^F\, p^{J_3-\frac{r}{2}}\, q^P\, y^Q\, e^{-\beta\delta}$$

$\delta = \{\mathcal{Q},\mathcal{Q}^\dagger\} = 0$ on BPS states

space of states on $HS^2 \times S^1$ for b.c. a

momentum on S^1

- Interpreted as $\mathcal{N} = (0,2)$ elliptic genus* for fixed m

Closset-Shamir 13
*Gadde-Razamat-Willet 15

Evaluation

- Naively:

$$\mathcal{B}_R^a(z_i;\sigma,\tau) = \begin{cases} \text{D:} \prod_{m=0}^{\infty} \left(y^{-1} p^{m+1-R/2}; q\right)_\infty \left(y p^{-m-1+R/2} q; q\right)_\infty \\ \text{R:} \prod_{m=0}^{\infty} \dfrac{1}{(y p^{m+R/2}; q)_\infty (y^{-1} p^{-m-R/2} q; q)_\infty} \end{cases}$$

BPS operator-state

Fermi multiplets

$$\left\{\partial_+^n \left(\chi_+^{(m)}, \bar{\chi}_+^{(-m)}\right)\right\}$$

2d chiral multiplets

$$\left\{\partial_+^n \left(\rho_+^{(-m)}, \bar{\rho}_+^{(m)}\right)\right\}$$

- Divergent for any $p \in \mathbb{C}$.

$$(y;q)_\infty = \prod_{n=0}^{\infty} (1 - y q^n)$$

- Source: arbitrarily positive and negative J_3.

Fix

- Impose D on chiral and R on anti-chiral (or vice versa)

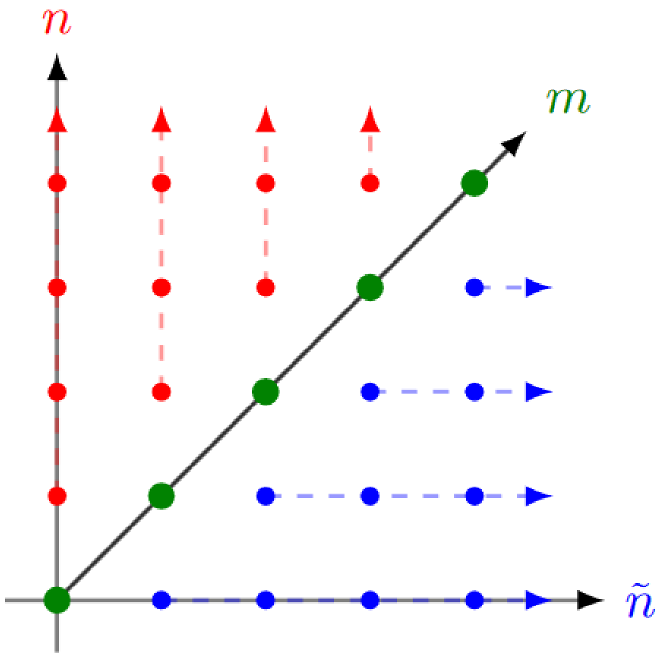
$$\mathcal{B}_R^a(z;\sigma,\tau) = \begin{cases} +: \displaystyle\prod_{m=0}^{\infty} \frac{\left(y^{-1}p^{m+1-R/2};q\right)_\infty}{(yp^{m+R/2}q;q)_\infty} = \frac{1}{\Gamma\left(-z+\left(1-\frac{R}{2}\right)\sigma;\sigma,\tau\right)} & \left\{\partial_+^n\left(\chi_+^{(m)},\bar{\rho}_+^{(m)}\right)\right\} \\ -: \displaystyle\prod_{m=0}^{\infty} \frac{\left(y^{-1}p^{-m+\frac{R}{2}-1}q;q\right)_\infty}{(yp^{-m-R/2};q)_\infty} = \Gamma\left(-z-\frac{R}{2}\sigma;-\sigma,\tau\right) & \left\{\partial_+^n\left(\rho_+^{(-m)},\bar{\chi}_+^{(-m)}\right)\right\} \end{cases}$$

- This confirms claim 1: $\mathcal{I}_R(z;\sigma,\tau) = \mathcal{B}_R^+(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}_R^+(z;\tau+\sigma,\sigma)$
  - $\widetilde{\mathcal{B}}_R^a(z;\sigma,\tau) \equiv \mathcal{B}_R^a(z-\tau;\sigma,\tau)$

Back to factorization index

- Hilbert space picture:
  - m labels 2d multiplet
  - $n, \tilde{n}$ left-moving momenta

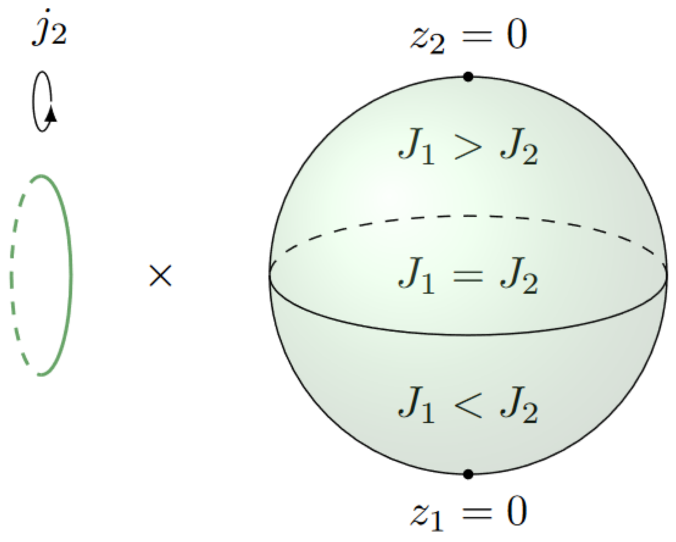
HS^2	J_3	P
N	$J_2 = m$	$2j_2 = \tilde{n}$
S	$J_1 = m$	$2j_2 = -n$



$$\Rightarrow \mathcal{H}_{\mathrm{BPS}}(S^3) = \mathcal{H}_{j_2 \le 0}(S^3) \otimes \mathcal{H}_{j_2 > 0}(S^3) = \bigotimes_{m=0}^{\infty} \mathcal{H}_{+}^{(m)}(S^1_H) \otimes \widetilde{\mathcal{H}}_{+}^{(m)}(S^1_H)$$

Intuitive picture

- Wavefunctions "localize" on HS^2_N or HS^2_S depending on $\operatorname{sgn} j_2$



- Picture consistent with $\mathcal{N} = 2$ SCFT/VOA correspondence $(J_2 = 0)$

Lagrangian perspective

- To understand modularity, a Lagrangian picture is useful:

$$\mathcal{I}_R(z;\sigma,\tau) = \mathcal{Z}_R[S^3 \times S^1(z;\sigma,\tau)]$$

$$\mathcal{B}_R^a(z;\sigma,\tau) = \mathcal{Z}_R^a[HS^2 \times T^2(z;\sigma,\tau)]$$

- Chemical potentials turn into complex structure and line bundle moduli

Closset-Dumitrescu et al 13

Bottom line

- Geometric interpretation of $\mathcal{I}_R(z;\sigma,\tau) = \mathcal{B}_R^{\alpha}(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}_R^{\alpha}(z;\tau+\sigma,\sigma)$

"splitting of a Hopf surface"

$$S^3 \times S^1(\sigma,\tau)$$
$$\downarrow$$
$$HS^2 \times_{\sigma+\tau} T^2_{\tau} \overset{g}{\cup} HS^2 \times_{\tau+\sigma} T^2_{\sigma}$$

$T^2 = S^1_H \times S^1_x$ in each patch

spatial cycle

temporal cycle

Prediction

- A modular transformation on $T^2 \cong S^1_H \times S^1_x$ predicts

$$\mathcal{I}_R(z;\sigma,\tau) = \mathcal{B}^+_R(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}^+_R(z;\tau+\sigma,\sigma)$$

$$\stackrel{?}{=} \mathcal{B}^+_R\left(\frac{z}{c\tau+d};\frac{\sigma+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\widetilde{\mathcal{B}}^+_R\left(\frac{z}{-c\sigma+d};\frac{\tau+\sigma}{-c\sigma+d},\frac{a\sigma-b}{-c\sigma+d}\right)$$

Proof

$$\mathcal{I}_R(z;\sigma,\tau) = \Gamma(z + R/2(\sigma+\tau);\sigma,\tau)$$

$$\mathcal{B}_R^+(z;\sigma,\tau) = \frac{1}{\Gamma(-z+(1-R/2)\sigma;\sigma,\tau)}, \qquad \widetilde{\mathcal{B}}_R^+(z;\sigma,\tau) = \Gamma(z + R/2\sigma;\sigma,\tau)$$

- Elliptic Gamma function satisfies:

Jejjala-Lei-SvL-Li 22

$$\Gamma(z+R/2(\sigma+\tau);\sigma,\tau) = e^{-i\pi Q_{(c,d)}(z+R/2(\sigma+\tau);\sigma,\tau)} \frac{\Gamma\left(\frac{z+R/2(\sigma+\tau)}{-c\sigma+d}; \frac{\tau+\sigma}{-c\sigma+d}, \frac{a\sigma-b}{-c\sigma+d}\right)}{\Gamma\left(\frac{-z+(1-R/2)(\sigma+\tau)}{c\tau+d}; \frac{\sigma+\tau}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)}$$

  - Holds for arbitrary $R \in \mathbb{R}$.
  - $Q_{(c,d)}(z;\sigma,\tau)$ is cubic polynomial in z.

4d modularity

- "Modular factorization":

Jejjala-Lei-SvL-Li 22

$$\mathcal{B}_R^+(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}_R^+(z;\tau+\sigma,\sigma) \cong \mathcal{B}_R^+\left(\frac{z}{c\tau+d};\frac{\sigma+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\widetilde{\mathcal{B}}_R^+\left(\frac{z}{-c\sigma+d};\frac{\tau+\sigma}{-c\sigma+d},\frac{a\sigma-b}{-c\sigma+d}\right)$$

- Natural specialization: $\sigma = -\bar{\tau}$

- How does this compare to $Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) \cong Z(\tau,\bar{\tau})$?

Conclusion

- Bottom-up argument for modular factorization:

$$\mathcal{B}_R^+(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}_R^+(z;\tau+\sigma,\sigma) \cong \mathcal{B}_R^+\left(\frac{z}{c\tau+d};\frac{\sigma+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\widetilde{\mathcal{B}}_R^+\left(\frac{z}{-c\sigma+d};\frac{\tau+\sigma}{-c\sigma+d},\frac{a\sigma-b}{-c\sigma+d}\right)$$

- Main ingredient: patch-wise reduction from $S^3\times S^1 \to T^2 \cong S_H^1 \times S_x^1$

Future directions

- Extension to non-trivial SCFTs, e.g., $SU(2)$ $\mathcal{N}=4$ SYM?

Pan-Peelaers 21

- Extension to other dimensions?

Cheng-Chun-Ferrari et al 18

- Generalization of $\mathcal{N}=2$ SCFT/VOA correspondence?
  - Non-chiral $\sigma=-\bar{\tau}$?

cf. Budzik-Gaiotto et al 23

Razamat 12, Beem et al 13
Dedushenko-Fluder 19
Pan-Peelaers 21
Beem-Razamat-Singh 21

Extra slides

$(H)S^2 \times S^1$ quantization

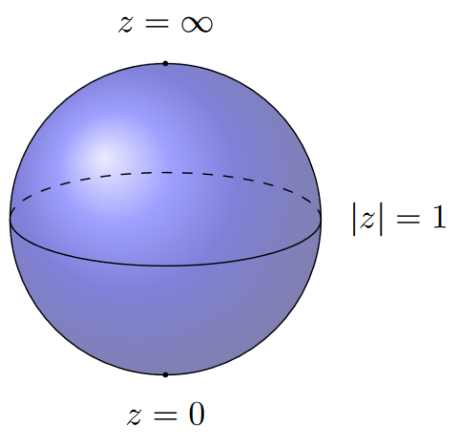
Chiral multiplet on $S^2 \times S^1$

- Two supercharges preserved on S^2 when:

Dumitrescu-Festuccia-Seiberg 12

$$A_R = -\frac{i}{2(1+|z|^2)}(\bar{z}dz - zd\bar{z}), \qquad ds^2 = \frac{4dzd\bar{z}}{(1+|z|^2)^2}$$

R-symmetry monopole potential



  - $U(1)_R$ topological twist on S^2
  - Due to magnetic flux, R-charge fields is quantized: $R \in \mathbb{Z}$

Shortening conditions on S^2

$$D_{\bar{z}}\psi_+ = D_z\phi = D_z\psi_- = 0$$

($z \leftrightarrow \bar{z}$ for $\bar{\phi}$ and $\bar{\psi}_\pm$)

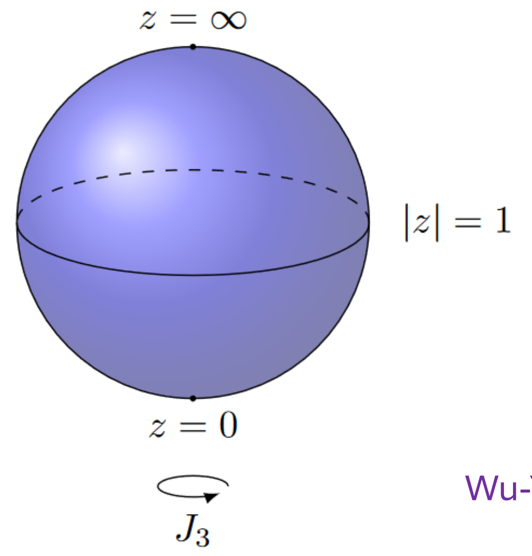
Closset-Shamir 13

covariant derivatives on S^2
in monopole background

- Solutions with $J_3 - \frac{r}{2}$ eigenvalue $m \in \mathbb{Z}\left(+\frac{1}{2}\right)$:

$$f_{\psi_+}^{(m)} = \frac{z^{m+\frac{R}{2}-1}}{(1+|z|^2)^{\frac{R}{2}-1}}, \qquad f_{\phi}^{(m)} = f_{\psi_-}^{(m)} = \frac{(1+|z|^2)^{\frac{R}{2}}}{\bar{z}^{m+\frac{R}{2}}}$$

  - $\Leftrightarrow$ zero-modes of S^2 "monopole harmonics"
  - m labels (non-trivial) $SU(2)$ representation, whose dimension depends on R



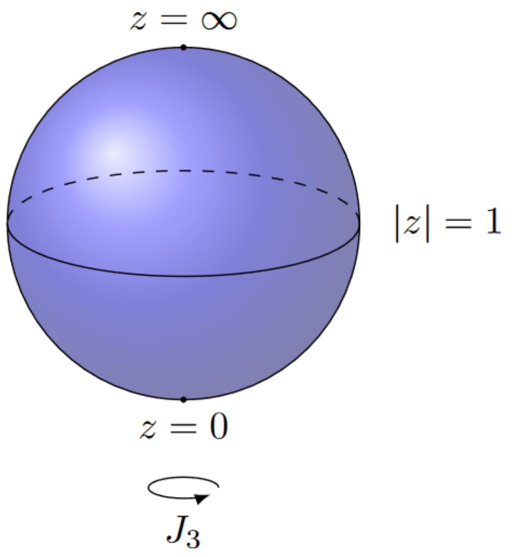
Wu-Yang 76

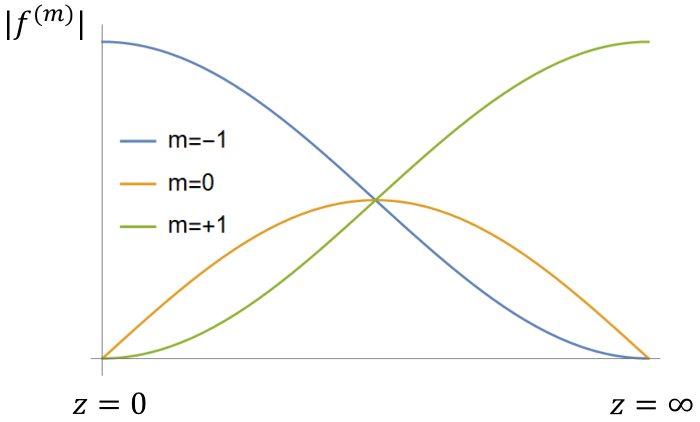
Normalizability

- The modes have to be regular at the poles

- This restricts the allowed modes depending on R:
  - $R > 1$: only $f_{\psi_+}^{(m)}$ for $1 - \frac{R}{2} \leq m \leq \frac{R}{2} - 1$
  - $R = 1$: no modes
  - $R < 1$: only $f_{\phi/\psi_-}^{(m)}$ for $-\frac{|R|}{2} \leq m \leq \frac{|R|}{2}$

- $\operatorname{sgn} m$ determines on which HS^2 the mode "localizes"





S^2 reduction

- Zero-modes of 4d chiral → short multiplets of 2d $\mathcal{N} = (0,2)$ susy:

Closset-Shamir 13

  - $R > 1: \{\psi_+\} \cup \{\bar{\psi}_+\} \to \left\{\chi_+^{(m)}\right\} \cup \left\{\bar{\chi}_+^{(-m)}\right\}$ $1 - \frac{R}{2} \leq m \leq \frac{R}{2} - 1$

    Fermi multiplets:
    left-moving complex fermion

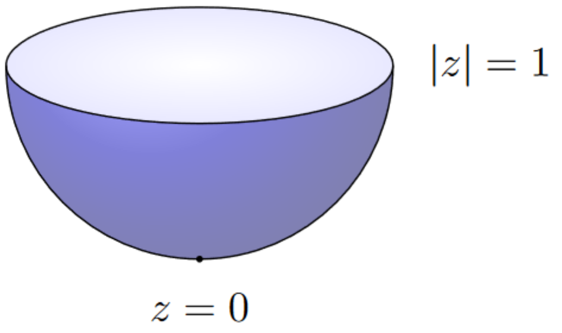
  - $R = 1:\ \emptyset$
  - $R < 1: \{\phi, \psi_-\} \cup \{\bar{\phi}, \bar{\psi}_-\} \to \left\{\rho^{(m)}, \chi_-^{(m)}\right\} \cup \left\{\bar{\rho}^{(-m)}, \bar{\chi}_-^{(-m)}\right\}$ $-\frac{|R|}{2} \leq m \leq \frac{|R|}{2}$

    2d chiral multiplets:
    complex boson and
    right-moving complex fermion

- Schematically: $\mathcal{H}_{\mathrm{BPS}}(S^2 \times S^1) = \prod_{|m| \leq g(R)} \mathcal{H}^{(m)}(S^1)$

Chiral multiplet on $HS^2 \times S^1$

- Restrict background to $|z| \leq 1$: hemi-sphere HS^2
- Replace regularity at $z = \infty$ by a (BPS) boundary condition at $|z| = 1$



Boundary conditions

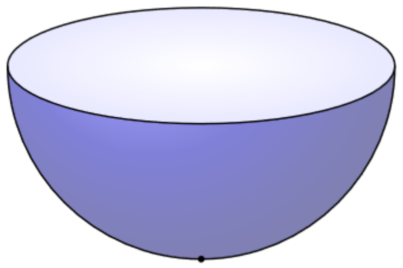
- Two (known) BPS boundary conditions:

most modes localize at boundary

Longhi-Nieri-Pittelli 19

Dirichlet $\quad \mathrm{D:}\ \phi/\psi_-\Big|_\partial = 0, \qquad D_{\bar z}\psi_+\Big|_\partial = 0$

Robin-like $\quad \mathrm{R:}\ D_z\phi/\psi_-\Big|_\partial = 0, \qquad \psi_+\Big|_\partial = 0$



- Regularity at $z = 0$ and b.c. at $|z| = 1$ yields:

$D_{\bar z}\psi_+ = D_z\phi = D_z\psi_- = 0$

$$\{\phi, \psi_\alpha\} \cup \{\bar\phi, \bar\psi_{\dot\alpha}\} \to \begin{cases} \mathrm{D:} & \left\{\chi_+^{(m)}\right\} \cup \left\{\bar\chi_+^{(-m)}\right\} \\ \mathrm{R:} & \left\{\rho^{(-m)}, \chi_-^{(-m)}\right\} \cup \left\{\bar\rho^{(m)}, \bar\chi_-^{(m)}\right\} \end{cases} \quad m \geq 0$$

$$f_{\psi_+}^{(m)} = \frac{z^m}{(1+|z|^2)^{\frac{R}{2}-1}},$$

$$f_\phi^{(-m)} = f_{\psi_-}^{(-m)} = \frac{(1+|z|^2)^{\frac{R}{2}}}{\bar z^{-m}}$$

Comments

- Match expressions obtained through localization

Longhi-Nieri-Pittelli 19

- Implicitly, included fermion zero-modes in $m \geq 0$

- If included in $m \leq 0$, the expressions are: $\widetilde{\mathcal{B}}_R^a(z;\sigma,\tau) = \mathcal{B}_R^a(z-\tau;\sigma,\tau)$

- Mixed b.c. reminiscent of BPS b.c. on a $\mathcal{N} = 2$ hypermultiplet
  - "Thimble boundary condition"

Dimofte-Gaiotto 12
Bullimore et al 16/20/21
Dedushenko-Nekrasov 21

Generalizations

Gauge theories

- Extension to gauge theories possible but non-trivial.

- In general:

chiral multiplets in gauge and flavor reps

$$\mathcal{I}(\vec{z};\sigma,\tau) = \frac{1}{|W|}\oint \prod_{i=1}^{r} \frac{dx_i}{2\pi i x_i} \Delta_G(\vec{u}) \mathcal{I}_{V_G}(\vec{u};\sigma,\tau) \prod_j \mathcal{I}_{\chi_j}(\vec{u},\vec{z}_j;\sigma,\tau)$$

vector multiplet and Vandermonde
determinant combine into elliptic Gamma

- Contour integral prevents naive factorization: Gauss law constraint.

- Getting rid of contour integral generally tricky. There exist two methods.

Higgs branch method

Benini-Cremonesi 12
Benini-Peelaers 13
Yoshida 14, Peelaers 14
Nieri-Pasquetti 15

- Applies to gauge theories whose gauge group can be completely Higgsed.
  - FI parameter and appropriate matter content.
- Since SCI is an RG invariant, expect an expression without gauge integral.
- Equivalent to residue sum of contour integral:

$$\mathcal{I}(z_i;\sigma,\tau) \sim \sum_a \mathcal{B}^a \widetilde{\mathcal{B}}^a\,, \qquad \mathcal{B}^a = \mathcal{B}^a_{\mathrm{cl}} \mathcal{B}^a_{\mathrm{pert}} \mathcal{B}^a_{\mathrm{vortex}}$$

- In this case, modular factorization goes through.

Jejjala-Lei-SvL-Li 22

Bethe ansatz method

- Also computes the residue sum.

Benini-Milan 18

- Applies to more general gauge theories, but:
  - Requires (in simplest case): $\sigma = \tau$.
  - Not fully understood for rank> 1 gauge theories.

Arabi-Ardehali-Hong-Liu 19
Gonzales Lezcano-Hong et al 21
Benini-Rizi 21

- Modular factorization when $\sigma = \tau$:

$$\mathcal{B}_R^a(z; 2\tau, \tau)\widetilde{\mathcal{B}}_R^a(z; 2\tau, \tau) \cong \mathcal{B}_R^a\left(\frac{z}{c\tau + d}; \frac{2\tau}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)\widetilde{\mathcal{B}}_R^a\left(\frac{z}{-c\tau + d}; \frac{2\tau}{-c\tau + d}, \frac{a\tau - b}{-c\tau + d}\right)$$

- Consistent with $\Gamma^0(n) \subset SL(2, \mathbb{Z})$ action when $R = \frac{m}{n}$.

Further generalizations

- SCFTs in even dimensions: $\mathcal{I} \sim \mathcal{Z}[S^{2d-1} \times S^1]$.

- S^{2d-1} admits Hopf fibration: $S^1 \hookrightarrow S^{2d-1} \to \mathbb{CP}^{d-1}$
  - BPS Hilbert space factorizes into d factors.
  - $\mathbb{CP}^{d-1}$ has d $\mathbb{C}^{d-1}$ patches where the fibration trivializes.
  - Consistent with factorization properties of multiple Gamma functions.

Gadde 20
Lei-SvL 24

- SCFTs in odd dimensions?
  - Hilbert space factorization still applies.
  - Factor of automorphy may be more complicated.

Cheng-Chun-Ferrari et al 18

3d limit

$SL(2,\mathbb{Z})$ family of 3d limits

- It is natural to shrink the T^2 spatial cycle, $S^1_{(c,d)}$.

$$\tilde{h} S_{23} \mathcal{O} h^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -c & a & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

- This corresponds to a Heegaard splitting of the lens space $L(c,1)$.

- At the level of the moduli: $\tau \to -\frac{c}{d} + \epsilon\hat{\tau}$, $\sigma \to \frac{c}{d} + \epsilon\hat{\sigma}$, $z \to \epsilon\hat{z}$ for $\epsilon \to 0$.

$$\mathcal{I}_R(z;\sigma,\tau) = \mathcal{B}_R^a\left(\frac{z}{c\tau+d};\frac{\sigma+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\widetilde{\mathcal{B}}_R^a\left(\frac{z}{-c\sigma+d};\frac{\tau+\sigma}{-c\sigma+d},\frac{a\sigma-b}{-c\sigma+d}\right)$$

$$\downarrow$$

$$\mathcal{Z}_{3d}[L(c,1)] \sim \mathcal{B}_{3d}^a\left(\frac{\hat{z}}{c\hat{\tau}};\frac{\hat{\sigma}+\hat{\tau}}{c\hat{\tau}}\right)\widetilde{\mathcal{B}}_{3d}^a\left(-\frac{\hat{z}}{c\hat{\sigma}};\frac{\hat{\tau}+\hat{\sigma}}{c\hat{\sigma}}\right)$$

Dimofte 14

Geometry

$D^2 \times T^2$

- Metric: $ds^2 = \frac{4dzd\bar{z}}{(1+|z|^2)^2} + dwd\bar{w}$

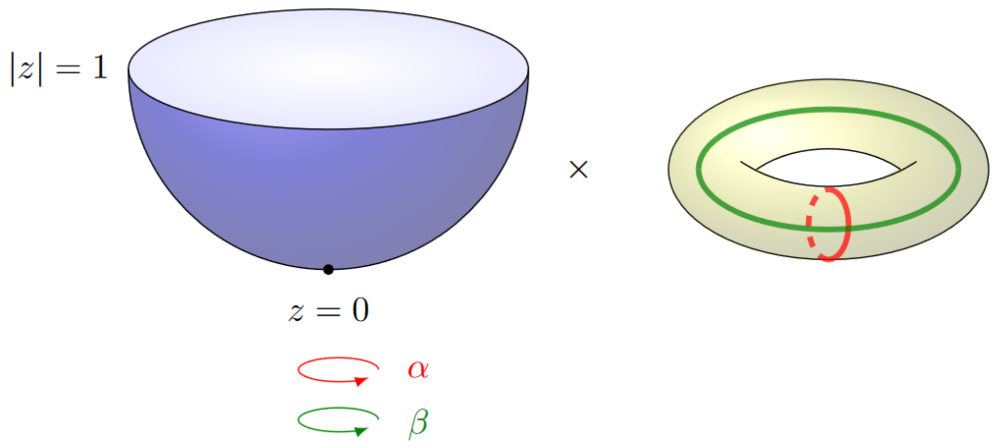
- Twisted b.c. implemented by:

$\mathcal{B}^{\alpha}(z;\sigma,\tau) = \mathrm{tr}_{\mathcal{H}^{\alpha}_{\mathrm{BPS}}(D^2\times S^1)}(-1)^F\, p^{J_3-\frac{r}{2}}\, q^P\, y^Q$

$$(z,w) \sim \left(e^{2\pi i\alpha}z, w+2\pi\right) \sim \left(e^{2\pi i\beta}z, w+2\pi\tau\right)$$

- Combine (α,β) into $\sigma = \beta - \alpha\tau$.

specifies torus fibration over disk

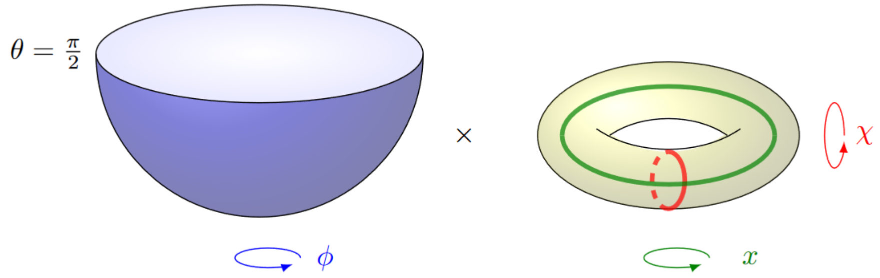


Large diffeomorphisms

- In real coordinates: $ds^2 = d\theta^2 + \sin^2\theta\,(d\phi + \alpha d\chi + \beta dx)^2 + (d\chi + \tau_1 dx)^2 + \tau_2 dx^2$
- Moduli space $T^3 = \frac{GL(3,\mathbb{R})}{O(3,\mathbb{R})\times\mathbb{R}^\times}$

$$\mathcal{M} = \begin{pmatrix} \tau_2 & \tau_1 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

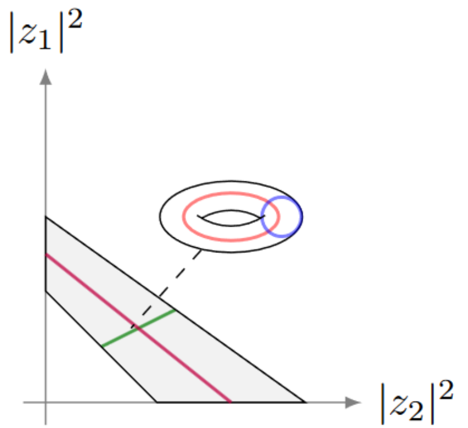
T^3 at $\theta = \frac{\pi}{2}$



- $SL(3,\mathbb{Z})$ = large diffeomorphisms of T^3
- $H \subset SL(3,\mathbb{Z})$ = large diffeomorphisms of $D^2 \times T^2$: $h = \begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & 1 \end{pmatrix}$
- Projective action: $(z;\sigma,\tau) \xrightarrow{h} \left(\frac{z}{c\tau+d};\frac{\sigma}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)$

$S^3 \times S^1$

fundamental domain

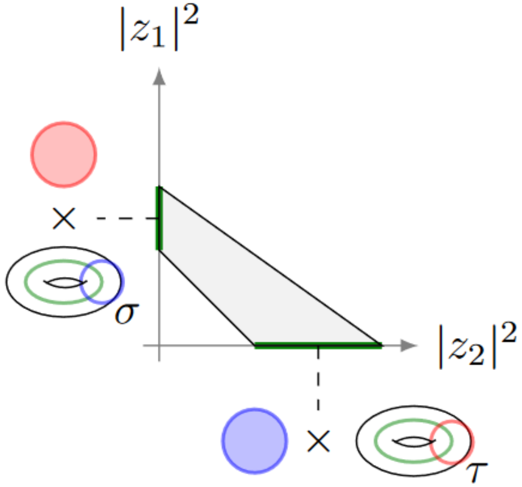


- Primary Hopf surface: for $(z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$

$$(z_1, z_2) \sim (pz_1, qz_2), \qquad |p|, |q| \leq 1$$

- Real metric:

$$ds^2 = \frac{1}{4} d\theta^2 + \sin^2\frac{\theta}{2}(d\phi + \sigma_1 dx) + \cos^2\frac{\theta}{2}(d\chi + \tau_1 dx) + \tau_2^2 dx^2$$



$\tau \leftrightarrow \sigma$ between patches

Splitting the Hopf surface

Closset-Shamir 13

- Zoom in on $D^2 \times T^2$ patch $|z_1| < |z_2|$ through:

$$z = \frac{z_1}{z_2}, \qquad w = -i \log z_2$$

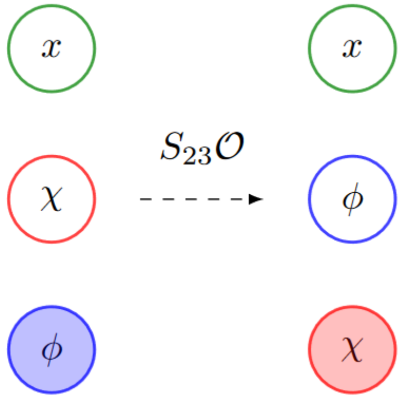
other patch

$$z' = \frac{1}{z}, \qquad w' = w - i \log z$$

$$S^3 \times S^1(\sigma,\tau)$$
$$\downarrow$$
$$D^2 \times T^2(\sigma,\tau) \overset{S_{23}\mathcal{O}}{\cup} D^2 \times T^2(\tau,\sigma)$$

- Read off $D^2 \times T^2$ moduli:

$$M = \begin{pmatrix} \tau_2 & \tau_1 & \sigma_1 - \tau_1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \tau_2 & \sigma_1 & \tau_1 - \sigma_1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$



- Related by $SL(3,\mathbb{Z})$: $\widetilde{M} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} M$

expresses exchange of $\chi \leftrightarrow \phi$ between patches

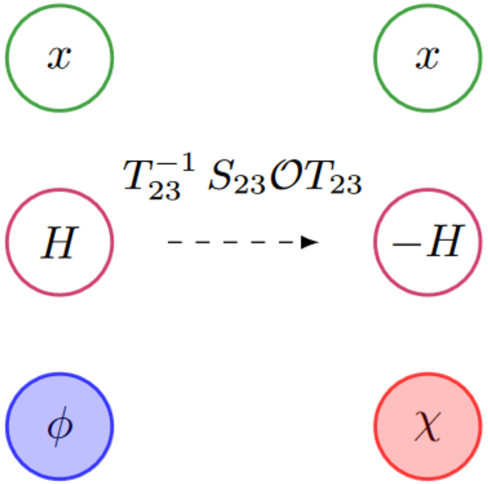
Geometry underlying factorization

- Return to: $\mathcal{I}_R(z;\sigma,\tau) = \mathcal{B}_R^{\alpha}(z;\sigma+\tau,\tau)\widetilde{\mathcal{B}}_R^{\alpha}(z;\tau+\sigma,\sigma)$

- Reflects an alternative splitting of the <u>same</u> Hopf surface:

$$S^3 \times S^1(\sigma,\tau) \to D^2 \times T^2(\sigma+\tau,\tau) \overset{T_{23}^{-1}S_{23}\mathcal{O}\,T_{23}}{\cup} D^2 \times T^2(\tau+\sigma,\sigma)$$

$M' = T_{23}^{-1}M$ $\qquad$ $\widetilde{M}' = T_{23}^{-1}\widetilde{M}$ $\qquad$ $T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in H$



$H = \chi - \phi$

Not all splittings are equal

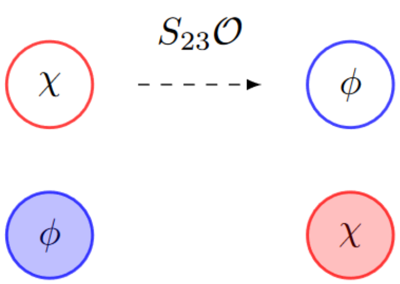
$$\mathcal{I}_R(z;\sigma,\tau) \stackrel{?}{=} \mathcal{B}_R^{\alpha}(z;\sigma,\tau)\widetilde{\mathcal{B}}_R^{\alpha}(z;\tau,\sigma)$$

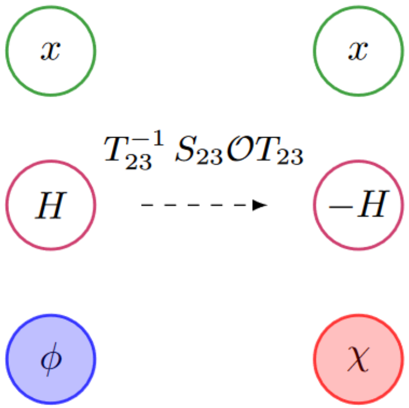
- Does not hold. What is the distinction?

$$\begin{array}{c} S^3 \times S^1(\sigma,\tau) \\ \downarrow \\ D^2 \times T^2(\sigma,\tau) \overset{S_{23}\mathcal{O}}{\cup} D^2 \times T^2(\tau,\sigma) \end{array}$$

$$\begin{array}{c} S^3 \times S^1(\sigma,\tau) \\ \downarrow \\ D^2 \times T^2(\sigma+\tau,\tau) \overset{T_{23}^{-1} S_{23}\mathcal{O}\, T_{23}}{\cup} D^2 \times T^2(\tau+\sigma,\sigma) \end{array}$$

no consistent disk reduction





consistent disk reductions!

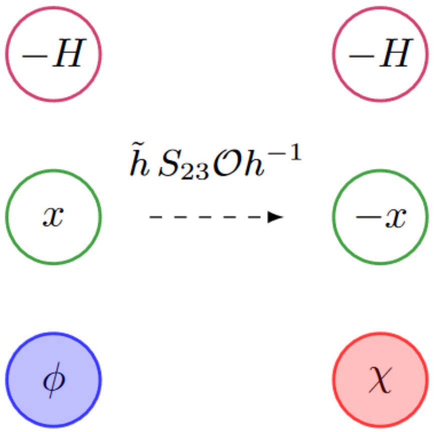
Other consistent splittings

- Consider: $M' = hM,\ \widetilde{M}' = \tilde{h}\widetilde{M}$

$$S^3 \times S^1(\sigma,\tau) \to D^2 \times T^2(h\cdot(\sigma,\tau)) \overset{\tilde{h}S_{23}\mathcal{O}h^{-1}}{\cup} D^2 \times T^2(\tilde{h}\cdot(\tau,\sigma))$$

- Splitting is consistent for any basis of cycles on $T^2 = S^1_H \times S^1_x$

- <u>Example</u>:



temporal circle S^1_x interpreted
as spatial cycle of T^2!

Index version modular factorization

Prediction

- Predicts: $\mathcal{J}_R(z;\sigma,\tau) \overset{?}{=} \mathcal{B}_R^a\left(\frac{z}{c\tau+d};\frac{\sigma+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\widetilde{\mathcal{B}}_R^a\left(\frac{z}{-c\sigma+d};\frac{\tau+\sigma}{-c\sigma+d},\frac{a\sigma-b}{-c\sigma+d}\right)$

- In more detail:

$$\mathcal{J}_R(z;\sigma,\tau) = \prod_{m=0}^{\infty}\left(\mathrm{tr}_{\mathcal{H}^{(m)}(S_H^1)}(-1)^F\,(pq)^{J_1-\frac{r}{2}}\,q^{-2j_2}\,y^Q\right)\times\left(\mathrm{tr}_{\widetilde{\mathcal{H}}^{(m)}(S_H^1)}(-1)^F\,(qp)^{J_2-\frac{r}{2}}\,p^{2j_2}\,y^Q\right)$$

$$\cong \prod_{m=0}^{\infty}\left(\mathrm{tr}_{\mathcal{H}^{(m)}(S_{(c,d)}^1)}(-1)^F\,(\tilde{p}\tilde{q})^{J_1-\frac{r}{2}}\,\tilde{q}^{-2j_2'}\,\tilde{y}^Q\right)\times\left(\mathrm{tr}_{\widetilde{\mathcal{H}}^{(m)}(S_{(c,d)}^1)}(-1)^F\,(\hat{q}\hat{p})^{J_2-\frac{r}{2}}\,\hat{p}^{2j_2'}\,\hat{y}^Q\right)$$

allow for
"factor of automorphy"

$\tilde{p} = e^{2\pi i\frac{\sigma+\tau}{c\tau+d}}, \tilde{q} = e^{2\pi i\frac{a\tau+b}{c\tau+d}}, \tilde{y} = e^{2\pi i\frac{z}{c\tau+d}}$ $\quad$ $\hat{q} = e^{2\pi i\frac{\tau+\sigma}{-c\sigma+d}}, \hat{p} = e^{2\pi i\frac{a\sigma-b}{c\sigma+d}}, \hat{y} = e^{2\pi i\frac{z}{c\tau+d}}$