Basic group representation theory

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1 Finite groups

A **group** is a set G of elements and a **group operation** $\cdot$ with the following properties.

- G is **closed** under the group operation: $a \cdot b \in G$ for each $a, b \in G$.
- The group operation is **associative**: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for each $a, b, c \in G$.
- G contains an **identity** element e with the property that $a \cdot e = e \cdot a = a$ for each $a \in G$.
- For each $a \in G$ there exists an **inverse** element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

The number of elements in the group is called the order of the group. We will write $[G]$ to denote the **order** of group G. A finite group is a group of finite order.

A **subgroup** $H \subset G$ is a subset of G which itself obeys the group axioms. Clearly if H is a subgroup of G and $g \in H$, then $g^{-1} \in H$. H must contain the identity. Associativity is inherited from G. Closure is the main issue in determining whether a subset of a group is a subgroup.

If $g' = h^{-1}gh$ then g and g' are called **conjugate** with respect to h. The **conjugacy class** of g is the set of all elements conjugate to g.

2 Group representations

Groups are abstract objects. In order to be usefully employed in calculations, and also to help elucidate properties of the groups themselves, we map group elements to linear operators, preserving the relationships implied by the group operation. We can then apply the familiar tools of linear algebra. Linear operators act on a vector space of particular dimension. Hence we must specify also the dimension of a group representation.

A **representation** of dimension n of the group G is a homomorphism

$$D : G \to GL(n, \mathbb{C}).$$

You should convince yourself that all of the group properties are preserved in the set $\{D(g)\}_{g \in G}$ with the group operation being usual matrix multiplication. These matrices can be thought of as acting on a vector space V. We say that V carries the representation D. Often we will say rep instead of representation.

If T is a linear operator on vector space V, and $|i\rangle$ is a basis for V then the matrix D realizing the operator T in this basis is given by

$$D_{ij} = \langle i| T |j\rangle$$

and

$$D |i\rangle = \sum_j |j\rangle \langle j| D |i\rangle = \sum_j D_{ji} |j\rangle .$$

This suggests some basis-dependence in our definition of a rep. In fact, if $D\vec{u} = \vec{v}$ in the basis $|i\rangle$ and $D'\vec{u}' = \vec{v}'$ in the basis $|i'\rangle$, where the two bases are related by the non-singular transformation S such that $\vec{u}' = S\vec{u}$ and $\vec{v}' = S\vec{v}$, then

$$\vec{v}' = S\vec{v} = S(D\vec{u}) = (SDS^{-1})\vec{u}'$$

and hence

$$D' = SDS^{-1}.$$

We would like to factor out this basis-dependence so that anything we can say about a rep in one basis is true in all bases. The above relation suggests that we consider two n-dimensional reps, D and D' to be **equivalent** if there is an S so that

$$D'(g) = SD(g)S^{-1} \quad \forall\, g \in G.$$

Henceforth I will usually say T to refer to an abstract, basis-independent linear operator and D to refer to the related matrix in some basis.

A rep is said to be **unitary** if the linear operator for each element $g \in G$ is unitary. Unitarity depends on the choice of inner product on the space V. This choice is not otherwise restricted by our notion of a rep. Given some inner product $(\cdot,\cdot)$, a linear operator is said to be unitary if

$$(T\vec{u}, T\vec{v}) = (\vec{u}, \vec{v}) \quad \forall \vec{u}, \vec{v} \in V.$$

Convince yourself that this reduces to the usual relation $D^\dagger = D^{-1}$ in an orthonormal basis (with respect to this inner product). Hint: write $\vec{v} = T^{-1}\vec{w}$.

Now we define the **group-invariant inner product**, which ensures the existence of a basis in which any given rep is unitary:

$$\{\vec{u}, \vec{v}\} = \frac{1}{[G]} \sum_{g\in G} (T(g)\vec{u}, T(g)\vec{v}).$$

With this definition we can check that, for any $h \in G$,

$$\begin{aligned}
\{T(h)\vec{u}, T(h)\vec{v}\} &= \frac{1}{[G]} \sum_{g\in G} (T(g)T(h)\vec{u}, T(g)T(h)\vec{v}) \\
&= \frac{1}{[G]} \sum_{g\in G} (T(gh)\vec{u}, T(gh)\vec{v}) \\
&= \frac{1}{[G]} \sum_{g'\in G} (T(g')\vec{u}, T(g')\vec{v}) \\
&= \{\vec{u}, \vec{v}\}
\end{aligned}$$

where in the third line we changed the sum into and equivalent sum over $g' = gh$. This step is only possible for finite or compact groups. Now if we use a basis orthonormal with respect to this group-invariant inner product, then we will have a unitary matrix rep. This means that any rep is related to a unitary rep by a change of basis, and hence all reps are unitary-equivalent.

3 Invariant subspaces and reducibility

Suppose that the space V carries an $n+m$-dimensional rep of G. Let's think of V as being composed of two subspaces U and W with respective bases $\{\vec{e}_i | i = 1, \ldots, m\}$ and $\{\vec{e}_j | j = m+1, \ldots, m+n\}$. U is

m-dimensional and W is n-dimensional. Further, suppose the rep on V has the property that

$$D(g)\vec{u} \in U \quad \forall \vec{u} \in U,\, g \in G.$$

We say that the subspace U is **invariant** (or closed) under the action of D, and that the rep is **reducible**.

What do the matrices $D(g)$ look like in this case? Considering how U and W are contained inside V we infer that, for U to be closed under $D(g)$, the matrices must take the form

$$D(g) = \begin{bmatrix} A(g) & C(g) \\ \underline{0} & B(g) \end{bmatrix}$$

where $A \sim m \times m$, $B \sim n \times n$, $C \sim m \times n$. Note that this form only ensures that U is closed. Specifically, if $C(g) \neq \underline{0}$ then W will not be closed.

If $C(g) = \underline{0}$, then W is also closed under D. All matrices $D(g)$ are block-diagonal and we say that the rep is **completely reducible**. Look closely at these matrices on the diagonal:

$$\begin{aligned} D(gh) &= D(g)D(h) \\ &= \begin{bmatrix} A(g)A(h) & \underline{0} \\ \underline{0} & B(g)B(h) \end{bmatrix}. \end{aligned}$$

This implies

$$\begin{aligned} A(gh) &= A(g)A(h) \\ B(gh) &= B(g)B(h), \end{aligned}$$

which means that A and B are respectively m- and n-dimensional reps of G. We have learned that if an n-dimensional rep is completely reducible, then it can be written as a direct sum of smaller reps whose dimensions sum to n.

If a rep T on V is unitary and there is an invariant subspace $U \subset V$, then if $\vec{w} \in W = V$
U we have

$$\begin{aligned} (T(g)\vec{w}, \vec{u}) &= (\vec{w}, T^{-1}(g)\vec{u}) \\ &= (\vec{w}, \vec{u}'), \quad \vec{u}' \in U \\ &= 0, \\ \implies \quad T(g)\vec{w} \in W. \end{aligned}$$

That is, the rep T is completely reducible.

Now we can understand **Maschke's Theorem**, which says that all reducible reps of a finite or compact group are completely reducible. For a finite or compact group we can define the group-invariant inner product. Consequently all reps are unitary equivalent. And for unitary reps, reducibility implies complete reducibility.

4 Irreducible representations

There are no limits on the number or dimension of reducible reps of a group. Given some reducible rep on V, we can find an invariant subspace and decompose the rep into a direct sum of two reps, on orthogonal subspaces of V. For each of these two reps we can do the same, and so on, breaking the original rep into a direct sum of successively smaller reps. Eventually, none of the reps on the diagonal will contain invariant subspaces. A rep with no invariant subspace is called an **irreducible representation**, or irrep. We say we have decomposed the rep on V into its irreps.

Any rep is either irreducible, or a direct sum of irreps. For this reason irreps are natural objects to study in representation theory. From their properties, the properties of any rep can be inferred. To learn more about irreps, we use two lemmas due to Schur.

Schur's first lemma: Given irrep T and some linear operator $B : U \to U$,

$$BT(g) = T(g)B \;\forall g \in G \implies B = \lambda \mathbb{I},$$

for $\lambda \in \mathbb{C}$ and $\mathbb{I} : \vec{u} \to \vec{u}$.

Proof: Consider an eigenvector $\vec{b}$ of B: $B\vec{b} = \lambda\vec{b}$. Then

$$BT(g)\vec{b} = T(g)B\vec{b} = \lambda[T(g)\vec{b}],$$

which means that $T(g)\vec{b}$ is also an eigenvector of B, with eigenvalue λ. So the eigenspace of B is closed under T. But T is an irrep and contains no invariant subspaces. Therefore, the eigenspace of B must be U itself. Similarly, all eigenvectors of B must have the same eigenvalue: subspaces with different eigenvalues closed under T would violate the irrep condition. If all the eigenvalues of B are equal, and B spans U, then one can see from the form of the characteristic equation that $B = \lambda\mathbb{I}$.

Schur's second lemma: Consider two inequivalent irreps T on U, and R on W, of dimension m and n, respectively. If $B : U \to W$ is a linear operator, then

$$BT(g) = R(g)B \;\forall g \in G \implies B = \underline{0}.$$

Proof: Consider $m < n$. For $\vec{u} \in U$,

$$R(g)(B\vec{u}) = BT(g)\vec{u} \in BU \;\forall g \in G.$$

So BU, the image of B, is an invariant subspace of W under R. But R is an irrep, so either $BU = W$ or $BU = \underline{0}$. The dimension of BU must be less than or equal to m, since U has only m basis vectors. By assumption, $m < n = \mathrm{dim}W$. So $BU \neq W$ and hence $BU = \underline{0}$.
Now consider $n > m$. The kernel of B is

$$K \equiv \{\vec{k} \in U | B\vec{k} = \vec{0}\}.$$

For $\vec{k} \in K$,

$$BT(g)\vec{k} = R(g)B\vec{k} = \vec{0},$$

so $T(g)K = K$. Hence $K \subset U$ is closed under T. But T is an irrep, so $K = \vec{0}$ or $K = U$. Our assumption implies $\mathrm{dim}BU < \mathrm{dim}U$ and so B must kill some non-zero vectors: $K \neq \vec{0}$. Thus, $K = U$ and $B = \underline{0}$.
Finally consider $m = n$, again thinking about the kernel K of B. If $K = \vec{0}$ then

$$B\vec{u} = B\vec{u}' \Rightarrow \vec{u} - \vec{u}' \in K \Rightarrow \vec{u} = \vec{u}',$$

so B is one-to-one and hence invertible. Then $T(g) = B^{-1}R(g)B$, which contradicts the inequivalence predicate. Hence $K = U$ and $B = \underline{0}$.

We're going to change notation now. We'll use $\Gamma_R(g)$ to denote the matrix corresponding to group element g in irrep R. The corresponding space carrying this irrep R will be denoted U_R. This will be useful to make our statements more compact. We'll use the symbol δ_{RS} as a Kronecker-delta analogue, equal to 1 if R and

S are equivalent as irreps, and 0 otherwise. $\dim_R$ is the dimension of irrep R.

In terms of this new notation we can combine Schur's lemmas into a single statement. Given two reps, R and S, of G and a linear operator $B : U_R \to U_S$,

$$B\Gamma_R(g) = \Gamma_S(g)B \;\forall\; g \in G \implies B = \lambda_R \delta_{RS}\mathbb{I}, \quad \lambda \in \mathbb{C}.$$

The matrix elements of irreps have a special property. Let U_R and U_S carry irreps of G, and let $A : U_S \to U_R$. Consider the operator

$$B \equiv \sum_{g\in G} \Gamma_R(g) A \Gamma_S(g^{-1}).$$

B satisfies the condition for Schur's lemma:

$$\begin{aligned}
\Gamma_R(h)B &= \sum_{g\in G} \Gamma_R(hg) A \Gamma_S(g^{-1}) \\
&= \sum_{g'\in G} \Gamma_R(g') A \Gamma_S(g'^{-1}h), \quad g' = hg,\, g^{-1} = g'^{-1}h \\
&= \sum_{g'\in G} \Gamma_R(g') A \Gamma_S(g'^{-1})\Gamma_S(h) \\
&= BT_S(h).
\end{aligned}$$

So $B = \lambda_R^{(A)} \delta_{RS}\mathbb{I}$.

We will choose $A_{\alpha\beta} = \delta_{\alpha,r}\delta\beta, s$ for some r, s. Check that this gives

$$\sum_{g\in G} \Gamma_R(g)_{ir} \Gamma_S(g^{-1})_{sj} = \lambda_R^{(rs)} \delta_{ij}\delta_{RS}.$$

To calculate $\lambda_R^{(rs)}$ set $R = S$ and trace over $i = j$. Convince yourself that this gives

$$[G]\delta_{rs} = \dim_R \lambda_R^{(rs)}.$$

This condition on the matrix elements of an irrep is called the **fundamental orthogonality relation**,

$$\sum_{g\in G} \Gamma_R(g)_{ir} \Gamma_S(g^{-1})_{sj} = \frac{[G]}{\dim_R} \delta_{ij}\delta_{rs}\delta_{RS}.$$

We can use this relation to understand how many irreps there can be for some group G. Since we can think of the Γs as unitary we can write the relation as

$$\sum_{g\in G} \Gamma_R(g)_{ir} \Gamma_S^*(g)_{sj} = \frac{[G]}{\dim_R} \delta_{ij}\delta_{rs}\delta_{RS}.$$

For some particular $R = S$ we can look at this as an inner product between two $[G]$-dimensional vectors. The vectors are labelled by two indices, each of which can take $\dim_R$ values. For each choice of indices, the vectors are orthogonal, so they can be thought of as a basis spanning this $[G]$-dimensional space. There can be at most $[G]$ such vectors, and hence

$$\sum_R (\dim_R)^2 \leq [G].$$

From this we learn that the number of irreps is bounded. To find out exactly how many there are we need to turn to one of the most powerful notions in group representation theory.

5 Characters

If $\Gamma_R(g)$ is a rep of G, then the **character** of g in R is

$$\chi_R(g) = \mathrm{tr}\left(\Gamma_R(g)\right).$$

Try to prove these simple properties of characters:

- Equivalent reps have the same characters.
- Conjugate elements have the same characters.
- If $\Gamma^\dagger = \Gamma^{-1}$ then $\chi(g^{-1}) = \chi^*(g)$. This is always true for finite or compact group reps.

By appropriately tracing the fundamental orthogonality relation, show that for irreps we have

$$\frac{1}{[G]}\sum_{g\in G}\chi_R(g)\chi_S(g^{-1}) = \delta_{RS},$$

or, for finite or compact groups,

$$\frac{1}{[G]}\sum_{g\in G}\chi_R(g)\chi_S^*(g) = \delta_{RS} \equiv \langle\chi_S, \chi_R\rangle.$$

This second relation looks like an inner product between $[G]$-dimensional vectors. Viewed in this way, group characters of irreps are orthonormal.

Let's work out how many irreps there are of a group G. As noted, the characters of all elements within a conjugacy class are equal. Suppose there are k conjugacy classes in G, each with k_i elements. Then

$$\frac{1}{[G]}\sum_i k_i\chi_R^{(i)}\chi_S^{(i)*} = \delta_{RS},$$

which expresses the orthogonality of k-dimensional vectors $\sqrt{k_i}\chi_R^{(i)}$. But there are most k such vectors, and so the number of irreps r is bounded by $r \le k$. We also can prove the orthogonality relation

$$\frac{1}{[G]}\sum_R \sqrt{k_i}\chi_R^{(i)}\sqrt{k_j}\chi_R^{(j)*} = \delta_{ij},$$

which describes k orthogonal r-dimensional vectors. There can be at most r such vectors, so $k \le r$. Hence $r = k$. The number of irreps of a group is equal to the number of conjugacy classes of the group.

6 Decomposition into irreducible representations

Finite and compact groups have the property that any reducible rep can be brought to block diagonal form, with irreps on the diagonal. For example, some 5-dimensional rep might have a single 1-dimensional irrep and two copies of the same 2-dimensional irrep on.

For an arbitrary rep Γ, we can write a direct sum over irreps:

$$\Gamma(g) = \bigoplus_R \Gamma_R(g)^{\oplus a_R},$$

where a_R is the **multiplicity** of the irrep R in the decomposition. The multiplicity can be determined by using the orthogonality property of characters. Trace both sides of the equation above to get

$$\chi(g) = \sum_R a_R \chi_R(g).$$

The character of a general rep is a sum over characters of its constituent irreps times a multiplicity factor. For any particular irrep,

$$\langle \chi, \chi_S \rangle = \frac{1}{G} \sum_R a_R \sum_{g \in G} \chi_R(g) \chi_S^*(g) = a_S.$$

Characters can also tell us when a rep is irreducible. For any rep with character $\chi(g)$,

$$\langle \chi, \chi \rangle = \frac{1}{[G]} \sum_R \sum_S a_R a_S \sum_{g \in G} \chi_R(g) \chi_S^*(g) = \sum_R a_R^2.$$

If χ is an irrep, then only a single a_R is non-zero, and it is equal to one. For a reducible representation, this inner product would be greater than 1. This is a much easier condition to check than looking for a change of basis which diagonalizes all of the Γs, or ruling one out.

7 The regular representation

We can create a vector space V of dimension $[G]$ by using group elements to label orthonormal basis vectors. That is, V is spanned by the basis

$$|g_i\rangle \text{ for } i = 1, 2, \ldots, [G].$$

This basis admits a natural action for group elements, terms of $[G] \times [G]$ matrices:

$$\Gamma(g) |g_i\rangle = |g g_i\rangle = \sum_j \Gamma(g)_{ji} |g_j\rangle \ \forall \ g \in G.$$

The basis vectors are orthonormal, so $\Gamma(g)_{ji}$ is non-zero only for a single j which satisfies $g g_i = g_j$. So for each column i, only a single row j has a non-zero entry, and that entry is one. For each i, $g g_i$ is a unique element, so each row has exactly one non-zero entry. Since $g g_i = g_i$ only for the identity element e, no matrix will have diagonal entries except for $\Gamma(e)$. This rep is called the **regular representation**.

Consider the vector $\vec{v} \in V$ such that $\vec{v} = \sum_i |g_i\rangle$. For any $g \in G$,

$$\Gamma(g)\vec{v} = \sum_i |g g_i\rangle = sum_i |g_i\rangle = \vec{v},$$

so $\vec{v}$ defines an 1-dimensional invariant subspace of V. This is an irrep called the symmetric rep. Clearly the regular rep is reducible. What lives on the other subspace? Certainly at least one more irrep. Consider that the regular rep includes a basis vector for every possible action of a group element. Hence every possible irrep must be some subspace of V. Since we know that this rep can be block-diagonalized these must be orthogonal subspaces. We can explore this using characters. Consider a decomposition of Γ into all possible irreps of G,

$$\Gamma(g) = \bigoplus_R \Gamma_R(g)^{\oplus a_R}.$$

From the form of Γ we know that all characters are zero, except $\chi(e) = [G]$. Furthermore

$$a_R = \langle \chi, \chi_R \rangle = \frac{1}{[G]} \sum_{g \in G} \chi_R(g) \chi^*(g) = \chi_R(e) = \dim_R,$$

so in the decomposition of the regular rep, each possible irrep of G appears with multiplicity equal to its dimension. From this we can also refine a previous result:

$$\sum_R \dim_R = [G].$$

This property of the regular rep, that it contains all possible irreps, is not all that useful in practice for groups of larger order. It becomes quite difficult to perform the decomposition. We'll look at an example later, after studying the symmetric group.

8 The symmetric group

The **symmetric group** S_n is the group of permutations of n objects. We'll think of these objects as the numbers $\{1, 2, \dots, n\}$. We'll write particular elements of the symmetric group in cycle notation. Look at some examples in S_4:

$$\begin{aligned}
(1\,3) &: \{1,2,3,4\} \to \{3,2,1,4\} \\
(1\,4\,3) &: \{1,2,3,4\} \to \{4,2,1,4\} \\
(1\,2\,3\,4) &: \{1,2,3,4\} \to \{2,3,4,1\} \\
(1\,3)(2\,4) &: \{1,2,3,4\} \to \{3,4,1,2\} \\
(1\,2)(1\,3) &: \{1,2,3,4\} \to \{3,1,2,4\}
\end{aligned}$$

Note in the second last example that $(1\,3)(2\,4) = (2\,4)(1\,3)$. $(1\,3)$ and $(2\,4)$ are called disjoint cycles. Disjoint cycles commute. Cycles which are not disjoint can be rewritten. In the last example, $(1\,2)(1\,3) = (1\,3\,2)$. As a further example, $(1\,2\,3)(2\,3\,4) = (1\,2)(3\,4)$. The 2-cycles are also called **transpositions**. Note that $(1\,2) = (2\,1)$ and that any transposition squares to the identity. A neighbouring transposition is one that can be written in the form $(k,\, k+1)$. Any element $\sigma \in S_n$ can be written as a product of neighbouring transpositions, for example,

$$\begin{aligned}
(1\,3) &= (1\,2)(2\,3)(1\,2) \\
(1\,2\,3) &= (1\,2)(2\,3).
\end{aligned}$$

Try some operations with cycles for yourself, to become familiar with the patterns. What does the inverse of an element look like?

When written in terms of disjoint cycles, every element of of S_n has a particular **cycle structure**; the number of 1-, 2-, 3-, ... cycles that make up the element. Numbers which don't appear to 2- or longer cycles can be thought of as sitting in 1-cycles. For example, in S_4 you can think of $(1\,2) = (1\,2)(3)(4)$ as being an element comprised of one 2-cycle and two 1-cycles.

Conjugation in S_n preserves cycle structure, and permutes the labels inside the conjugated elements by the action of the conjugating element. For example, in S_5 with $\sigma = (1\,2\,3)$ and $\sigma^{-1} = (1\,3\,2)$,

$$\begin{aligned}
\sigma(1\,5)(2\,3\,4)\sigma^{-1} &= (2\,5)(3\,1\,4) \\
\sigma(4\,5)(1\,2)\sigma^{-1} &= (4\,5)(2\,3).
\end{aligned}$$

So conjugacy classes in S_n are defined by cycle structure. Note that for any $\sigma \in S_n$ the sum of the lengths of cycles in σ equals n. Hence conjugacy classes, and therefore irreps, of S_n are in one-to-one correspondence with partitions of n.

We can associate with each $\sigma \in S_n$ a **signature** which is 1 if σ can be written as the product of an even number of neighbouring transpositions and -1 otherwise. We'll denote it $\mathrm{sgn}(\sigma)$. Then we can form a simple

1-dimensional irrep by the map $\Gamma(\sigma) = \text{sgn}(\sigma)$. This is called the **alternating representation**. The **trivial representation** has $\Gamma(\sigma) = 1$.

S_n subgroups are collections of permutations which hold some subset of $\{1, 2, \ldots, n\}$ fixed, or only permute some elements amongst themselves. For example, there are several possible subgroups $S_3 \subset S_6$ or even $S_4 \times S_2 \subset S_6$. Try to count how many possible subgroups there are of each form. Can you generalize your reasoning?

The symmetric group is the most important finite group. **Cayley's theorem** states that every finite group G is isomorphic to a subgroup of $S_{[G]}$. So if we understand the representation theory of the symmetric group and its subgroups, we can apply our understanding to any other finite group.

9 Example: S_2

S_2 has 2 elements, $\sigma_1 = (1)(2)$ and $\sigma_2 = (1\,2)$. Its regular rep is therefore 2-dimensional. We'll take as orthonormal basis vectors

$$|\sigma_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; |\sigma_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The group structure is simply $\sigma_1\sigma_i = \sigma_i\sigma_1 = \sigma_i$ and $\sigma_2\sigma_2 = \sigma_1$. From this the representation matrices follow trivially:

$$\Gamma(\sigma_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \Gamma(\sigma_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We simply need to diagonalize $\Gamma(\sigma_2)$ in order to block-diagonalize the rep and obtain the irreps. The characteristic equation for $\Gamma(\sigma_2)$ is $\lambda^2 - 1 = 0$ which has eigenvalues $\lambda = \pm 1$. We then find normalized eigenvectors, which we'll label by the eigenvalue:

$$|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \; |-1\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

So the diagonalizing similarity transformation is

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \; S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then compute the transformed matrices:

$$S\Gamma(\sigma_1)S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; S\Gamma(\sigma_2)S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

On the top diagonal we see the trivial rep, and on the second we see the alternating rep. Each appears once, as expected, since each has dim=1.

Look closely at the space carrying the trivial rep. Its single basis vector is $\frac{1}{\sqrt{2}} |\sigma_1\rangle + \frac{1}{\sqrt{2}} |\sigma_2\rangle$, a sum over each element with equal coefficients. For this reason it is also called the **symmetric rep**.

Now consider the space carrying the alternating rep. It has a single basis vector

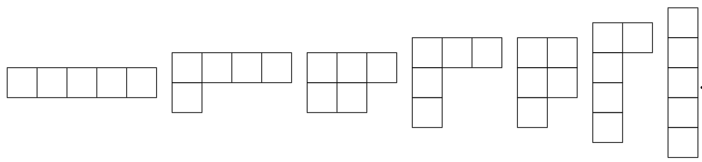
$$\frac{1}{\sqrt{2}} |\sigma_1\rangle - \frac{1}{\sqrt{2}} |\sigma_2\rangle = \text{sgn}(\sigma_1)\frac{1}{\sqrt{2}} |\sigma_1\rangle + \text{sgn}(\sigma_2)\frac{1}{\sqrt{2}} |\sigma_2\rangle,$$

and is therefore called the **antisymmetric rep**.

You should construct the regular rep for S_3. Think about how to decompose it. Can you easily find an appropriate similarity transformation? What about S_4? Clearly the applicability of the regular rep to finding irreps is of limited value. We will now turn our attention to some powerful tools of symmetric group representation theory which will allow us, among other nifty things, to write down representations matrix element by matrix element.

10 Young diagrams

As discussed, irreps of S_n are in one-to-one correspondence with conjugacy classes and hence with partitions of n. Let's describe a partition of n by drawing rows of boxes. The total number of boxes is n. The number of rows is the size of the partition. The number of boxes in each row gives a set of integers which sum to n and describe the partition. In order to avoid over-counting, enforce the restriction that the number of boxes in a row is nonincreasing from top to bottom. For example, consider the partitions of 5:



From this we conclude that there are seven possible irreps of S_5. These diagrams are called **Young diagrams**. They are more than a convenient means of labelling irreps. Many properties of irreps are encoded in the structure of Young diagrams, and they can be used to construct a natural basis in which it becomes easier to construct the irreps explicitly.

A useful way to gain more insight into the rep theory of the symmetric group is the process of **subduction**. When we restrict an irrep of some group to a subgroup, then the irrep **subduces** a rep on the subgroup. More precisely, if $H \subset G$ is a subgroup, and we have an irrep R of G, then we can subduce a rep $R|_H$ on H:

$$\Gamma_{R|_H}(h) = \Gamma_R(h) \; \forall \; h \in H.$$

A natural question is whether this subduced rep is reducible. If R is an irrep, then there is no transformation S such that $S\Gamma_R(g)S^{-1}$ has the same block-diagonal structure for each $g \in G$. That does not preclude the possibility of finding S such that $S\Gamma_R(g)S^{-1}$ *is* block-diagonal for each $g \in H$. So in general we must assume that the subduced rep is reducible. In that case the natural question is, which irreps of the subgroup H appear in the decomposition of $R|_H$? We'll answer this for the symmetric group in particular.

Suppose R is an irrep of S_{n+m}. R is a Young diagram with $n+m$ boxes. Consider a general $S_n \times S_m$ subgroup of S_{n+m}. What does an irrep of $S_n \times S_m$ look like? We can pick any irrep of S_n for that factor, and any irrep of S_m. Suppose we choose S and T as irreps, respectively. S is a Young diagram with n boxes and T is a Young diagram with m boxes. We write the irrep of $S_n \times S_m$ as $S \times T$.

If we restrict the rep R of S_{n+m} to $S_n \times S_m \subset S_{n+m}$, the subduced rep is reducible. That is, the rep can be written as a direct sum of irreps of $S_n \times S_m$, each irrep appearing with some multiplicity:

$$\Gamma_R = \bigoplus_{S,T} (\Gamma_{(S\times T)})^{\oplus f_{RST}}.$$

The multiplicity factors f_{RST} are called **Littlewood-Richardson** numbers. They can be determined by the **Littlewood-Richardson rule**.

We begin by studying the Littlewood-Richardson rule for $S_{n-1} \times S_1 \subset S_n$ subgroups. S_1 is trivial and obviously $S_{n-1} \times S_1 \cong S_{n-1}$. The only conjugacy class, and hence irrep, of S_1 is $\square$, so that is the only possible choice for T in the decomposition of the subduced rep. Convince yourself that $\square$ is a 1-dimensional irrep that must map the only element in S_1 to 1. The Littlewood-Richardson rule says that the irreps of S_{n-1} appearing in the rep subduced from R of S_n are those which can be obtained by removing a single box from R to leave a valid Young diagram. A few examples will illustrate. I'll only draw the Young diagrams, but you should think of them as labelling irreps, and think of the equations as describing direct sums or matrices.

$$
\begin{aligned}
\ydiagram{2,1,1} &= \ydiagram{1,1,1} \times \ydiagram{1} + \ydiagram{2,1} \times \ydiagram{1} \\
\ydiagram{3,2,1} &= \ydiagram{2,2,1} \times \ydiagram{1} + \ydiagram{3,1,1} \times \ydiagram{1} + \ydiagram{3,2} \times \ydiagram{1} \\
\ydiagram{2,2} &= \ydiagram{2,1} \times \ydiagram{1}
\end{aligned}
$$

Convince yourself that for any subduction of a rep onto $S_{n-1} \subset S_n$ the Littlewood-Richardson numbers can only be 0 or 1.

We can use the Littlewood-Richardson rule to learn how the Young diagrams encode the dimension of the irreps they label. In the second example above we have irrep $\ydiagram{3,2,1}$ of S_6. The subgroup representation matrices can be brought to block-diagonal form with irreps of S_5 on the diagonal:

$$
\begin{bmatrix}
\ydiagram{2,2,1} & \underline{0} & \underline{0} \\
\underline{0} & \ydiagram{3,1,1} & \underline{0} \\
\underline{0} & \underline{0} & \ydiagram{3,2}
\end{bmatrix}.
$$

Now imagine we restrict further, to an S_4 subgroup of the $S_5 \subset S_6$. The appropriate matrices (which ones are they?) can be block-diagonalized with S_4 irreps on the diagonal. Which irreps? Well, the Littlewood-Richardson rule acts on the irreps of S_5 on the diagonal. The above structure becomes

$$
\begin{bmatrix}
\ydiagram{2,1,1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \ydiagram{2,2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \ydiagram{2,1,1} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \ydiagram{3,1} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \ydiagram{2,2} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \ydiagram{3,1}
\end{bmatrix},
$$

where the first two irreps come from $\ydiagram{2,2,1}$, the second two from $\ydiagram{3,1,1}$ and the last two from $\ydiagram{3,2}$. Notice that we're starting to see some multiplicity. Now you should continue this chain of subduction until you are left with S_1 irreps on the diagonal. Only one matrix has that form - the identity in the original S_6 irrep. So if we could count the number of $\square$s we would know the dimension of the irrep $\yng(3,2,1)$.

So how many $\square$s are there? At each restriction to a subgroup the decomposition of irreps was given by all ways of pulling off a block to leave a valid Young diagram. Each of these irreps appeared on a diagonal block. Convince yourself that as a result, the number of single boxes appearing is equal to the number of possible sequences of pulling boxes off the original Young diagram to leave a single box, where at each step only valid Young diagrams are kept. This is equal to the dimension of the irrep. Show that $\yng(3,1)$ is 3-dimensional.

For larger Young diagrams, this isn't a practical technique. Happily, the diagrams encode the dimension of the irreps they label in another way. We can associate a number with each box in the diagram in several ways. One way is to use the **hook length**. The hook length of a box is the sum of the number of boxes beneath it plus the number of boxes to its right, plus one (for the box itself). This is called the hook length because it counts the number of boxes an L-shape flipped about the horizontal axis, with its corner in the box in question, passes through. Example diagrams with hooks filled in:

$$\young(531,31,1)\,, \quad \young(421,1).$$

We will write hooks_R for the product of hook lengths for all boxes in the diagram R. Then the dimension of the S_n rep R is

$$\dim_R = \frac{n!}{\text{hooks}_R}.$$

So far we have discussed the Littlewood-Richardson rule for the case of subducing representations on S_{n-1} subgroups of S_n. Now we'll consider subducing reps on $S_n \times S_m \subset S_{n+m}$. This is based on a procedure called **induction**, which we will only touch upon, in the specific manner in which it is used here. We know how to subduce a rep on $S_n \times (S_1)^m \subset S_{n+m}$: irrep R on S_{n+m} decomposes into a sum over irreps $S \times T$, always with T of the form

$$T = \underbrace{\square \times \square \times \cdots \times \square}_{m \text{ boxes}},$$

where we think of the m boxes of T as having been pulled off the diagram successively (their ordering in T doesn't matter since the product operation commutes). We want to **induce** a rep on S_m from $(S_1)^m$. The irrep $S \times T$ appears in the decomposition of the subduced rep f_{RST} times. Inducing a rep on $S_n \times S_m$ involves a change of basis that mixes these f_{RST} copies. The induced rep is reducible, and the irreps can be found by the rule for multiplying the m boxes of T: take the first box, and attached the second in all possible ways to form valid Young diagrams. Then take the third box, and attach it to each of the 2-box

Young diagrams in all possible ways that form Young diagrams, and so on. For example:

$$
\begin{aligned}
\ydiagram{1}\times\ydiagram{1}\times\ydiagram{1}\times\ydiagram{1} \quad \text{induces} \quad & \left(\ydiagram{2}+\ydiagram{1,1}\right)\times\ydiagram{1}\times\ydiagram{1}\\
= \quad & \left(\ydiagram{3}+\ydiagram{2,1}+\ydiagram{2,1}+\ydiagram{1,1,1}\right)\times\ydiagram{1}\\
= \quad & \ydiagram{4}+\ydiagram{3,1}+\ydiagram{3,1}+\ydiagram{2,2}+\ydiagram{2,1,1}+\ydiagram{3,1}\\
& +\ydiagram{2,2}+\ydiagram{2,1,1}+\ydiagram{2,1,1}+\ydiagram{1,1,1,1}\\
= \quad & \ydiagram{4}+3\,\ydiagram{3,1}+2\,\ydiagram{2,2}+3\,\ydiagram{2,1,1}+\ydiagram{1,1,1,1}.
\end{aligned}
$$

There is one important caveat. Recall that in the irrep $S \times T$ subduced from R we think of T as a product of boxes pulled off of R. When forming the product as in the example above, any boxes that appeared in the same row or column in R must appear in the same row or column in the irrep of S_m. It can be useful to label boxes when computing products, until you are used to this. Simply multiply the boxes together as in the example above, and then keep only those diagrams whose boxes have the correct relationship. Here is an example. Note that the constant 2 on the right hand side in the first line is absorbed during induction; the two copies of the space mix to form irreps of $S_3 \times S_3$.

$$
\begin{aligned}
\begin{ytableau} \; & \; & a \\ d & b & c \end{ytableau} \quad &= \quad 2\,\ydiagram{2,1}\times\begin{ytableau} a \end{ytableau}\times\begin{ytableau} b \end{ytableau}\times\begin{ytableau} c \end{ytableau}+\ydiagram{3}\times\begin{ytableau} d \end{ytableau}\times\begin{ytableau} b \end{ytableau}\times\begin{ytableau} c \end{ytableau}\\
&= \quad \ydiagram{2,1}\times\begin{ytableau} c & a \\ b \end{ytableau}+\ydiagram{3}\times\begin{ytableau} d & b & c \end{ytableau} \quad \text{(we have dropped several invalid diagrams)}\\
&= \quad \ydiagram{2,1}\times\ydiagram{2,1}+\ydiagram{3}\times\ydiagram{3}.
\end{aligned}
$$

The dimension of a rep $S \times T$ is $dim_S \times dim_T$. Show that the dimensions on the left and right hand side of the example above agree. Would they agree if we did not preserve row and column relationships of boxes? Convince yourself by trying a few examples that if $S \times T$ appears f_{RST} times in a subduction from R, then R appears f_{RST} times in the product $S \times T$. This is called **Frobenius-Schur duality**.

11 Yamanouchi basis and Young tableaux

In the previous section we discovered that the number of sequences of pulling all the boxes off a Young diagram, at each step leaving a valid Young diagram, is equal to the dimension of the irrep labelled by the diagram. Suppose we numbered the n boxes of a diagram with 1 to n, to indicate at which step we would remove each box, for example

$$\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau} \quad \text{or} \quad \begin{ytableau} 4 & 3 & 1 \\ 2 \end{ytableau}.$$

Such a numbered diagram is called a **Young tableau**. Not every numbering of a diagram is a valid tableau. For example,

$$\begin{ytableau} 2 & 3 & 1 \\ 2 \end{ytableau}$$

is invalid. For each diagram there as many valid tableaux as the dimension of the irrep. (Write down the tableaux for S_3 and compare with what you know about the dimensions of S_3 irreps). We can therefore assume that the Young tableaux for a particular diagram label a set of orthonormal basis vectors spanning the space carrying the irrep.

If R is an irrep of S_n and $\sigma \in S_n$ then, for some appropriate action of σ, the matrix element

$$\Gamma_R(\sigma)_{ij} = \langle R_i | \, \sigma \, | R_j \rangle \, ,$$

where R_i and R_j are Young tableaux. For this to be useful in practice we need to things: an ordering on the tableaux and a convenient action of a permutation on a tableau.

A suitable ordering on Young tableaux is to compare two tableaux in terms of the row in which the smallest number occurs. The tableau with the number 1 in the lowest position comes first. If the tableaux have 1 in the same row, then look at the position of 2, and so on. Some examples:

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} < \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

There is no comparison for tableaux of different shape. Note that this ordering places all vectors in the subspace obtained by removing a particular block together. That guarantees a block diagonal structure for appropriate permutations when subducing reps for $S_{n-k} \times (S_1)^k \subset S_n$. Which permutations are appropriate? In other words, suppose we remove the box labelled 1 to get a rep on $S_{n-1} \subset S_n$: there are n such subgroups. The elements of all of them can't be block-diagonal, because then all elements of S_n would be block-diagonal. In fact, only one subgroup has the block-diagonal structure. We will establish the convention that it is the subgroup of S_n which holds n fixed. Hence, removing the box labelled 1 corresponds to restricting the subgroup which holds n fixed. Similarly, removing the box labelled 2 restricts to S_{n-2} which holds n and $n-1$ fixed.

We still need an appropriate action of a permutation on a Young tableaux. To that end, we now define the **axial distance** between two boxes on a tableaux. Given two boxes on the tableaux, say 1 and 3, the axial distance between them is the number of steps which must be taken (horizontally or vertically) from box labelled 1 to get to box labelled 3. Steps down or to the left count +1 and steps up or to the right count -1. We will write η_{ij} for the axial distance between box labelled i and box labelled j. Here are some examples:

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \; : \;\; \eta_{13} = -1, \; \eta_{12} = -2,$$

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \; : \;\; \eta_{13} = 1, \; \eta_{14} = 2, \; \eta_{12} = 3.$$

Next we define the action of an adjacent 2-cycle on a Young tableau. Because of the subgroup chain we use in subductions, the 2-cycle $(n-k, n-k-1)$ acts naturally on the tableau labels $k+1$ and $k+2$. It's action is to map a tableau (basis vector) to the same tableau times a **no-swap factor** plus the tableau obtained by swapping the two labels (only if this is valid tableau) time a **swap factor**. The no-swap and swap factors are defined in terms of the axial distance between the boxes with the relevant labels on the tableau being acted upon:

$$\text{noswap}_{ij} = \frac{1}{\eta_{ij}}, \;\; \text{swap}_{ij} = \sqrt{1 - \frac{1}{(\eta_{ij})^2}}.$$

The no-swap factor is antisymmetric in its labels. When calculating the no-swap factor between adjacent labels k and $k+1$, we use $\text{noswap}_{k,k+1}$. Some examples to illustrate (the permutation are implicitly in the representation given by the shape of the tableaux):

$$
\begin{aligned}
(2\,3)\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle &= \text{noswap}_{12}\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle + \text{swap}_{12}\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle \\
&= -\frac{1}{2}\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle + \frac{\sqrt{3}}{2}\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
(3\,4)\left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle &= \text{noswap}_{12}\left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle \quad \text{(swap factor term not a valid tableau)} \\
&= \left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
(2\,3)\left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle &= \text{noswap}_{23}\left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle + \text{swap}_{23}\left|\,\begin{ytableau} 4 & 3 & 1 \\ 2 \end{ytableau}\right\rangle \\
&= \frac{1}{2}\left|\,\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau}\right\rangle + \frac{\sqrt{3}}{2}\left|\,\begin{ytableau} 4 & 3 & 1 \\ 2 \end{ytableau}\right\rangle
\end{aligned}
$$

With this action, a permutation can never change the shape of a tableau. Think about why that is necessary. For general permutations, we must first express the permutation in terms of a product of adjacent 2-cycles. Recall the relationship between the action of a linear operator on an orthonormal basis and its matrix representation in that basis. Compute the matrices for S_2 in both possible irreps. Compare your result with what we obtained using the regular rep. Calculate a few S_3 irrep matrices as well. This is a very convenient way to build irreps for any finite group.

This action on the Young tableaux creates an orthogonal representation. Check that the matrices you calculated above are indeed orthogonal ($M^T = M^{-1}$). In the next section we cover a graphical notation for summarizing calculations of matrix elements in the Yamanouchi basis.

12 Strand diagrams

We are interested in calculating matrix elements of the form (the permutation is implicitly in the rep $\ydiagram{3,2,1}$)

$$
\left\langle\,\begin{ytableau} \; & \; & 1 \\ \; & 2 \\ 3 \end{ytableau}\,\right|(4\,6)\left|\,\begin{ytableau} \; & \; & 3 \\ \; & 1 \\ 2 \end{ytableau}\,\right\rangle
$$

in the orthogonal representation as described in the previous section. To make the calculation more convenient we use a graphical notation called **strand diagrams**. These diagrams capture the process of acting with successive adjacent 2-cycles on a tableau and taking the overlap of the resulting vector with another tableau.

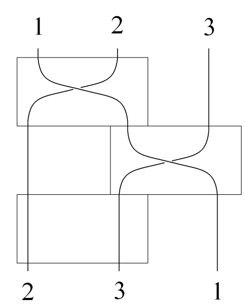
In the present example, $(4\,6)$ is not an adjacent 2-cycle. We first need to express it as a product of such. Convince yourself that

$$(4\,6) = (4\,5)(5\,6)(4\,5).$$

So we need to compute

$$\left\langle \ytableaushort{{}{}1,{}2,3} \right| (4\,5)(5\,6)(4\,5) \left| \ytableaushort{{}{}3,{}1,2} \right\rangle .$$

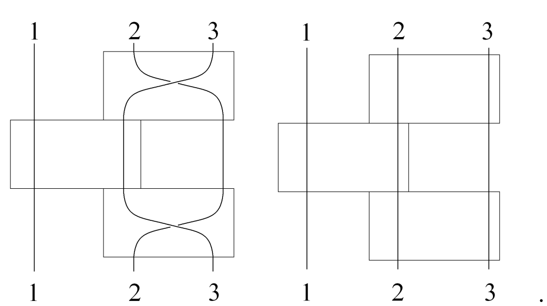
Here's how to draw a strand diagram. We'll think of the 2-cycles as acting to the left on the row labels. By the relation between the subduction chain and tableau labels, this permutation will affect the labels 1, 2 and 3. As a result, the diagram will have three columns. In the first column, write the label of the first box to be removed, 1. In the second column the label of the second box to be removed, 2. In the third column, 3. The first 2-cycle to act on the left is $(4\,5)$, which acts on labels 2 and 3. So draw a box underneath the labels which spans the columns of 2 and 3. The next 2-cycle which acts on the left is $(5\,6)$, which acts on labels 1 and 2. In a row beneath the previous box, draw a box spanning the columns of 1 and 2. In the next row draw the final 2-cycle: a box spanning the columns with labels 2 and 3. The top of the columns carry row labels. At the bottom of the columns, write the column labels. In the first column, instead of writing 1, write the label of the box in the bra which corresponds the box labelled 1 in the ket. Similarly for the second and third column. Now we draw strands connecting labels at the top with the corresponding labels at the bottom. Strands are identified by the label at the top where they originate. Strands move down the diagram, and each time a pair of strands enters a box, they can either go straight through, or swap columns in the box. If the mth and nth strands enter a box (reading from left to right), and don't swap, they contribute a factor noswap_{mn}. If they do swap, they contribute a factor swap_{mn}. The factors for each diagram are multiplied together, and the matrix element is a sum over all possible diagrams (there may be several ways to connect the strands). For our example, the only diagram is:



Convince yourself that this gives $\frac{\sqrt{3}}{2}\frac{\sqrt{15}}{4}\frac{1}{2} = \frac{3\sqrt{5}}{16}$. Now try to compute

$$\left\langle \ytableaushort{{}{}1,{}2,3} \right| (4\,6) \left| \ytableaushort{{}{}1,{}2,3} \right\rangle .$$

You will find two diagrams:



This powerful technique lets us calculate any matrix element of any permutation in any irrep. It is among the most efficient ways to generate representations.

13 Casimirs and projectors

When we restrict representations to subgroups $S_{n-k}\times(S_1)^k\subset S_n$, the subduced representation is reducible. How do we extract the irreps of S_{n-k} on the diagonal? We use linear operators called projectors. As their name suggests, projectors isolate subspaces of a linear space. For example, we know that the space carrying S_6 $\ydiagram{3,2,1}$ decomposes under restriction to S_5 as $\ydiagram{2,2,1}+\ydiagram{3,1,1}+\ydiagram{3,2}$. We want an operator that takes us to some particular subspace, say

$$P_{\ydiagram{3,2,1}\to\ydiagram{3,1,1}}\left|\ydiagram{3,2,1}\right\rangle = P_{\ydiagram{3,2,1}\to\ydiagram{3,1,1}}\left(\left|\ydiagram{2,2,1}\right\rangle\oplus\left|\ydiagram{3,1,1}\right\rangle\oplus\left|\ydiagram{3,2}\right\rangle\right)=\left|\ydiagram{3,1,1}\right\rangle.$$

How could we build such an operator? Suppose we had another operator $\mathcal{O}$, which had the same eigenvalue for any vector in the space carrying R. In other words, if we use the notation $|R,i\rangle$ with i running over the dimension of R to label basis vectors,

$$\mathcal{O}\,|R,i\rangle=\lambda_R\,|R,i\rangle\ \forall\, i.$$

Then we could build a projector as follows:

$$P_{\ydiagram{3,2,1}\to\ydiagram{3,1,1}}=\mathcal{N}\left(\mathcal{O}-\lambda_{\ydiagram{2,2,1}}\right)\left(\mathcal{O}-\lambda_{\ydiagram{3,2}}\right).$$

Check that this operator behaves in the desired fashion. Show that the normalization must be

$$\mathcal{N}=\frac{1}{\left(\lambda_{\ydiagram{3,1,1}}-\lambda_{\ydiagram{2,2,1}}\right)\left(\lambda_{\ydiagram{3,1,1}}-\lambda_{\ydiagram{3,2}}\right)}.$$

Construct the projectors onto the other two subspaces carrying S_5 irreps in this example.

The operator $\mathcal{O}$ is called a Casimir. For use in constructing projectors it is important that the Casimir have a different eigenvalue for each Young diagram (why?). Note that

$$\mathcal{O}\,|R,i\rangle=\lambda_R\,|R,i\rangle\ \forall\, i$$

implies

$$\Gamma_R(\sigma)\mathcal{O}\,|R,i\rangle=\Gamma_R(\sigma)\lambda_R\,|R,i\rangle=\mathcal{O}\Gamma_R(\sigma)\,|R,i\rangle\ \forall\, i$$

since $\Gamma_R(\sigma)$ cannot change the shape of the tableau. Consequently, $\mathcal{O}$ commute with the whole group S_n and by Schur's lemma must be proportional to the identity. We can define several such operators. We will select one with an eigenvalue that is easy to compute directly from the Young diagram. It is a sum over all possible 2-cycles:

$$\mathcal{O}=\sum_{i<j}(i\,j).$$

Show that this commutes with any element in S_n. The eigenvalue for this operator is equal to the sum of the number of pairs of boxes in each row, minus the sum over number of pairs of boxes in each column. For example:

$$\begin{aligned}\mathcal{O}\left|\ydiagram{3,2}\right\rangle &= (3+1-1-1)\left|\ydiagram{3,2}\right\rangle,\\ \mathcal{O}\left|\ydiagram{2,2,1}\right\rangle &= (1+1-3-1)\left|\ydiagram{2,2,1}\right\rangle,\\ \mathcal{O}\left|\ydiagram{3,1,1}\right\rangle &= (3-3)\left|\ydiagram{3,1,1}\right\rangle.\end{aligned}$$