Operator mixing and three-point functions in AdS/CFT.

Based on:

George Georgiou, Valeria L. Gili and R. R. arXiv:0810.0499
George Georgiou, Valeria L. Gili and R. R. arXiv:0907.1576

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Queen Mary - London

Joburg Workshop, 30/04/2010

Aim of the talk

- $\mathcal{N} = 4$ SYM is a non-trivial 4-dimensional CFT: the crucial data we want to derive are the spectrum and couplings.
- In the recent years, we have seen impressive progress in deriving the conformal dimensions.
- However, in order to derive the couplings we need to know also the precise expression of the states ("wavefunctions").
- This is a non-trivial problem, because states with the same quantum numbers mix.
- These remarks apply also to the string side of the AdS/CFT duality. We need to provide an explicit dictionary between the gauge and the string descriptions in the non-BPS sector.
- We can test our results in various non-trivial ways: $U(1)$ bonus symmetry, Ward identities, . . .

$\mathcal{N} = 4$ SYM

- The elementary fields are: a gauge boson A_μ, four Weyl fermions ψ^A_α and 3 complex scalars Z_i (or Φ_{AB}). The observables are
  - the gauge invariant operators build with A_μ, ψ^A_α and Φ_{AB},
  - correlation functions among these gauge invariant operators.
- We will focus on the Konishi operator $O_K \sim \mathrm{Tr}[\Phi_{AB}\Phi^{AB}]$ and its "Kaluza-Klein" generalization ($2\Phi_{34} = Z$ and $n > 0$): [Beisert 2002]

$$\mathcal{O}_n^{(0)J} \sim \sum_{p=0}^{J} \cos \frac{\pi n(2p+3)}{J+3} \mathrm{Tr}[\Phi_{AB} Z^p \Phi^{AB} Z^{J-p}]$$

  - These are non-BPS operators: the 1-loop correction to the conformal dimension Δ is $\lambda \sin^2[\pi n/(J+3)]$.
  - In the large λ, J limit (with $\lambda' = \lambda/J^2$ fixed), there is a simple string description for these operators [BMN 2002].

The operator mixing

- On the SYM side this question is usually addressed by computing some correlators and checking whether they take the form dictated by conformal invariance.
- In the case at hand, it is more efficient to use the basic definition of a primary
$$[\mathcal{S}, \mathcal{O}_n^{(0)J}(x=0)] \stackrel{?}{=} 0$$
$\mathcal{S} = (S_\alpha, \bar{S}^{\dot{\alpha}})$ is any superconformal charge of the $\mathcal{N} = 4$ SYM.
- It is easy to check this at the classical level ($g_{YM} \to 0$), but what about quantum corrections ?
- We can employ the same approach also on the string side of the duality: just use the corresponding string supercharges. Then the string h.w.s. is defined by $\mathcal{S}|hws\rangle = 0$, $\forall \mu\alpha$.
- We certainly need explicit expressions for $\mathcal{S}$ ...

The SYM charges

- We can follow BSS (1977) and derive from the 10D supercurrent

$$Q^M = \frac{\mathrm{i}}{2}[\Gamma^R, \Gamma^N]\,\mathrm{Tr}(F_{RN}\Gamma^M\lambda), \quad M = 0, \ldots, 9$$

  the (standard) 4D supersymmetry variations. For instance:

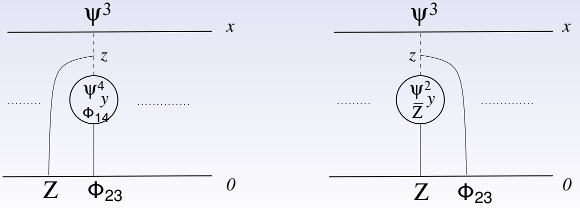
$$Q^\alpha_C \Phi_{AB} = \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{ABCD}\,\psi^D_\alpha\,, \quad \bar{Q}^{C\,\dot\alpha}\Phi_{AB} = \frac{\mathrm{i}}{\sqrt{2}}\left[-\delta^C_A\,\bar\psi^{\dot\alpha}_B + \delta^C_B\,\bar\psi^{\dot\alpha}_A\right]$$

- In 4D something special happens: we can construct another set of (onshell) conserved supercurrents $S, \bar{S}$:

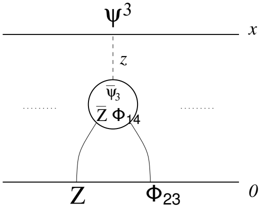
$$\bar{S}^{\mu\dot\alpha}_{\;A} = 2x_\tau(\bar\sigma^\tau)^{\dot\alpha\alpha}\mathrm{Tr}[(\sigma^{\rho\nu})^\beta_\alpha F_{\rho\nu}\sigma^\mu_{\beta\dot\beta}\bar\psi^{\dot\beta}_A + 2\sqrt{2}D_\rho\Phi_{AB}\sigma^\rho_{\alpha\dot\beta}\bar\sigma^{\mu\,\dot\beta\beta}\psi^B_\beta +$$
$$- 4\mathrm{i}g[\Phi_{AC}, \Phi^{CB}]\sigma^\mu_{\alpha\dot\beta}\bar\psi^{\dot\beta}_B] + 8\sqrt{2}\mathrm{Tr}[\phi_{AB}(\bar\sigma^\mu)^{\dot\alpha\alpha}\psi^B_\alpha],$$

- At the classical level we have that $[\mathcal{S}(0), \Phi_{AB}(0)] = 0$ and thus $\mathcal{O}^{(0)J}_n$ is a h.w.s. at leading order in λ.

A diagrammatic derivation in SYM



$\Rightarrow \quad 0$



This diagram yields a non-trivial contribution, which is consistent with the Ward identities if

$$\bar{S}^{\dot{\alpha}}_{A}\Phi_{BC}\Phi_{DE}(0) = \mathrm{i}\,\frac{gN}{32\pi^2}\left(\epsilon_{ABC[D}\bar{\psi}^{\dot{\alpha}}_{E]}(0) - \epsilon_{ADE[B}\bar{\psi}^{\dot{\alpha}}_{C]}(0)\right)$$

The SYM highest weight state

- The KK Konishi-like operators are at $\mathcal{O}(g^2)$

$$
\begin{aligned}
\mathcal{O}_n^J \quad \sim \quad & \mathcal{Z} \sqrt{\frac{N_0^{-J-2}}{(J+3)}} \sum_{p=0}^{J} \cos \frac{\pi n(2p+3)}{J+3} \mathrm{Tr}(Z_i Z^p \bar{Z}_i Z^{J-p}) \\
+ \quad & \frac{g\sqrt{N}}{4\pi} \sin \frac{\pi n}{J+3} \sqrt{\frac{N_0^{-J-1}}{(J+3)}} \sum_{p=0}^{J-1} \sin \frac{\pi n(2p+4)}{J+3} \mathrm{Tr} \psi^{1\alpha} Z^p \psi^2_\alpha Z^{J-1-p} \\
- \quad & \frac{g\sqrt{N}}{4\pi} \sin \frac{\pi n}{J+3} \sqrt{\frac{N_0^{-J-1}}{(J+3)}} \sum_{p=0}^{J-1} \sin \frac{\pi n(2p+4)}{J+3} \mathrm{Tr} \bar{\psi}_{3\dot{\alpha}} Z^p \bar{\psi}_4^{\dot{\alpha}} Z^{J-1-p} \\
+ \quad & \frac{g^2 N}{16\pi^2} \sin^2 \frac{\pi n}{J+3} \sqrt{\frac{N_0^{-J}}{(J+3)}} \sum_{p=0}^{J-2} \cos \frac{\pi n(2p+5)}{J+3} \mathrm{Tr} D_\mu Z Z^p D^\mu Z Z^{J-p-2}
\end{aligned}
$$

- $\mathcal{Z} = 1 + \ldots$ is a scheme-dependent renormalization and $N_0 = N/(8\pi^2)$.
- The same result was re-derived in 0910.3390 [Xiao].

BMN duality

- Type IIB string theory on the maximally supersymmetric PP-wave background: in the light-cone gauge we have eight towers of bosonic and fermionic harmonic oscillators ($a_n^\dagger$, $b_n^\dagger$).
  - $a_n^{i'\dagger}$: vector rep. of $SO(4) \subset SO(6)$
  - $a_n^{i\dagger}$: vector rep. of $SO(4) \subset SO(2,4)$
  - $\mathbb{P}^\pm b^\dagger$: spinor rep. of $SO(4) \times SO(4)$;
    $\mathbb{P}^\pm = (1 \pm \Pi)/2$ and Π is the appropriate 16×16 block of $\prod_{i'=1}^4 \Gamma^{i'}$
- The physical spectrum is the subset of the Fock space satisfying the level matching condition.
  - The light-cone momentum $\alpha = \alpha' p^+$ corresponds to the field theory quantum number J
  - Δ is related ($\lambda' = 1/(\mu\alpha)^2$) to the string Hamiltonian: $\Delta - J = H$

$$H = \frac{1}{\mu\alpha} \sum_{n=-\infty}^{\infty} \omega_n \left[a_n^\dagger a_n + b_n^\dagger b_n \right] \;, \quad \omega_n = \sqrt{n^2 + (\mu\alpha' p^+)^2} \;.$$

Supercharges

- There are 16 kinematic charges involving only zero-modes.
- The other 16 supercharges dynamical such as

$$\mathbb{P}^+\bar{Q}^- = \sqrt{2}\left[\gamma^i a_0^i \mathbb{P}^- b_0^\dagger + \gamma^{i'} a_0^{i'\,\dagger} \mathbb{P}^+ b_0\right] + \frac{1}{\sqrt{\mu|\alpha|}} \sum_{n=1}^{\infty} \sqrt{n} \Big\{$$

$$\gamma^{i'}\left[a_n^{i'\dagger}\mathbb{P}^+ b_n + \mathrm{i} a_{-n}^{i'\dagger}\mathbb{P}^+ b_{-n}\right] U_n^{-\frac{1}{2}} + \gamma^{i'}\left[a_n^{i'}\mathbb{P}^+ b_{-n}^\dagger - \mathrm{i} a_{-n}^{i'}\mathbb{P}^+ b_n^\dagger\right] U_n^{\frac{1}{2}}$$

$$+ \gamma^i\left[a_n^i \mathbb{P}^- b_{-n}^\dagger - \mathrm{i} a_{-n}^i \mathbb{P}^- b_n^\dagger\right] U_n^{-\frac{1}{2}} + \gamma^i\left[a_n^{i\dagger}\mathbb{P}^- b_n + \mathrm{i} a_{-n}^{i\dagger}\mathbb{P}^- b_{-n}\right] U_n^{\frac{1}{2}} \Big\},$$

where $\rho_n = \frac{\omega_n - n}{\mu\alpha}$ and $U_n^{\pm 1} \equiv \frac{1 \mp \rho_{n(1)}}{1 \pm \rho_{n(1)}} \sim \left[\frac{n}{2\mu\alpha}\right]^{\pm 1}$.

- From the small λ' limit, we can read the following dictionary

$$Q_{1,2} \leftrightarrow \mathbb{P}^+ Q^+ \,,\; Q_{3,4} \leftrightarrow \mathbb{P}^+ Q^- \,,\; \bar{Q}^{1,2} \leftrightarrow \mathbb{P}^- \bar{Q}^- \,,\; \bar{Q}^{3,4} \leftrightarrow \mathbb{P}^- \bar{Q}^+ \,,$$

$$S^{1,2} \leftrightarrow \mathbb{P}^+ \bar{Q}^+ \,,\; S^{3,4} \leftrightarrow \mathbb{P}^+ \bar{Q}^- \,,\; \bar{S}_{1,2} \leftrightarrow \mathbb{P}^- Q^- \,,\; \bar{S}_{3,4} \leftrightarrow \mathbb{P}^- Q^+ \,.$$

The string highest weight state

- According to the standard BMN correspondence we have

$$\mathcal{O}_n^{(0)J} \leftrightarrow |n^{(0)}\rangle \equiv \sum_{i'=1}^{4} \left[a^{\dagger}{}_{n}^{i'} a^{\dagger}{}_{n}^{i'} + a^{\dagger}{}_{-n}^{i'} a^{\dagger}{}_{-n}^{i'} \right] |\alpha\rangle \ ,$$

- The string state $|n\rangle$ is annihilated by $\mathbb{P}^{+}\bar{Q}^{-}$ only in the limit $\lambda' \to 0$.
- The 2-impurity h.w.s. of BMN string theory are states annihilated by both $\mathbb{P}^{+}\bar{Q}^{-}$ and $\mathbb{P}^{-}Q^{+}$. They are:

$$|n\rangle = \frac{1}{4(1+U_n^2)} \Big[\ a^{\dagger}{}_{n}^{i'} a^{\dagger}{}_{n}^{i'} \ + \ a^{\dagger}{}_{-n}^{i'} a^{\dagger}{}_{-n}^{i'} + 2U_n b^{\dagger}_{-n} \, \Pi \ b^{\dagger}_{n} \\ - \quad U_n^2 \left(a^{\dagger}{}_{n}^{i} a^{\dagger}{}_{n}^{i} \ + \ a^{\dagger}{}_{-n}^{i} a^{\dagger}{}_{-n}^{i} \right) \Big] |\alpha\rangle,$$

- Notice that in the small λ' limit this is the same pattern seen in field theory (in the large J limit).

A first correlator (SYM side).

- We focus on correlators among one non-BPS and two BPS states.

$$\langle \bar{Q}^1_{\dot\alpha} \bar{Q}^{2\dot\alpha} \bar{\mathcal{O}}_0^{J_1+2}(x_1)\, \bar{Q}^1_{\dot\beta} \bar{Q}^{2\dot\beta} \bar{\mathcal{O}}_0^{J_2+2}(x_2)\, \mathcal{O}_n^{J_3}(x_3)\rangle,$$

where $J_3 = J_1 + J_2 + 2$.

- The operators in x_1 and x_2 are at level two in the BPS multiplet.
- According to Intriligator and Skiba (IS) this amplitude should vanish at all orders of $1/N$ and λ!
- This selection rule is called "bonus" $U(1)_Y$ symmetry.
- This is consistent with the constraints from $\mathcal{N} = 4$ superspace.
- However, the perturbative check ($\mathcal{O}(g)$) provided by IS is valid only for $n = 0$, which does not describe a non-BPS primary state ...
- It turns out that the non-zero result obtained in $n \neq 0$ case is exactly cancelled by a "tree-level" diagram involving the terms with ψ^1, ψ^2 in the corrected h.w.s.!

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A first correlator (string side).

- Can we perform a check at strong coupling ($\lambda \to \infty$)? The only technique at our disposal is to focus on the BMN limit.
- The correlator is proportional the cubic Hamiltonian $|H_3\rangle$ evaluated on the three string states corresponding to the SYM operators.
- The coefficient of proportionality can be computed exactly in the BPS case [Yoneya et al.]
- In our case we do not need the details and want to check whether $|H_3\rangle$ vanishes on our three states.
- One needs to use the full form of $|H_3\rangle$ [S. Lee and RR], a few properties of the BMN Neumann matrices and the h.w.s. $|n\rangle$; then it can be checked that the amplitude vanishes $\forall \lambda'$!
- However, by looking at the details of the string computation, one can see that the parallelism with the SYM computation is not perfect . . .

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Other effects of the mixing.

- The operator mixing is crucial for ensuring superconformal invariance in many other cases.
- We considered a correlator with a level 3 descendant of the h.w.s.
- Superconformal invariance constrains this correlator to vanish.
- In an explicit computation this happens only after including the corrections to the non-BPS operator due to the mixing.
- The mixing is important also in non-vanishing correlators.
- 3-point functions with non-BPS states have been considered by different authors [Okuyama and Tseng, Alday et al.]
- For instance, the correlator between the non-BPS h.w.s. and two 2-impurity BPS states is corrected by the terms due to the mixing and yields a non-trivial result.

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A second correlator.

- Let us consider a descendant of level four of the non-BPS 2-impurity h.w.s. $(Q_3^2 Q_4^2 \mathcal{O}_n)$

$$
{}^{[4]}\mathcal{O}_n^J \propto \sum_{p=0}^{J} \sin \frac{\pi n(2p+2)}{J+2} \mathrm{Tr}\psi^{1\alpha} Z^p \psi^2_\alpha Z^{J-p} +
$$

$$
-2\sqrt{2} g \sin \frac{\pi n}{J+2} \sum_{p=0}^{J+1} \cos \frac{\pi n(2p+1)}{J+2} \mathrm{Tr}\Phi_{AB} Z^p \Phi^{AB} Z^{J-p+1} +
$$

$$
+\frac{gN}{8\sqrt{2}\pi^2} \sin \frac{\pi n}{J+2} \sum_{p=0}^{J-1} \cos \frac{\pi n(2p+3)}{J+2} \mathrm{Tr} D_\mu Z Z^p D^\mu Z Z^{J-p-1} + \ldots .
$$

- We consider the correlator among ${}^{[4]}\mathcal{O}_n^J$ and two BPS states with 2 bosonic impurities.
- This correlator should vanish thanks to the $U(1)_Y$ selection rule!

Figure: On the left, the diagram contributing to the contraction of the leading term of the level-four long state with the two BPS and a Yukawa coupling. On the right, the planar diagram contributing to the tree-level contraction of the subleading term with flavour impurities of the level-four long state with two BPS.

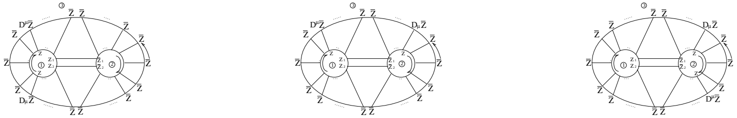

Figure: Diagrams originating from the free contractions of the subleading term of the long operator having two vector impurities and the two BPS operators. In the diagram on the left, both impurities of the long state are contracted with the operator sitting at x_1. In the diagram in the middle, one impurity is contracted with the operator at x_1 and the other with the operator at x_2. Finally, in the diagram on the right both impurities are contracted with the operator at x_2.

A mismatch.

- The field theory amplitude vanishes at order g, thanks to a non-trivial cancellation among the different diagrams.
- The corresponding (BMN) string result for the cubic Hamiltonian is non-zero.
- Apparently the cubic Hamiltonian discussed does not have room for a contribution corresponding to the field theory diagrams where DZ is contracted with $\bar{Z}$.
- This seems to be the main source of mismatch.
- A similar mismatch appears also in the correlator involving the level 3 non-BPS descendant. So how do we realise the superconformal WI on the string side?
- The BMN limit is defined by using the global coordinates and this makes the comparison with our SYM less direct.

Conclusions

- The precise form of the SYM primary (and descendant) operators can be computed by using the supersymmetry algebra.
- There is a striking similarity between the string and the field theory mixing patterns.
- The knowledge of the precise form of the SYM operators is crucial for computing SYM correlators.
- Mixing is a crucial feature also in the string descriptions.
- Possible developments:
  - Understand/solve the current mismateches between string and gauge theory results.
  - Can we compute the correlation functions by using some semi-classical approximation? [Janik et al.]
  - Can we say something more on how the quantum SYM dynamics is encoded in string world-sheet Lagrangian ?

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An alternative derivation: using the spin chain

- The superconformal algebra is $PSU(2,2|4)$ requires

$$\left\{ \bar{Q}^{A\dot{\alpha}}, \bar{S}_B^{\dot{\beta}} \right\} = \epsilon^{\dot{\alpha}\dot{\beta}} \delta_B^A 2\Delta + \ldots$$

- When acting on scalars of different flavor we have at order g^2

$$\Delta \to \mathbb{H} = \frac{g^2 N}{8\pi^2} (\mathbb{I} - \mathbb{P})$$

where $\mathbb{I}$ is the identity and $\mathbb{P}$ is the permutation operator.

- Up to order g, the action of $\bar{Q}$ can be read directly from the classical currents
- So from the algebra we have

$$\left\{ \bar{Q}^{1\dot{\alpha}}, \bar{S}_1^{\dot{\beta}} \right\} \Phi_{24}\Phi_{34} = \bar{Q}^{1\dot{\alpha}} \bar{S}_1^{\dot{\beta}} \Phi_{24}\Phi_{34} = \epsilon^{\dot{\alpha}\dot{\beta}} \frac{g^2 N}{4\pi^2} [\Phi_{24}, \Phi_{34}] .$$

- By using the action of $\bar{Q}$ on $\bar{\psi}$, we get the same results for $\bar{S}\Phi\Phi$.