An Introduction to Schur Polynomials and Restricted Schur Polynomials

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1 Introduction

We now turn our attention to Schur polynomials and restricted Schur polynomials, their properties and correlation functions in the context of quantum field theory. The first question we address is: why study these polynomials?

- Schur polynomial and Restricted Schur polynomial technology brings together a number of aspects of symmetric group representation theory and as such their study will deepen ones understanding of these aspects.
- Both Schur polynomials and restricted Schur polynomials encapsulate some very interesting physics. They are relevant in the study of the AdS/CFT correspondence from the quantum field theory perspective. We elaborate on this in section 5.

We begin with a brief review of Schur polynomials.

2 Schur Polynomials

The Schur polynomial is defined as follows:

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)Z^{i_1}_{i\sigma(1)}Z^{i_2}_{i\sigma(2)}\dots Z^{i_{n-1}}_{i\sigma(n-1)}Z^{i_n}_{i\sigma(n)}.$$

The label R is a Young diagram of n boxes. Young diagrams of n boxes are in one-to-one correspondence with the irreducible representations of the symmetric group S_n and thus a Schur polynomial labeled by R is associated with a particular irreducible representation of the symmetric group. For this general definition, Z is an arbitrary matrix (note that if $Z \in$ SU(N), then Schur polynomials also have an interpretation in Unitary group theory, $\chi_R(Z)$ gives the character of Z in the SU(N) irreducible representation specified by the Young diagram R). The factor $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in the irreducible representation R. Here the indices $i_k,\ k = 1,\dots,n$ range over $1,\dots,m$ where m sets the size of the matrix Z, i.e. Z is an $m\times m$ matrix. Z^l_m represents the matrix element in the lth row and mth column of the matrix Z. Note that in accordance with the Einstein summation convention repeated indices are summed, for example Z^m_m would represent the trace of the matrix Z, $\mathrm{Tr}\,Z$. In the Schur polynomial definition, the lower indices of the matrices in each term of the sum are a particular permutation of the indices $i_1\dots i_n$ specified by σ.

Consider the following multi-trace factors for $n = 4$ for example:

1. $\sigma = \mathbf{1}$.

$$\begin{aligned} Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} Z^{i_4}_{i\sigma(4)} &= Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_3} Z^{i_4}_{i_4} \\ &= (\mathrm{Tr}\, Z)^4 \,. \end{aligned}$$

2. $\sigma = (123)$.

$$\begin{aligned} Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} Z^{i_4}_{i\sigma(4)} &= Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} Z^{i_4}_{i_4} \\ &= \left(\mathrm{Tr}\, Z^3\right) (\mathrm{Tr}\, Z) \,. \end{aligned}$$

3. $\sigma = (1234)$.

$$\begin{aligned} Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} Z^{i_4}_{i\sigma(4)} &= Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_4} Z^{i_4}_{i_1} \\ &= \mathrm{Tr}\, Z^4. \end{aligned}$$

Clearly, obtaining the explicit form of a Schur polynomial involves determining the character of each group element in irreducible representation R as well as the associated multi-trace factor appearing in each term of the Schur polynomial. The procedure can be considerably simplified in realizing that all group elements with a certain cycle structure (a cycle of a particular length or products of cycles of particular lengths) belong to the same conjugacy class and have the same character for a given irreducible representation. Moreover it is clear that all multi-trace factors are equal for group elements belonging to a particular conjugacy class. As you have learnt, the number of conjugacy classes matches the number of irreducible representations of S_n. The number of conjugacy classes is vastly smaller than the number of elements in S_n (the order of S_n is $n!$) even when n is only moderately large. For example the number of conjugacy classes for $n = 6$ is 11 whereas the number of elements in S_6 is $6! = 720$. To explicitly obtain the required characters one can construct the matrices representing the group elements of S_n in irreducible representation R and then take the trace. For this, the method of constructing representations in the Yamanouchi basis presented previously can be utilized. Alternatively, the strand diagram technology furnishes a neat, graphical way to determine the characters. There are also recursive formulae that can be applied to calculate characters which will not be presented here.

2.1 Examples of Schur Polynomials

Explicit expressions for all the possible Schur polynomials for n=1, 2, 3 are:

$$
\begin{aligned}
\chi_{\ydiagram{1}}(Z) &= \mathrm{Tr}\, Z, \\
\chi_{\ydiagram{2}}(Z) &= \frac{1}{2}\Big((\mathrm{Tr}\, Z)^2 + \mathrm{Tr}\, Z^2\Big), \\
\chi_{\ydiagram{1,1}}(Z) &= \frac{1}{2}\Big((\mathrm{Tr}\, Z)^2 - \mathrm{Tr}\, Z^2\Big), \\
\chi_{\ydiagram{1,1,1}}(Z) &= \frac{1}{6}\Big((\mathrm{Tr}\, Z)^3 - 3\,(\mathrm{Tr}\, Z)\Big(\mathrm{Tr}\, Z^2\Big) + 2\mathrm{Tr}\, Z^3\Big), \\
\chi_{\ydiagram{2,1}}(Z) &= \frac{1}{3}\Big((\mathrm{Tr}\, Z)^3 - \mathrm{Tr}\, Z^3\Big), \\
\chi_{\ydiagram{3}}(Z) &= \frac{1}{6}\Big((\mathrm{Tr}\, Z)^3 + 3\,(\mathrm{Tr}\, Z)\Big(\mathrm{Tr}\, Z^2\Big) + 2\mathrm{Tr}\, Z^3\Big).
\end{aligned}
$$

3 Restricted Schur Polynomials: Distinguishable Matrix Words

Restricted Schur polynomials are the multi-matrix generalization of the single matrix Schur polynomials described previously. Restricted Schur polynomials are constructed from more than one type of matrix, individually or combined into matrix words. These matrix words are simply products of matrices, which we term letters. In the case of distinguishable matrix words, these matrix words are all distinct. This is the case we consider presently.

The definition of a restricted Schur polynomial is as follows:

$$\chi^{(k)}_{R,R_1}(Z, W^{(1)}, \ldots, W^{(k)}) = \frac{1}{(n-k)!}\sum_{\sigma\in S_n} \mathrm{Tr}_{R_1}(\Gamma_R(\sigma))\mathrm{Tr}\,(\sigma Z^{\otimes n-k} W^{(k)}\ldots W^{(1)}). \tag{1}$$

In this definition, R_1 is an irreducible representation of $S_{n-k}\times(S_1)^k$ and we therefore associate it with a Young diagram with n-k boxes. $W^{(1)}\ldots W^{(k)}$ are all distinct. $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ is the restricted character where the trace is taken over the indices belonging to the subspace specified by R_1. These restricted characters could be calculated using projection operators built from Casimirs or by utilizing strand diagrams as you have learnt about previously. The following notation

$$\mathrm{Tr}\,(\sigma Z^{\otimes n-k} W^{(k)}\ldots W^{(1)}) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)}\ldots Z^{i_{n-k}}_{i\sigma(n-k)} (W^{(k)})^{i_{n-k+1}}_{i\sigma(n-k+1)}\cdots (W^{(1)})^{i_n}_{i\sigma(n)},$$

expresses the multi-trace factor obtained for a given permutation, σ, of the lower indices, $i_1\ldots i_n$.

Consider for example ($n=4$, $k=1$):

1. $\sigma = \mathbf{1}$.

$$\begin{aligned}\mathrm{Tr}\,(\sigma Z^{\otimes 3} W^{(1)}) &= Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} (W^{(1)})^{i_4}_{i\sigma(4)} \\ &= Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_3} (W^{(1)})^{i_4}_{i_4} \\ &= (\mathrm{Tr}\,Z)^3 \left(\mathrm{Tr}\,W^{(1)}\right).\end{aligned}$$

2. $\sigma = (123)$.

$$
\begin{aligned}
\mathrm{Tr}\,(\sigma Z^{\otimes 3} W^{(1)}) &= Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} (W^{(1)})^{i_4}_{i\sigma(4)} \\
&= Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} (W^{(1)})^{i_4}_{i_4} \\
&= \left(\mathrm{Tr}\, Z^3\right) \left(\mathrm{Tr}\, W^{(1)}\right).
\end{aligned}
$$

3. $\sigma = (1234)$.

$$
\begin{aligned}
\mathrm{Tr}\,(\sigma Z^{\otimes 3} W^{(1)}) &= Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} Z^{i_3}_{i\sigma(3)} (W^{(1)})^{i_4}_{i\sigma(4)} \\
&= Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_4} (W^{(1)})^{i_4}_{i_1} \\
&= \mathrm{Tr}\, Z^3 W^{(1)}.
\end{aligned}
$$

3.1 Examples of Restricted Schur Polynomials

$$
\begin{aligned}
\chi_{\tiny\begin{array}{|c|c|}\hline \ & 1 \\ \hline\end{array}}\left(Z, W^{(1)}\right) &= \mathrm{Tr}\, Z \mathrm{Tr}\, W^{(1)} + \mathrm{Tr}\, Z W^{(1)}, \\
\chi_{\tiny\begin{array}{|c|}\hline \ \\ \hline 1 \\ \hline\end{array}}\left(Z, W^{(1)}\right) &= \mathrm{Tr}\, Z \mathrm{Tr}\, W^{(1)} - \mathrm{Tr}\, Z W^{(1)}, \\
\chi_{\tiny\begin{array}{|c|c|}\hline \ & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}\left(Z, W^{(1)}, W^{(2)}\right) &= \mathrm{Tr}\, Z \mathrm{Tr}\, W^{(1)} \mathrm{Tr}\, W^{(2)} - \mathrm{Tr}\, Z W^{(2)} \mathrm{Tr}\, W^{(1)} + \frac{1}{2} \mathrm{Tr}\, Z W^{(1)} \mathrm{Tr}\, W^{(2)} \\
&+ \frac{1}{2} \mathrm{Tr}\, Z \mathrm{Tr}\, W^{(1)} W^{(2)} - \frac{1}{2} \mathrm{Tr}\, Z W^{(2)} W^{(1)} - \frac{1}{2} \mathrm{Tr}\, Z W^{(1)} W^{(2)}.
\end{aligned}
$$

3.2 Reduction Rule

In this section we will consider the action of

$$
\mathrm{Tr}\left(\frac{d}{dZ}\right) \equiv D_Z, \quad \text{and} \quad \mathrm{Tr}\left(\frac{d}{dW^{(k)}}\right) \equiv D_{W^{(k)}},
$$

on restricted Schur polynomials. These operations are important in evaluating quantum field theory theory correlators of restricted Schur polynomials. We call these "reductions" of the restricted Schur polynomial because the action of the operators removes boxes from the Young diagram label of the polynomial. The action of $D_{W^{(k)}}$ is very simply stated (provided we are reducing with respect to the word associated with the index to be fixed first in restriction). It removes the box associated with $W^{(k)}$ (the box labeled by k in the Young diagram) and multiplies the resultant polynomial by something called the weight of the removed box (the weight of the box in the ith row and jth column of the Young diagram is defined as $N-i+j$), provided the matrices from which the restricted Schur polynomials are built are rank N. We will henceforth assume this to be the case. The reduction rule can be stated as:

$$D_{W^{(1)}}\chi_{R,R'} = c_{R,R'}\chi_{R'},$$

where $c_{R,R'}$ is the weight of the removed box.

For example:

$$D_{W^{(1)}}\chi_{\ytableaushort{{}{}1,{}{}}}\left(Z,W^{(1)}\right) = (N+2)\chi_{\ydiagram{2,2}}(Z)\,.$$

One subtlety to consider is that if we have a restricted Schur with an off-diagonal restricted trace involving the matrix word with the smallest label and we reduce with respect to that matrix word then the result of the reduction vanishes i.e.

$$D_{W^{(1)}}\chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\1}}}}\left(Z,W^{(1)},W^{(2)}\right) = 0.$$

However, it is not true in general that:

$$D_{W^{(2)}}\chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\1}}}}\left(Z,W^{(1)},W^{(2)}\right) = 0.$$

Correctly evaluating the result of this reduction requires the application of the subgroup swap rule, discussed in the section 3.3.

Upon acting D_Z on an ordinary Schur polynomial, all Schur polynomials that can be obtained by removing a single box from the Schur polynomial it acts on are produced. Each of the polynomials produced are multiplied by the weight of the removed box. For example:

$$D_Z\chi_{\ydiagram{3,1}} = (N+2)\chi_{\ydiagram{2,1}} + (N-1)\chi_{\ydiagram{3}}$$

Finally, we will evaluate the result of acting D_Z on a restricted Schur polynomial. By explicitly evaluating the derivative, we have

$$\frac{d}{dZ^a_a}\chi^{(1)}_{R,R_1}(Z,W) \;=\; \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\Gamma_R(\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}\delta^{i_{n-1}}_{i_{\sigma(n-1)}}W^{i_n}_{i_{\sigma(n)}}$$

$$= \quad D_X \sum_\alpha \chi^{(2)}_{R,T_\alpha}(Z,X,W), \tag{2}$$

where in the restricted Schur polynomial $\chi^{(2)}_{R,T_\alpha}(Z,X,W)$, W is associated with the box that must be removed from R to obtain R_1 and X is associated with the box that must be removed from R_1 to obtain T_α. The δ^i_j appearing in the above expression is the usual Kronecker delta with definition $\delta^i_j = 1$ if $i = j$ and 0 otherwise. In this last formula, the representations T_α are all representations that can be obtained by removing a single box from R_1, so that

$$R_1 = \oplus_\alpha T_\alpha.$$

The reduction with respect to X in (2) is now easily computed using the subgroup swap rule (section 3.3).

3.2.1 General Proof of Reduction Rule for one matrix word

We will now discuss the general proof of the reduction rule (acting on a restricted Schur polynomial with one matrix word) since it is a good example of the concrete application of cosets and Casimirs of S_n.

For the case of a single matrix word, the result can be proved as follows:

Consider the matrix-valued function

$$\hat{\chi}_R(Z,W) = \frac{1}{(n-1)!} \sum_{\sigma\in S_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \Gamma_R(\sigma),$$

where $\Gamma_R(\sigma)$ is the matrix representing σ in irreducible representation R. There is a simple relation between this function and all the restricted Schur polynomials that can be obtained by restricting R to $S_{n-1} \times S_1$. Denote the possible irreducible representations which arise upon restriction by R_α. Then

$$\mathrm{Tr}_{R_\alpha}\left(\hat{\chi}_R(Z,W)\right) = \frac{1}{(n-1)!} \sum_{\sigma\in S_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \mathrm{Tr}_{R_\alpha}\Gamma_R(\sigma) = \chi^{(1)}_{R,R_\alpha}(Z,W).$$

The sum over S_n can be reorganized into a sum over an S_{n-1} subgroup and cosets of this subgroup. Let us digress for a moment to refresh our memory on cosets with a concrete example. Consider the set of permutations comprising S_3 (in what follows we use the cycle notation for permutations):

$$\sigma = \mathbf{1}, (12), (13), (23), (123), (132),$$

and the S_2 subgroup obtained by leaving $n = 3$ inert:

$$\tau = \mathbf{1}, (12).$$

Now, σ can be obtained by composing the elements in τ with the identity $\mathbf{1}$ and the two cycles (n,i) where $i = 1, \ldots, n-1$ (in this case the two cycles (13) and (23)). It would be a good idea to explicitly convince yourself that this is true. The set of permutations thus obtained is said to be a coset of the particular S_2 subgroup we considered in S_3.

Now if f is some function of σ:

$$\sum_{\sigma \in S_n} f(\sigma) = \sum_{\tau \in S_{n-1}} \left(f(\tau) + \sum_{i=1}^{n-1} f((n,i)\tau) \right).$$

Returning to the proof, we reorganize our sum as follows:

$$\begin{aligned} \hat{\chi}_R(Z,W) &= \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} \Big[Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \mathrm{Tr}\,(W) \Gamma_R(\sigma) \\ &+ (WZ)^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((1,n)\sigma) + Z^{i_1}_{i_{\sigma(1)}} (WZ)^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((2,n)\sigma) \\ &+ ... + Z^{i_1}_{i_{\sigma(1)}} \cdots (WZ)^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((n-1,n)\sigma) \Big]. \end{aligned}$$

The S_{n-1} subgroup is the subgroup of S_n comprising of the permutations σ that leave n inert, i.e. $\sigma(n) = n$. We will also need the definition of matrix differentiation

$$\frac{d}{dM^i_j} M^k_l = \delta^j_l \delta^k_i.$$

It is now straight forward to compute the reduction

$$\begin{aligned} D_W \hat{\chi}_R(Z,W) &= \frac{d}{dW^i_i} \hat{\chi}_R(Z,W) \\ &= \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} \Big[N Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R(\sigma) \\ &+ (Z)^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((1,n)\sigma) + Z^{i_1}_{i_{\sigma(1)}} (Z)^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((2,n)\sigma) \\ &+ ... + Z^{i_1}_{i_{\sigma(1)}} \cdots (Z)^{i_{n-1}}_{i_{\sigma(n-1)}} \Gamma_R((n-1,n)\sigma) \Big] \\ &= \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} \left[N + \sum_{i=1}^{n-1} \Gamma_R\left((i,n)\right) \right] \Gamma_R(\sigma) \end{aligned}$$

.

Note that the following Casimirs for S_n and the S_{n-1} subgroup can be defined:

$$O_{S_n}(2) = \sum_{i \neq j} \Gamma_R\left((i,j)\right), \; i,j = 1 \ldots n,$$

$$O_{S_{n-1}}(2)=\sum_{l\neq m}\Gamma_R\left((l,m)\right),\ l,m=1\dots n-1,$$

note that

$$\sum_{i=1}^{n-1}\Gamma_R\left((i,n)\right)=O_{S_n}(2)-O_{S_{n-1}}(2)\equiv O_{S_n/S_{n-1}}(2).$$

Tracing over the subspace of R corresponding to representation R_α we find

$$D_W\chi^{(1)}_{R,R_\alpha}(Z,W)=\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}_{R_\alpha}\left(\left[N+O_{S_n/S_{n-1}}(2)\right]\Gamma_R(\sigma)\right).$$

The Young diagram labeling R_α is obtained from R by removing a single box. Assume that the removed box lies in th a^{th} row and the b^{th} column. If R has r^R_i boxes in the i^{th} row and c^R_j boxes in the j^{th} column, then R_α will have

$$r^{R_\alpha}_i=r^R_i-\delta_{ia}$$

boxes in the i^{th} row and

$$c^{R_\alpha}_j=c^R_j-\delta_{jb}$$

boxes in the j^{th} column. Consequently, when acting on those states of irreducible representation R that span the R_α subspace, we obtain

$$O_{S_n}(2)=\sum_i\frac{r^R_i(r^R_i-1)}{2}-\sum_j\frac{c^R_j(c^R_j-1)}{2},$$

$$O_{S_{n-1}}(2)=\sum_i\frac{r^{R_\alpha}_i(r^{R_\alpha}_i-1)}{2}-\sum_j\frac{c^{R_\alpha}_j(c^{R_\alpha}_j-1)}{2},$$

$$O_{S_n/S_{n-1}}(2)=O_{S_n}(2)-O_{S_{n-1}}(2)=r^R_a-c^R_b.$$

Thus,

$$\begin{aligned}D_W\chi^{(1)}_{R,R_\alpha}(Z,W)&=\left[N+r^R_a-c^R_b\right]\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right)\\&=\left[N+r^R_a-c^R_b\right]\chi_{R_\alpha}(Z).\end{aligned}$$

Note that $\left[N+r^R_a-c^R_b\right]$ is the weight of the box that must be removed from R to obtain R_α. This proves that the reduction $D_W\chi^{(1)}_{R,R_\alpha}(Z,W)$ is computed by removing the box associated with W and multiplying by the weight of the removed box.

3.3 Subgroup Swap Rule

Consider the definition provided previously for a restricted Schur polynomial with k distinguishable matrix words, $W^{(1)}...W^{(k)}$. In order to reduce with respect to the word which does not have the smallest label we need to apply the subgroup swap rule. Thus far, we have assumed that in restricting to the $S_{n-k} \otimes (S_1)^k$ subgroup we have first restricted to the $S_{n-1} \otimes (S_1)$ subgroup that leaves the index of $W^{(1)}$ (i.e. n) inert, then further restricted to the $S_{n-2} \otimes (S_1)^2$ subgroup that leaves the index of $W^{(2)}$ (i.e. n-1) inert and so forth. This is denoted as:

$$\chi_{R,R^{(k)}}|_1|_2 \cdots |_k.$$

If however, we were to first restrict to the subgroup that leaves the index of $W^{(2)}$ inert and then further restrict to the subgroup that leaves the index of $W^{(1)}$ inert, we would in general obtain a different polynomial which is denoted:

$$\chi_{R,R^{(k)}}|_2|_1 \cdots |_k.$$

These two polynomials are related through the subgroup swap rule. In general, the subgroup swap rule relates any two restricted Schur polynomials that differ in the interchange of two adjacent indices indicating the order of restriction. Of course any ordering of the indices can be related to any other ordering by successive applications of the subgroup swap rule.

Returning to the importance of the subgroup swap rule to the reduction rule, note that the reduction rule only yields the particularly simple result of removing the appropriate box of the Young diagram labeling the restricted Schur polynomial and multiplying by the weight of the removed box if we are reducing with respect to the index that appears first in the order of restrictions. If not, we must apply the subgroup swap rule until the appropriate index appears first.

To state the rule we utilize the graphical notation introduced previously (in specifying the restricted character). Note that:

$$\chi_{\begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}|_1|_2 \equiv \chi_{\begin{array}{|c|c|}\hline & \\ \hline & \substack{1\\1} \\ \hline \substack{2\\2} \\ \cline{1-1}\end{array}}|_1|_2 .$$

Now, we denote the weight of the box with the index that is fixed first (before the swap) in the upper left hand corner as c_1^U. We denote the weight of the box with the index that is fixed first (before the swap) in the lower right hand corner as c_1^L. The weight of the box with the index that is fixed second (before the swap) in the upper left hand corner is denoted as c_2^U. The weight of the box with the index that is fixed second (before the swap) in the lower right hand corner is denoted as c_2^U. We define the following swap factors associated with swapping a pair of upper or lower indices:

$$S^U = \frac{1}{c_1^U - c_2^U}, \qquad S^L = \frac{1}{c_1^L - c_2^L}.$$

We define the following no swap factors associated with leaving a pair of upper or lower indices inert:

$$N^U = \sqrt{1 - \frac{1}{(c_1^U - c_2^U)^2}}, \qquad N^L = \sqrt{1 - \frac{1}{(c_1^L - c_2^L)^2}}.$$

The subgroup swap rule states that the result of swapping the order of two adjacent indices indicating the order of restriction is captured by implementing all possible swaps of the upper and lower labels corresponding to those indices and including the appropriate swap or no swap factors. For example:

$$\begin{aligned}
\chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{3} \\ \cline{1-3} \genfrac{}{}{0pt}{}{3}{2} \\ \cline{1-1}\end{array}}|_1|_3|_2 &= N^U N^L \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{3} \\ \cline{1-3} \genfrac{}{}{0pt}{}{3}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + N^U S^L \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \genfrac{}{}{0pt}{}{1}{3} \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{1} \\ \cline{1-3} \genfrac{}{}{0pt}{}{3}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&+ S^U N^L \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \genfrac{}{}{0pt}{}{3}{1} \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{3} \\ \cline{1-3} \genfrac{}{}{0pt}{}{1}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + S^U S^L \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 3 \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{1} \\ \cline{1-3} \genfrac{}{}{0pt}{}{1}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&= \frac{3\sqrt{2}}{5} \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{3} \\ \cline{1-3} \genfrac{}{}{0pt}{}{3}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + \frac{\sqrt{6}}{5} \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \genfrac{}{}{0pt}{}{1}{3} \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{1} \\ \cline{1-3} \genfrac{}{}{0pt}{}{3}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&+ \frac{\sqrt{3}}{10} \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \genfrac{}{}{0pt}{}{3}{1} \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{3} \\ \cline{1-3} \genfrac{}{}{0pt}{}{1}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + \frac{1}{10} \chi_{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 3 \\ \hline \; & \; & \genfrac{}{}{0pt}{}{2}{1} \\ \cline{1-3} \genfrac{}{}{0pt}{}{1}{2} \\ \cline{1-1}\end{array}}|_3|_1|_2 .
\end{aligned}$$

Where:

$$N^U = \sqrt{1 - \frac{1}{(c_1^U - c_3^U)^2}} = \frac{2\sqrt{6}}{5}, \quad N^L = \sqrt{1 - \frac{1}{(c_1^L - c_3^L)^2}} = \frac{\sqrt{3}}{2},$$

$$S^U = \frac{1}{(c_1^U - c_3^U)} = \frac{1}{5}, \quad S^L = \frac{1}{(c_1^L - c_3^L)} = \frac{1}{2}.$$

An example of a reduction operation requiring the application of the subgroup swap rule is:

$$\begin{aligned}
D_{W^{(3)}}\chi_{\begin{ytableau} \, & \, & \, & 1 \\ \, & \, & \substack{2\\3} \\ \substack{3\\2} \end{ytableau}}|_1|_3|_2 &= \frac{3\sqrt{2}}{5} D_{W^{(3)}}\chi_{\begin{ytableau} \, & \, & \, & 1 \\ \, & \, & \substack{2\\3} \\ \substack{3\\2} \end{ytableau}}|_3|_1|_2 + \frac{\sqrt{6}}{5} D_{W^{(3)}}\chi_{\begin{ytableau} \, & \, & \, & \substack{1\\3} \\ \, & \, & \substack{2\\1} \\ \substack{3\\2} \end{ytableau}}|_3|_1|_2 \\
&+ \frac{\sqrt{3}}{10} D_{W^{(3)}}\chi_{\begin{ytableau} \, & \, & \, & \substack{3\\1} \\ \, & \, & \substack{2\\3} \\ \substack{1\\2} \end{ytableau}}|_3|_1|_2 + \frac{1}{10} D_{W^{(3)}}\chi_{\begin{ytableau} \, & \, & \, & 3 \\ \, & \, & \substack{2\\1} \\ \substack{1\\2} \end{ytableau}}|_3|_1|_2 \\
&= 0 + 0 + 0 + \frac{1}{10}(N+3)\chi_{\begin{ytableau} \, & \, & \, \\ \, & \, & \substack{2\\1} \\ \substack{1\\2} \end{ytableau}}|_1|_2 \\
&= \frac{1}{10}(N+3)\chi_{\begin{ytableau} \, & \, & \, \\ \, & \, & \substack{2\\1} \\ \substack{1\\2} \end{ytableau}}|_1|_2 .
\end{aligned}$$

4 Restricted Schur Polynomials: Indistinguishable Matrix Words

We now consider the case where the matrix words comprising the restricted Schur polynomial are indistinguishable. For example we could have a polynomial constructed from two matrices, Z and X say. The definition of the restricted Schur polynomial in this case is:

$$\chi_{R,R_\alpha}(Z,X) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))\mathrm{Tr}(\sigma Z^{\otimes n} \otimes X^{\otimes m}),$$

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes X^{\otimes m}) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \dots Z^{i_n}_{i\sigma(n)} X^{i_{n+1}}_{i\sigma(n+1)} \dots X^{i_{n+m}}_{i\sigma(n+m)}.$$

In this definition, R is an irreducible representation of S_{n+m} and is associated with a Young diagram of $n+m$ boxes. R_α is an irreducible representation of $S_n \times S_m$ and is therefore specified by two Young diagrams, one comprised of n boxes and the other of m boxes ($R_\alpha \equiv (r_{\alpha 1}, r_{\alpha 2})$). The $S_n \times S_m$ subgroup is the subgroup for which S_n acts on the n indices of the Z's and S_m acts on the m indices of the X's. Again, $\mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))$ indicates taking a restricted trace of the group element $\sigma \in S_{n+m}$ in the irreducible representation R. Under restricting to the $S_n \times S_m$ subgroup, R will in general be reducible. We can decompose the carrier space of irreducible representation R according to the irreducible $S_n \times S_m$ representations that are subduced. The restricted trace corresponds to only tracing over the subspace corresponding to R_α. The restricted trace can be calculated by constructing projectors in a manner analogous to the previous algorithm, or via other techniques not presented here. Note that this definition is easily generalized to more than two types of indistinguishable matrices comprising the restricted Schur polynomial.

4.1 Examples

The construction of

$$\chi_{\square\square;\square\otimes\square} = \mathrm{Tr}\,(Z)\mathrm{Tr}\,(X) + \mathrm{Tr}\,(ZX), \qquad \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix};\square\otimes\square} = \mathrm{Tr}\,(Z)\mathrm{Tr}\,(X) - \mathrm{Tr}\,(ZX),$$

is particularly simple because we do not need a projector to implement the restricted trace. This follows because $\square \otimes \square$ is the only $S_1 \times S_1$ irreducible representation subduced from either $\square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$.

Consider next

$$\chi_{\yng(3);\yng(2)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(Z)^2\mathrm{Tr}\,(X) + \mathrm{Tr}\,(Z^2)\mathrm{Tr}\,(X) + 2\mathrm{Tr}\,(ZX)\mathrm{Tr}\,(Z) + 2\mathrm{Tr}\,(Z^2X)\right],$$

$$\chi_{\yng(1,1,1);\yng(1,1)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(Z)^2\mathrm{Tr}\,(X) - \mathrm{Tr}\,(Z^2)\mathrm{Tr}\,(X) - 2\mathrm{Tr}\,(ZX)\mathrm{Tr}\,(Z) + 2\mathrm{Tr}\,(Z^2X)\right].$$

For these two restricted Schur polynomials we again do not need a projector to implement the restricted trace.

5 The Physics of Schur Polynomials and Restricted Schur Polynomials

5.1 Role in the AdS/CFT correspondence

The AdS/CFT correspondence is a duality between a theory that incorporates gravity in its description and a theory that does not incorporate gravity. It is a concrete realization of the holographic principle which posits that the physical description of a volume of space is encoded on the boundary of the region. As such, the dimensionality of the theory on the boundary is one dimension less than the theory on the interior of the space. The AdS/CFT correspondence is a conjecture that a string theory (a theory of quantum gravity) defined on a d+1 dimensional Anti-de Sitter space (a space with constant negative scalar curvature) is dual to a quantum field theory (which does not incorporate gravity) defined on the d-dimensional boundary of this space. The word "dual" here is a statement about the full dynamical equivalence between the two theories. In other words, the theories are different languages in which to describe the same dynamics. Some processes are easier to describe and solve in the string theory, others in the quantum field theory. In fact the particular usefulness of the correspondence is that in general (aside from some special cases which are useful for testing the correspondence) calculations in the string theory are easy when the calculations in the field theory are hard and vice versa. We thus need a dictionary allowing us to translate from the one language to the other when desired.

Schur polynomials and restricted Schur polynomials are part of this dictionary. The Schur polynomials and restricted Schur polynomials are gauge invariant field theory operators. Gauge invariant operators are those operators which have a physical interpretation in the quantum field theory. The matrices from which they are built are complex linear combinations of scalar fields present in the quantum field theory. The Schur polynomials are dual to different objects in the string theory depending on how many matrix fields comprise the polynomial. If they are comprised of $O(N)$ matrix fields (here N is a parameter of the quantum field theory matching the rank of the matrix fields) then the Schur polynomials are dual to what are termed giant gravitons (extended higher dimensional membranes). If they are comprised of $O(N^2)$ matrix fields then they are dual to new background geometries on the string theory side of the correspondence. Restricted Schur polynomials with $O(N)$ of one type of field are dual to excited giant gravitons (giant gravitons with open strings attached).

5.2 A Short Digression on Field Theory Correlators

The correlation functions of operators in quantum field theory fully capture the dynamics of the theory. Correlation functions (correlators) can be utilized to learn about a number of different aspects of the quantum field theory. For one thing, a suitably normalized correlator of a given product of operators gives the interaction strength / amplitude for those operators. In the context of the AdS/CFT correspondence studying the gauge theory correlators of operators identified with certain degrees of freedom in the string theory gives us information about the string theory dynamics.

The correlators that we will study are those of the quantum field theory known as $\mathcal{N}=4$ Super Yang-Mills theory (SYM), a four dimensional supersymmetric quantum field theory with gauge group $U(N)$. It is sufficient for our purposes to note that in this theory, there are six real (space-time) scalar fields transforming in the adjoint of $U(N)$ and as such can be regarded as $N \times N$ matrices. We combine these six real scalar fields into the following complex matrix fields:

$$Z = \Phi_1 + i\Phi_2,\ Y = \Phi_3 + i\Phi_4,\ X = \Phi_5 + i\Phi_6$$

These matrix fields constitute the matrices from which we build the Schur and restricted Schur polynomials. We define the contractions (two point functions) of these fields as follows:

$$\left\langle Z_{ij} Z^{\dagger}_{kl} \right\rangle = \delta_{il}\delta_{jk},\ \left\langle Y_{ij} Y^{\dagger}_{kl} \right\rangle = \delta_{il}\delta_{jk},\ \left\langle X_{ij} X^{\dagger}_{kl} \right\rangle = \delta_{il}\delta_{jk}. \tag{3}$$

Note that we have dropped spacetime dependence of these two point functions which plays no role in our discussions. Correlators of operators built from these fields are obtained by contracting all the fields present in all possible ways. As a very simple example:

$$\begin{aligned}
\langle \chi_{\square}(Z)\, \chi^{\dagger}_{\square}(Z) \rangle &= \langle \mathrm{Tr}\, Z\, \mathrm{Tr}\, Z^{\dagger} \rangle \\
&= \langle Z_{ii} Z^{\dagger}_{jj} \rangle \\
&= \delta_{ij}\delta_{ij} \\
&= \delta_{ii} \\
&= N
\end{aligned}$$

5.3 Two Point Functions

We will now state, without proof, the two point functions of Schur and restricted Schur polynomials.

5.3.1 Schur Polynomials

The exact two point correlation function of Schur polynomials is as follows:

$$\langle \chi_R\,(Z)\,\chi_S^{\dagger}\,(Z)\rangle = \delta_{RS} f_R.$$

The correlator is diagonal in the young diagram labels and is only non-zero when $R = S$. In this result, the quantity f_R denotes the product of weights of the boxes of the Young diagram labeling the Schur polynomial (these are not Dynkin weights). The weight of a box in the ith row and jth column is given by $N-i+j$ and thus the product of weights of all the boxes comprising the young diagram is:

$$f_R = \prod_{i,j}(N-i+j).$$

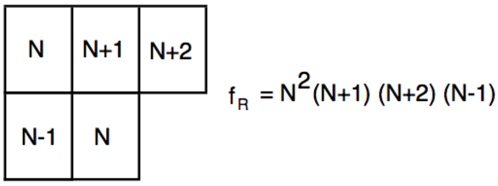

Figure 1: Example of the product of weights of the boxes of a Young diagram.

An example of the application of this result is:

$$\langle \chi_{\tiny\yng(3,1)}\,(Z)\,\chi^{\dagger}_{\tiny\yng(3,1)}\,(Z)\rangle = N(N^2-1)(N+2).$$

5.3.2 Restricted Schur polynomials: Distinguishable Matrix Words

We will now present the result of evaluating the two point correlators of restricted Schur polynomials with distinguishable matrix words. From this point forward we will specialize to the case where the matrix words are identified with open string excitations of giant gravitons on the string theory side of the AdS/CFT correspondence. To be identified with open string excitations, the matrix words must be comprised of $O(\sqrt{N})$ letters each of which could in principle be any of the fields in $\mathcal{N} = 4$ SYM or covariant derivatives of these fields (we restrict ourselves to the complex scalar fields above though i.e. Z,X,Y). Henceforth we refer to these matrix words as open string words. An example of an open string word comprised of two of the three complex scalar fields defined in the previous section is:

$$W^i_j = (YZZZY)^i_j$$

In the definition (1) of the restricted Schur polynomial presented previously, the identification with an excited giant graviton (a giant graviton with open strings attached) is only sensible if n is O(N) and the number of open string words, k is O(1).

The correlator result presented below is valid as long as the number of Z's comprising the restricted Schur polynomial operator is less than $O(N^2)$ and the number of Z's in any open string word is $O(1)$. In this regime contractions between the open string word and the rest of the operator can be neglected. In the case where the number of Z's comprising the restricted Schur polynomial operator is $O(N^2)$, these contractions can no longer be neglected. Indeed, the interpretation of the restricted Schur polynomial operators changes and they are now identified with new classical backgrounds with string excitations on the string theory side of the AdS/CFT correspondence.

For each open string word the most general form that the two point function can take is:

$$\langle (W^a)^i_j (W^{a\dagger})^k_l \rangle = F^a_0 \delta^i_l \delta^k_j + F^a_1 \delta^i_j \delta^k_l. \tag{4}$$

Where F^a_0 and F^a_1 are dependent on the precise composition of the open string word. In evaluating the result of correlators involving multiple open string words, contractions which mix four or more words (corresponding to string interactions) are dropped and only pairwise contractions are considered. Moreover, it is assumed that:

$$\langle (W^a)^i_j (W^{b\dagger})^k_l \rangle \propto \delta^{ab}$$

which will mean that only a single pairing contributes.

Now, given the form of the open string two point function (4), it is easy to see that upon contracting a particular open string word, the correlator can be split into a sum of terms, one with a coefficient of F^a_0 and the other with a coefficient of F_1^a. The first of these contributions corresponds to implementing the delta function index structure associated with F_0^a in the open string two point function. The latter contribution corresponds to implementing the delta function index structure associated with F_1^a in the open string two point function. We will now discuss an algorithm for keeping track of and ultimately evaluating the resultant contributions of both types for one or more open string words. The F_0^a contribution is said to correspond to the case where the open string word W^a has been 'glued'. The F_1^a contribution turns out to correspond to taking a reduction with respect to the open string word W^a. For restricted Schur correlators involving multiple open string words, the F_0 and F_1 type contributions from each open string must be evaluated, in what is essentially a recursive procedure that is applied until we are left with correlators in which all strings are glued or no strings remain after successive reductions. In the latter case, the two point correlator result for ordinary Schur polynomials can be applied. In the case of correlators involving one or more glued strings, the following result is applied:

For restricted Schur polynomials involving restricted traces where we trace over the row and column indices of a particular subspace:

$$\langle \chi_{R,R^{(n)}} \chi^{\dagger}_{S,S^{(n)}} \rangle|_{\text{glued}} = \frac{(\text{Hooks})_R}{(\text{Hooks})_{R^{(n)}}} f_R \delta_{R\to R^{(n)},S\to S^{(n)}}.$$

For restricted Schur polynomials involving restricted traces where we trace over row and column indices of different subspaces:

$$\langle \chi_{R,R^{(n)}T^{(n)}} \chi^{\dagger}_{S,S^{(n)}U^{(n)}} \rangle|_{\text{glued}} = \frac{(\text{Hooks})_R}{(\text{Hooks})_{R^{(n)}}} f_R \delta_{R\to(R^{(n)}T^{(n)}),S\to(S^{(n)}U^{(n)})}.$$

These results are applicable for n glued strings. The delta function $\delta_{R\to R^{(n)},S\to S^{(n)}}$ indicates that, in addition to R matching S and $R^{(n)}$ matching $S^{(n)}$, all the irreducible representations appearing in intermediate stages of subducing $R^{(n)}$ from R must precisely match the irreducible representations appearing in intermediate stages of subducing $S^{(n)}$ from S. Similarly for $\delta_{R\to(R^{(n)}T^{(n)}),S\to(S^{(n)}U^{(n)})}$.

For example, bringing it all together (note that if an open string is glued it is denoted graphically by an arrow above the associated open string label in the Young diagram) :

$$
\begin{aligned}
\langle \chi_{\young(~1,~,2)} \chi^{\dagger}_{\young(~1,~,2)} \rangle &= F_0^1 \langle \chi_{\young(~\vec{1},~,2)} \chi^{\dagger}_{\young(~\vec{1},~,2)} \rangle + F_1^1 \langle D_{W^{(1)}} \chi_{\young(~1,~,2)} (D_{W^{(1)}} \chi_{\young(~1,~,2)})^{\dagger} \rangle \\
&= F_0^1 F_0^2 \langle \chi_{\young(~\vec{1},~,\vec{2})} \chi^{\dagger}_{\young(~\vec{1},~,\vec{2})} \rangle + F_0^1 F_1^2 \langle D_{W^{(2)}} \chi_{\young(~\vec{1},~,2)} (D_{W^{(2)}} \chi_{\young(~\vec{1},~,2)})^{\dagger} \rangle \\
&+ F_1^1 F_0^2 \langle D_{W^{(1)}} \chi_{\young(~1,~,\vec{2})} (D_{W^{(1)}} \chi_{\young(~1,~,\vec{2})})^{\dagger} \rangle \\
&+ F_1^1 F_1^2 \langle D_{W^{(2)}} D_{W^{(1)}} \chi_{\young(~1,~,2)} (D_{W^{(2)}} D_{W^{(1)}} \chi_{\young(~1,~,2)})^{\dagger} \rangle \\
&= F_0^1 F_0^2 (4N(N^2-1)(N-2)) \\
&+ F_0^1 F_1^2 (\frac{4}{3} N(N-2)^2(N^2-1) + \frac{1}{3} N(N^2-1)(N+1)(N-2)) \\
&+ F_1^1 F_0^2 (3N(N^2-1)(N+1)(N-2)) \\
&+ F_1^1 F_1^2 (N(N^2-1)(N-2)).
\end{aligned}
$$

5.3.3 Restricted Schur polynomials: Indistinguishable Matrix Words

The two point correlation function of restricted Schur polynomials with indistinguishable matrix words is as follows:

$$
\begin{aligned}
\langle \chi_{R,(r_{\alpha_1}, r_{\alpha_2})} \chi^{\dagger}_{S,(s_{\beta_1}, s_{\beta_2})} \rangle &= \delta_{RS} \delta_{r_{\alpha_1} s_{\beta_1}} \delta_{r_{\alpha_2} s_{\beta_2}} \frac{(\text{hooks})_R}{(\text{hooks})_{R_\alpha}} f_R \\
&= \delta_{RS} \delta_{r_{\alpha_1} s_{\beta_1}} \delta_{r_{\alpha_2} s_{\beta_2}} \frac{(\text{hooks})_R}{(\text{hooks})_{r_{\alpha_1}} (\text{hooks})_{r_{\alpha_2}}} f_R \,.
\end{aligned}
$$

For example:

$$
\langle \chi_{\yng(2,1,1);\yng(2)\otimes\yng(1,1)} (Z,X)\, \chi^{\dagger}_{\yng(2,1,1);\yng(2)\otimes\yng(1,1)} (Z,X) \rangle = 2N(N^2-1)(N-2)
$$

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