Free particles from Brauer algebras in complex matrix models

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QMUL

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Based on arXiv:0911.4408
with Yusuke Kimura and Sanjaye Ramgoolam

Motivations

1. Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills: holomorphic, $U(N)$ singlet sector of a free $N \times N$ complex matrix model.
2. Description in terms of N free fermions - eigenvalues

Corley, Jevicki, Ramgoolam

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5. Free particle descriptions in other sectors? Non-holomorphic sectors?

Overview

1. Review of free particles in matrix models and AdS/CFT
2. Introduction to Brauer algebra basis
3. Emergence of free particles in complex matrix models
4. Counting and stringy exclusion principle
5. Open Questions

Free particles in unitary matrix quantum mechanics

Consider the free Unitary matrix quantum mechanics with Hamiltonian

$$H = \mathrm{tr}\left(U\frac{\partial}{\partial U}\right)^2$$

- $U(N)$ symmetry $U \to gUg^\dagger, \qquad g \in U(N)$.
- An orthonormal basis of $U(N)$ invariant wavefunctions is given by $U(N)$ characters.

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$U(N)$ representations built from tensor products of the fundamental are specified by a Young diagram R with $c_1(R) \leq N$ and their characters are the Schur polynomials:

$$\chi_R(U) = \sum_{\sigma \in S_n} \chi_R(\sigma) U^{i_1}{}_{i_{\sigma_1}} \cdots U^{i_n}{}_{i_{\sigma n}} \ .$$

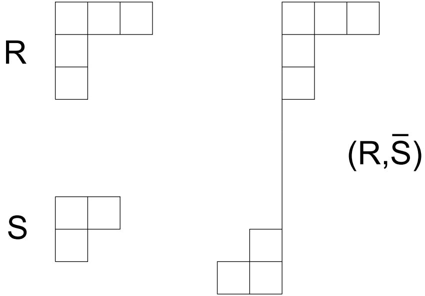
The same is true for tensor copies of the antifundamental with $U \leftrightarrow U^\dagger$.

Free particles in unitary matrix quantum mechanics

More general representations are specified by a *composite Young diagram* $(R, \bar{S})$, where

- R controls the fundamental indices
- S controls the antifundamental indices

For $U(N)$ (and everywhere in this talk) a composite Young diagram has N rows so we require $c_1(R) + c_1(S) \leq N$:



Free particles in unitary matrix quantum mechanics

Use symmetry $U \to gUg^\dagger$ to diagonalise U:

$$U = gDg^\dagger, \qquad D = diag(e^{i\theta_1}, \ldots, e^{i\theta_N}), \qquad g \in U(N) \ .$$

This introduces jacobian $\Delta(u) = \prod_{i<j}(e^{i\theta_i} - e^{i\theta_j})$.

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This introduces jacobian $\Delta(u) = \prod_{i<j}(e^{i\theta_i} - e^{i\theta_j})$.
The Hamiltonian becomes

$$H \ = \ -\sum_i \left[\frac{1}{\tilde{\Delta}}\frac{d^2}{d\theta_i^2}\tilde{\Delta}\right] - \frac{1}{12}N(N^2-1) \ + \ \text{off-diag} \ ,$$

where

$$\tilde{\Delta} \ = \ \prod_{i<j} \sin\frac{\theta_i - \theta_j}{2} \ = \ \frac{\Delta(u)}{(\det U)^{\frac{N-1}{2}}} \ .$$

Douglas '93

Free particles in unitary matrix quantum mechanics

Absorb $\tilde{\Delta}$ into wavefunctions and Hamiltonian:

$$\psi_f = \tilde{\Delta}\psi \ , \qquad H_f \ = \ \tilde{\Delta} H \frac{1}{\tilde{\Delta}} \ = \ \sum_i \frac{\partial}{\partial \theta_i^2} - \frac{1}{12} N(N^2 - 1)$$

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Singlet eigenfunctions are Slater determinants - N fermion wavefunctions,

$$\psi_{\vec{p}} \ = \ \det_{j,k} e^{i\theta_j p_k}$$

which are related to Schur polynomials via

$$\Psi^f_{\vec{p}} \ = \ \Delta(u)\chi_R(U)$$

where if r_j are the rows of the Young diagram R,

$$p_j = r_j + (n_F + 1 - j) \ , \qquad n_F = \frac{N-1}{2} \ .$$

- This sector is thus equivalent to N free fermions on a circle.

Douglas '93

Fermions on a circle

- States of fermion on a circle: quantised momentum $p \in \mathbb{Z}$
- Energy $E = p^2$
- N fermions: Fermi sea with two Fermi levels.

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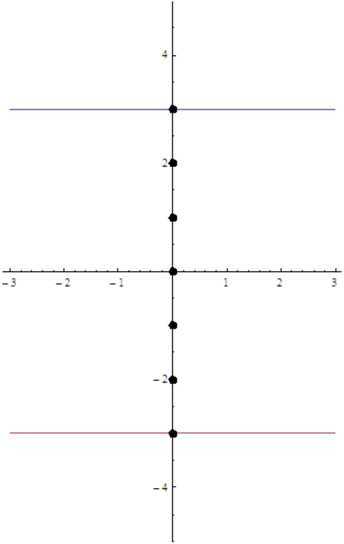
- Excitations labelled by composite Young diagram $(R, \bar{S})$: length of row j is excitation energy of fermion j
- Natural interpretation of $c_1(R) + c_1(S) \leq N$.



Free particles in hermitian matrix models

Consider the Gaussian hermitian matrix quantum mechanics with Lagrangian

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\Phi^2\right)$$

which is invariant under the global $U(N)$ action

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$$\Phi \to g\Phi g^\dagger \ , \qquad\qquad g \in U(N) \ .$$

Decompose Φ into diagonal and off-diagonal d.o.f.:

$$\Phi = U\Lambda U^\dagger, \qquad \Lambda = diag(\lambda_1, \ldots, \lambda_N) \ , \qquad U \in U(N) \ .$$

The jacobian is $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$ and the Hamiltonian becomes

$$H_\Lambda = \frac{1}{2}\sum_i \left(-\frac{1}{\Delta}\frac{\partial^2}{\partial\lambda_i^2}\Delta + \lambda_i^2\right) \ + \ \text{off-diag} \ ,$$

Free particles in hermitian matrix models

Absorb Δ into wavefunctions and Hamiltonian:

$$
\begin{aligned}
\Psi^f(\lambda) &= \Delta\Psi(\lambda) \\
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Singlet eigenfunctions are Slater determinants - N fermion wavefunctions,

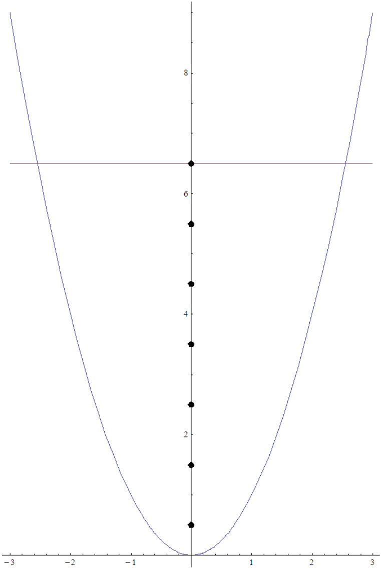
$$\Psi^f_{\vec{\mathcal{E}}} = \det_{i,j} \lambda_i^{\mathcal{E}_j}\, e^{-\frac{1}{2}\operatorname{tr}\Phi^2}$$

which are related to Schur polynomials as in the UMM via

$$\Psi^f_{\vec{\mathcal{E}}} = \Delta\mathcal{O}_R(\Phi)e^{-\frac{1}{2}\operatorname{tr}\Phi^2}\ , \qquad \mathcal{E}_i = r_i + (N-i)$$

where r_i are the rows of the Young diagram R.

Fermions in 1D SHO



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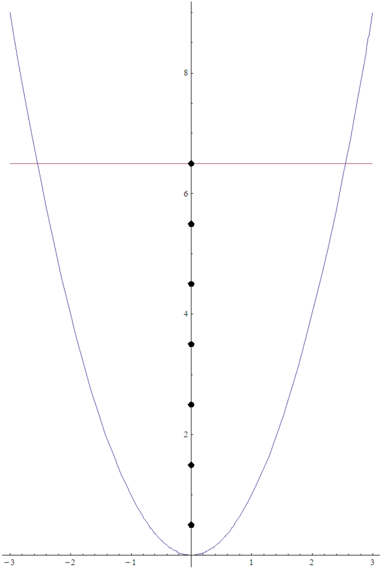

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- Natural interpretation of $c_1(R) \leq N$

Free particles and AdS/CFT

$\mathcal{N}=4$ SYM Gauge group $U(N)$ 't Hooft coupling $\lambda = g_{YM}^2 N$	$\longleftrightarrow$	IIB String Theory on $AdS_5 \times S^5$ Radius L, F_5 flux N

Parameters:

$$\sqrt{\lambda} \longleftrightarrow \frac{L^2}{\alpha'}$$

$$\frac{\lambda}{N} \longleftrightarrow g_s$$

- Strong form of conjecture: equivalence for all λ, N.
- This talk: $\lambda = 0$, N finite.

Field Content of $\mathcal{N} = 4$ SYM

- Gauge field, 4 Weyl fermions
- 3 Complex scalars X, Y, Z
- All fields in adjoint of $U(N)$.

Restrict attention to one complex scalar - say Z.

Holomorphic polynomials in Z are $\frac{1}{2}$-BPS operators: they preserve half of the supersymmetries.

- Relevant part of the Lagrangian is:

$$\mathcal{L}_Z = \mathrm{tr}\left(D_\mu Z^\dagger D^\mu Z\right)$$

Spherical Harmonics and Dimensional Reduction

Consider $\mathcal{N} = 4$ SYM on $S^3 \times \mathbb{R}$:

- $\frac{1}{2}$-BPS states correspond to s-wave modes.
- Fields $Z(t)$, $A_0(t)$.
- A_0 non-dynamical - imposes Gauss's Law (gauge invariance).

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Dimensionally reduced Lagrangian: extra term from conformal coupling to curvature of S_3 - absorbing constants, this becomes

$$\mathcal{L} = \mathrm{tr}\left(\dot{Z}\dot{Z}^\dagger - ZZ^\dagger\right)$$

$\rightarrow U(N)$ singlet sector of complex matrix quantum mechanics in a simple harmonic oscillator potential.

Hashimoto '00, Corley, Jevicki, Ramgoolam '01

Schur polynomials (yet again)

The Schur polynomials generalised to a complex matrix, $\mathcal{O}_R(\Phi)$ are polynomials of degree n labelled by a representation R of S_n, where the first column of R has length at most N:

$$\mathcal{O}_R(\Phi) = \sum_{\sigma \in S_n} \chi_R(\sigma) \Phi^{i_1}{}_{i_{\sigma_1}} \cdots \Phi^{i_n}{}_{i_{\sigma n}}$$

In this basis the two-point function is diagonal:

$$\left\langle \mathcal{O}_R(\Phi)^\dagger \mathcal{O}_S(\Phi) \right\rangle = f_R \delta_{RS}$$

Corley, Jevicki, Ramgoolam '01

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Corley, Jevicki, Ramgoolam '01

At $n = 2$ the Schur polynomials are

$$\begin{aligned} \mathcal{O}_{[2]}(\Phi) &= \frac{1}{2}\left(\operatorname{tr}\Phi \operatorname{tr}\Phi + \operatorname{tr}\Phi^2\right) \\ \mathcal{O}_{[1^2]}(\Phi) &= \frac{1}{2}\left(\operatorname{tr}\Phi \operatorname{tr}\Phi - \operatorname{tr}\Phi^2\right) \end{aligned}$$

where $[2] = \square\square$ (symmetric), $[1^2] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (antisymmetric).

Schur Triangularisation

$U(N)$ not sufficient to diagonalise Z; use Schur Triangularisation:

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- z_i: diagonal entries - eigenvalues of Z
- t_{jk}: off-diagonal entries for $j < k$.

Since

$$\mathrm{tr}\, Z^p \ = \ \mathrm{tr}\, T^p \ = \sum_i {z_i}^p$$

the holomorphic GIOs are symmetric polynomials in the z_i, related to Schur polynomials as for the Hermitian matrix model.

Change of variables

Since the holomorphic GIOs are symmetric polynomials in the z_i, change variables

$$Z_{ij} \quad \longrightarrow \quad \{z_i, t_{jk}, U\}$$

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- Absorb Δ into wavefunctions - interpret the z_i as fermions.

- Fermions z_i are complex - target space is a plane.
- Holomorphic dynamics is effectively one - dimensional.

Thus the holomorphic, $U(N)$ singlet sector of the matrix SHO quantum mechanics is equivalent to a theory of N fermions in a 1D SHO potential.

Fermion phase space and LLM

- SHO Fermions on $\mathbb{R}$ have a 2D phase space plane
- Quantize : each fermion occupies area $\hbar$
- System occupies area $N\hbar$ of phase space

So phase space configurations are colourings of the plane into black/white regions.

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$\frac{1}{2}$-BPS solutions to IIB supergravity with $SO(4) \times SO(4)$ isometry: (LLM)

- Coordinates $t, y, x_1, x_2, S^3, \tilde{S}^3$
- Geometries determined by function $u(x_1, x_2)$
- Smoothness condition : $u(x_1, x_2) = 0 \text{ or } 1$
- $x_1 - x_2$ plane identified with fermion phase space above.

Lin, Lunin, Maldacena '04

Beyond the Holomorphic Sector

- So far: $U(N)$ singlet, holomorphic sector of Complex Matrix Model
- Natural extension: relax holomorphic constraint
- GIOs now functions of Z, $Z^\dagger$, equivalently z_i, t_{jk} - this takes us beyond eigenvalues.

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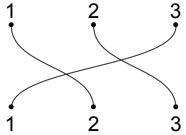
New features:

1. This sector is non-BPS - no non-renormalisation theorems.
2. However at zero coupling, Z, $Z^\dagger$ sector remains a consistent truncation of $\mathcal{N} = 4$ SYM
3. Is there a string dual of this sector at zero coupling?

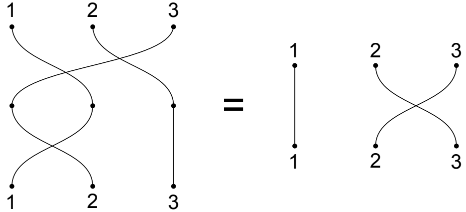
The symmetric group S_n in diagrams

Symmetric group elements may be represented by diagrams:



$$= \quad (123)$$

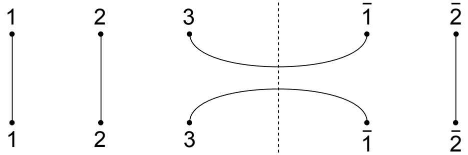
Products are obtained by stacking diagrams: e.g.



represents the product $(12)(123) = (23)$.

The Brauer algebra

The (walled) Brauer algebra $B_N(m, n)$ contains the group algebra of $S_m \times S_n$ along with 'contraction' elements, which cross a wall:



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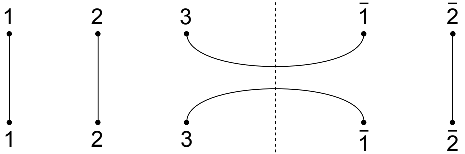
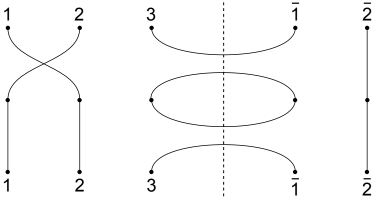
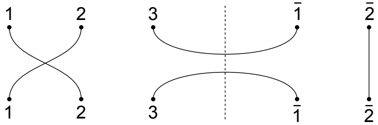


$= \quad C_{3\bar{1}}$

along with the rule that in a product, a closed loop is replaced by multiplication by the parameter N:



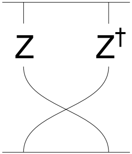
$= \quad N$



which represents the product $C_{3\bar{1}}\left[(12)C_{3\bar{1}}\right] = N\,(12)\,C_{3\bar{1}}$.

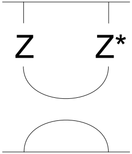
Application to GIOs

The index structure of $\mathrm{tr}\, ZZ^\dagger$ can be represented diagrammatically using Z and $Z^\dagger$, using a symmetric group element and a trace:



$$= Z^i{}_j Z^{\dagger j}{}_i = \mathrm{tr}\, ZZ^\dagger$$

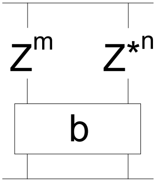
Alternatively, $\mathrm{tr}\, ZZ^\dagger$ can be represented with Z and Z^*, using a Brauer algebra contraction and a trace:



$$= Z^i{}_j Z^*{}_j{}^i = Z^i{}_j Z^{\dagger j}{}_i = \mathrm{tr}\, ZZ^\dagger$$

Application to GIOs

More generally, any GIO may be written using Z, Z^* and a Brauer algebra element b as



$$= \quad \mathrm{tr}\left(b\,\mathbf{Z}^m\mathbf{Z}^{*n}\right)$$

Brauer basis of operators

The Brauer algebra can be used to build an orthogonal basis, as follows:

- The representations of the Brauer algebra are labelled by $\gamma = (k, \gamma_+, \gamma_-)$ where
  - k is an integer in the range $0 \leq k \leq \min(m, n)$
  - (γ_+, γ_-) have $m - k$ and $n - k$ boxes respectively and form a form a composite Young diagram; $c_1(\gamma_+) + c_1(\gamma_-) \leq N$

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- A representation γ can be decomposed into irreps $A = (\alpha, \beta)$ of the $\mathbb{C}[S_m \times S_n]$ sub-algebra, where $\alpha \vdash m, \beta \vdash n$.

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- A representation γ can be decomposed into irreps $A = (\alpha, \beta)$ of the $\mathbb{C}[S_m \times S_n]$ sub-algebra, where $\alpha \vdash m, \beta \vdash n$.
- Let the irrep A appear with multiplicity M_A^γ, let i run over this multiplicity and let

  $$|\gamma; A, m_A; i\rangle$$

  be the state in the representation γ which transforms in the i th copy of the state m_A of the irrep A of the sub-algebra.

Brauer basis of operators

The Brauer basis, formed of particular linear combinations of such traces, is a generalisation of the Schur Polynomials to Z, $Z^\dagger$ operators,

$$\mathcal{O}^{\gamma}_{\alpha\beta;ij}(Z, Z^\dagger) = \mathrm{tr}\left(Q^{\gamma}_{\alpha\beta;ij}\mathbf{Z}^m\mathbf{Z}^{*n}\right)$$

where

$$Q^{\gamma}_{\alpha\beta;ij} = |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; j| \ .$$

This basis diagonalises the two-point function.

Kimura, Ramgoolam '07

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Kimura, Ramgoolam '07

For example, when $m = 1,\ n = 1$, suppressing non-essential labels:

$$\begin{aligned}
\mathcal{O}^{k=0}_{[1],[\bar{1}]} &= \mathrm{tr}\, Z \,\mathrm{tr}\, Z^\dagger - \frac{1}{N}\mathrm{tr}(ZZ^\dagger) \\
\mathcal{O}^{k=1}_{[1],[\bar{1}]} &= \frac{1}{N}\mathrm{tr}(ZZ^\dagger) \ .
\end{aligned}$$

- Note that the coefficients depend on N.

The $k = 0$ sector

The $k = 0$ operators are special:

- They do not require point-splitting regularisation
- In the $k = 0$ sector $\gamma = (0, \alpha, \beta)$ so operators are labelled simply by α and β.
- To connect with the notation of the unitary matrix model, we write $\alpha = R$ and $\beta = S$.

The $k = 0$ sector

If $S = \emptyset$, then the $k = 0$ operator is a holomorphic Schur polynomial:

$$\mathcal{O}^{k=0}_{R,\emptyset}(Z, Z^\dagger) = \chi_R(Z) \ .$$

If $R = \emptyset$, then the $k = 0$ operator is an anti-holomorphic Schur polynomial:

$$\mathcal{O}^{k=0}_{\emptyset,\bar{S}}(Z, Z^\dagger) = \chi_S(Z^\dagger)$$

The $k = 0$ sector

If $S = \emptyset$, then the $k = 0$ operator is a holomorphic Schur polynomial:

$$\mathcal{O}^{k=0}_{R,\emptyset}(Z, Z^\dagger) = \chi_R(Z) \ .$$

If $R = \emptyset$, then the $k = 0$ operator is an anti-holomorphic Schur polynomial:

$$\mathcal{O}^{k=0}_{\emptyset,\bar{S}}(Z, Z^\dagger) = \chi_S(Z^\dagger)$$

If both α and β are nontrivial, the leading order term in the expansion of $\mathcal{O}^{k=0}$ begins with the product of the holomorphic and antiholomorphic Schur polynomials:

$$\mathcal{O}^{k=0}_{R,\bar{S}}(Z, Z^\dagger) = \chi_R(Z)\chi_S(Z^\dagger) + \cdots \ ,$$

where the dots denote terms with at least one $ZZ^\dagger$ inside a trace.

The $k = 0$ sector

The $k = 0$ operators are the generalisations of the characters of the composite representations of unitary matrix model to a complex matrix - if we replace Z by a unitary matrix, we obtain:

$$\mathcal{O}^{k=0}_{R\bar{S}}(U, U^\dagger) = d_R d_S \chi_{R\bar{S}}(U) \ .$$

This gives an isomorphism between the $k = 0$ sector and the states of the Unitary matrix model.

- Motivation to look for free fermions on a circle in $k = 0$ sector.

Free particles from Brauer Algebra

Strategy:

- Seek free particle physics in the Brauer basis at particular values of k

Results:

- Free particle descriptions in two sectors: $k = 0$ and $k = m = n$
- $k = 0$ sector: Explicit expressions at $N = 2$ for momenta of free fermions on a circle in terms of combinations of z_i, t_{jk}, implicit generalisation to arbitrary N
- $k = m = n$ sector: map to free fermions in harmonic oscillator of hermitian matrix model for arbitrary N.

see also Masuku & Rodrigues, 0911.2846

Schur Triangularisation revisited

Let us examine more closely the Schur Decomposition,

$$Z = UTU^\dagger$$

where $t_{ii} = z_i$ and $t_{jk} = 0$ for $j < k$.
Residual symmetries:

- S_N permutes eigenvalues z_i (& transforms t_{jk})
- $U(1)^{N-1}$ acts on phases of the t_{jk}.

The parameter space of inequivalent adjoint $U(N)$ orbits, $\mathcal{M}_N$ can be obtained by fixing an ordering of z_i and setting $t_{j,j+1} \in \mathbb{R}$.

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At $N = 2$ setting $t_0 \in \mathbb{R}$ we have

$$T = \begin{pmatrix} z_1 & t_0 \\ 0 & z_2 \end{pmatrix} .$$

Differential Gauss's law

Recall the relevant part of the $\mathcal{N} = 4$ SYM Lagrangian

$$\mathcal{L}_Z = \mathrm{tr}\left(D_\mu Z^\dagger D^\mu Z \right) \ .$$

A convenient gauge choice is to set $A_0 = 0$. The e.o.m. for A_0 leads to Gauss's Law:

$$Z^\dagger \dot{Z} + Z \dot{Z}^\dagger - \dot{Z} Z^\dagger - \dot{Z}^\dagger Z = 0 \ .$$

Upon canonical quantization this leads to the differential form of Gauss's Law,

$$G = G_1 + G_2 + G_3 + G_4 = 0$$

where G_i are defined as:

$$(G_1)^i_j = {Z^\dagger}^i{}_k \left(\frac{\partial}{\partial Z^\dagger} \right)^k_j \qquad (G_2)^i_j = Z^i{}_k \left(\frac{\partial}{\partial Z} \right)^k_j$$

$$(G_3)^i_j = -{Z^\dagger}^k{}_j \left(\frac{\partial}{\partial Z^\dagger} \right)^i_k \qquad (G_4)^i_j = -Z^k{}_j \left(\frac{\partial}{\partial Z} \right)^i_k$$

Casimir Operators

Given generators $\{e_i\}$ of a Lie algebra $\mathcal{G}$, with

$$[e_i, e_j] = c_{ij}^k e_k \ , \qquad\qquad (*)$$

the algebra formed from linear combinations of products of the $\{e_i\}$, subject to $(*)$, is called the *universal enveloping algebra* $\mathcal{G}^U$ of $\mathcal{G}$.

- Elements in the centre of $\mathcal{G}^U$ are called *Casimir operators*.

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Given a representation of $\mathcal{G}$, $e_i \to \rho(e_i) = E_i$, then given a Casimir operator c,

$$C = \rho(c)$$

is called a *Casimir operator of the representation* ρ.

- By Schur's Lemma, Casimir operators of irreducible representations take constant values.

Casimir Operators

The Gauss Law operators G_i each form a representation of $U(N)$ on GIOs. There are thus associated Casimir operators: recalling the definitions

$$(G_2)^i_j \;=\; Z^i{}_k \left(\frac{\partial}{\partial Z}\right)^k_j \,, \qquad (G_3)^i_j \;=\; -\, Z^{\dagger k}{}_j \left(\frac{\partial}{\partial Z^\dagger}\right)^i_k$$

and defining $G_L = G_2 + G_3$, one may define

$$\begin{array}{lll} H_1 = \mathrm{tr}\, G_2 & H_2 = \mathrm{tr}\, G_2^2 & \\ \bar{H}_1 = \mathrm{tr}\, G_3 & \bar{H}_2 = \mathrm{tr}\, G_3^2 & H_L = \mathrm{tr}\, G_L^2 \,. \end{array}$$

- These can be thought of as Hamiltonians acting on GIOs.

Casimirs and Young diagrams

Recall that a composite Young diagram R with arbitrary integer row lengths r_i labels momenta p_i of N free fermions on a circle given in terms of the Fermi energy $n_F = \frac{N-1}{2}$ by

$$p_i = r_i + (n_F + 1 - i) \ .$$

Given a Young diagram R, the linear and quadratic Casimirs of the $U(N)$ representation R are expressible in terms of r_i or p_i:

$$\begin{aligned} C_1(R) &= \sum_i r_i = \sum_i p_i = n \\ C_2(R) &= nN + \sum_i r_i(r_i - 2i + 1) = \sum_{i=1}^{N} p_i^2 - \frac{N}{12}(N^2 - 1) \end{aligned}$$

Casimir Operators

Acting on a Brauer basis operator $\mathcal{O}^{\gamma}_{\alpha\beta;ij}(Z, Z^{\dagger})$,

- $H_1, \; H_2$ measure $C_1(\alpha), \; C_2(\alpha)$
- $\bar{H}_1, \; \bar{H}_2$ measure $C_1(\beta), \; C_2(\beta)$
- $H_1 - \bar{H}_1$ measures $C_1(\gamma)$, $\quad H_L$ measures $C_2(\gamma)$.

Generalized Casimir operators such as $\mathrm{tr}(G_2^2 G_3)$ are sensitive to the labels i, j.

Kimura, Ramgoolam '08

At $N = 2$ the labels i, j are trivial and it suffices to consider linear and quadratic Casimirs.

Free fermions in the $k = 0$ sector

At $N = 2$, a $k = 0$ operator is determined by the composite Young diagram γ which has two integer rows $r_1^\gamma,\ r_2^\gamma$.
We shift the row lengths to obtain fermion momenta:

$$p_1 = r_1 + \frac{1}{2}, \qquad p_2 = r_2 - \frac{1}{2} \ .$$

The linear and quadratc Casimirs at $N = 2$ become

$$\begin{aligned} C_1 &= p_1 + p_2 \\ C_2 &= p_1^2 + p_2^2 - \frac{1}{2} \end{aligned}$$

which may be inverted to

$$\begin{aligned} p_1 &= \frac{C_1}{2} + \sqrt{\frac{C_2}{2} - \frac{C_1^2}{4} + \frac{1}{4}} \\ p_2 &= C_1 - p_1 . \end{aligned}$$

Free fermions in the $k = 0$ sector

We have identified differential operators which measure Casimirs, in particular:

- $H_1 - \bar{H}_1$ measures $C_1(\gamma)$, H_L measures $C_2(\gamma)$.

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We may thus write the fermion momenta as differential operators:

$$\begin{aligned} \hat{p}_1 &= \frac{H_1 - \bar{H}_1}{2} + \sqrt{\frac{H_L}{2} - \frac{(H_1 - \bar{H}_1)^2}{4} + \frac{1}{4}} \\ \hat{p}_2 &= H_1 - \bar{H}_1 - \hat{p}_1 \end{aligned}$$

In terms of the matrix entries, $H_1, \ \bar{H}_1, \ H_L$ are combinations of the eigenvalues z_i and off-diagonal entries t_{jk}...

Hamiltonians in terms of matrix elements

Introducing for convenience $z_c = z_1 + z_2, \; z = z_1 - z_2$ and

$$L_1 = z_1 \frac{\partial}{\partial z_1} \qquad \bar{L}_1 = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}$$

$$L_2 = z_2 \frac{\partial}{\partial z_2} \qquad \bar{L}_2 = \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \qquad L_t = \frac{t_0}{2} \frac{\partial}{\partial t_0} \; ,$$

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Then $H_1, \; \bar{H}_1, \; H_L$ in terms of the entries of Z at $N = 2$ are:

$$H_1 = L_1 + L_2 + L_t, \qquad \bar{H}_1 = \bar{L}_1 + \bar{L}_2 + \bar{L}_t$$

$$\begin{aligned} H_L &= (L_1 - \bar{L}_1)^2 + (L_2 - \bar{L}_2)^2 + \frac{z_c}{z}(L_1 - L_2) + \frac{\bar{z}_c}{\bar{z}}(\bar{L}_1 - \bar{L}_2) \\ &- \frac{2}{|z|^2} \Big\{ t_0^2 (L_1 - L_2)(\bar{L}_1 - \bar{L}_2) + \frac{1}{t_0^2}(z_1\bar{z}_1 - z_2\bar{z}_2)^2 L_t^2 \\ &\quad - (z_1\bar{z}_1 - z_2\bar{z}_2)\left[(L_1 - L_2) + (\bar{L}_1 - \bar{L}_2)\right] L_t - (z_1\bar{z}_1 + z_2\bar{z}_2) L_t \Big\} \end{aligned}$$

Free fermions in the $k = 0$ sector

The construction carried out expicitly at $N = 2$ generalises to general N in a slightly weaker form.

At general N, we have N independent Casimirs leading to a degree N polynomial for the p_i.

- To obtain closed form expressions for the p_i in terms of the C_i would require one to solve arbitrary order polynomials, however for any specific values of the C_i one may solve for p_i.
- This gives an implicit map to free fermion momenta for any N.

Free fermions in the $k = m = n$ sector

The label k is related to the number of contractions in an operator.

- $k = m = n$: all terms in an operator involve the maximum number of contractions

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- Y is hermitian so we find a map to the N fermions of the hermitian matrix model...

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In this sector $\gamma = (k = m, \gamma_+ = \emptyset, \gamma_- = \emptyset)$ and $\alpha = \beta$ so operators in this sector labelled by α alone.

Free fermions in the $k = m = n$ sector

The operators may be written as

$$\mathcal{O}_{\alpha}^{k=m}(Z, Z^{\dagger}) = \frac{d_{\alpha}}{Dim\,\alpha}\,\mathrm{tr}_k(p_{\alpha} Y^{\otimes k})$$

where

- d_{α} is the dimension of the S_k representation α
- $Dim\,\alpha$ is the dimension of the $U(N)$ representation α.
- p_{α} is the projector onto the S_k representation α.

Thus operators in this sector are Schur polymonials constructed from Y.

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As discussed earlier, Schur polynomials in a hermitian matrix correspond to the states of N free fermions in a harmonic oscillator potential.

- The harmonic oscillator fermions observed here are a second emergence of free particles, distinct from those of the $k = 0$ sector.

Counting

It is important to check whether the Brauer basis matches the counting of operators at finite N described by other bases for two-matrix models.

Bhattacharyya, Collins, de Mello Koch & collaborators; Brown, Heslop, Ramgoolam

To count Brauer basis operators $\mathcal{O}^{\gamma}_{\alpha\beta;ij}(Z, Z^{\dagger})$, we must calculate the multiplicity M^{γ}_{A} of the restriction $\gamma \to A = (\alpha, \beta)$ of $S_m \times S_n$,

$$M^{\gamma}_{A} = \sum_{\delta \vdash k} \sum_{\delta} g(\gamma_+, \delta; \alpha) g(\gamma_-, \delta, \beta)$$

since the indices i, j run from 1 to M^{γ}_{A}. The number of operators in the Brauer basis is thus

$$Q^{N}_{b}(m, n) = \sum_{\gamma, A} \left(M^{\gamma}_{A}\right)^2 .$$

- For $m + n \leq N$, this formula counts multi-traces correctly.

What if N isn't big enough?

- For $m + n > N$, further constraints must be added to $c_1(\gamma_+) + c_1(\gamma_-) \leq N$
- At $N = 2$, we have a conjecture, as follows.

What if N isn't big enough?

- For $m + n > N$, further constraints must be added to $c_1(\gamma_+) + c_1(\gamma_-) \leq N$
- At $N = 2$, we have a conjecture, as follows.

Firstly, replace the reduction multiplicities by

$$M^{\gamma;N=2}_{\alpha,\beta} = \begin{cases} 1 & \text{if } M^{\gamma}_{\alpha,\beta} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and secondly, constrain $\alpha,\ \beta$ as follows:

1. $c_1(\alpha) + c_1(\beta) \leq N + k$
2. $[c_1(\alpha) + c_1(\beta)] + [c_2(\alpha) + c_2(\beta)] \leq 2N + k$

$\vdots$

and in general for each $p = 1, 2, \ldots, \min(m, n)$, constrain

$$\sum_{r=1}^{p} (c_r(\alpha) + c_r(\beta)) \leq pN + k.$$

Counting and the stringy exclusion principle

We can also express this as a constraint on k:

$$k \geq \min(r_2, \bar{r}_2) + \min\left(\min(r_1, \bar{r}_1),\ \max(r_2, \bar{r}_2)\right)$$

- We have numerically checked our conjecture up to $(m, n) = (15, 15)$.
- Is there a physical meaning to these extra constraints?

Summary

- We found free particle signatures in two sectors: $k = 0$ and $k = m = n$
- $k = 0$ sector: Explicit expressions at $N = 2$ for momenta of free fermions on a circle in terms of combinations of z_i, t_{jk}, implicit generalisation to arbitrary N
- $k = m = n$ sector: map to free fermions in harmonic oscillator of hermitian matrix model for arbitrary N.
- Brauer basis counts correctly for $N \geq m + n$; interesting subtleties for $m + n > N$.

Open questions

- Is it possible to realise these emergent fermions more explicitly?
  - We have their momenta - can we find the dual coordinates?
  - Can we express the wavefunctions as Slater determinants?

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- Is it possible to realise these emergent fermions more explicitly?
  - We have their momenta - can we find the dual coordinates?
  - Can we express the wavefunctions as Slater determinants?

- The label k seems to interpolate between degrees of freedom described by
  - Free fermions on a circle for $k = 0$
  - Free fermions on a line for $k = m = n$.

  Can these be interpreted as 'radial' and 'angular' degrees of freedom?

Open questions

- There is a family of bubbling $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS asymptotically $AdS_5 \times S^5$ geometries

  Chen, Cremonini, Donos et al

- Recent progress on identifying $\frac{1}{4}$-BPS operators at non-zero λ

  Brown; Kimura

- Can we find a description for $\frac{1}{4}$-BPS operators in terms of fermions?

Open questions

At zero coupling, this model is a consistent truncation of $\mathcal{N} = 4$ SYM.

- Does it have a string dual at zero coupling?

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Some speculations on this conjectured string dual:

- z_i: positions of N branes in two space dimensions.
- t_{ij}: strings connecting brane i to j

c.f. Witten '95

Here the triangular constraint $(t_{ij} = 0$ for $i > j)$ will make the dual qualitatively different from the standard system of strings and branes.

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- Does any of this physics survive at $\lambda \neq 0$, $\lambda \to \infty$ in SYM?
- Can it be compared to supergravity? Near-extremal AdS black holes?

Thanks!