Matrix models, Hurwitz spaces and strings over algebraic numbers

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with Robert de Mello Koch.

Introduction

Gauge-String Duality

$$\begin{array}{rcl} \text{Strings on } AdS_5 \times S^5 & \longleftrightarrow & \mathcal{N} = 4 \text{ SYM} \\ g_s & \longleftrightarrow & g_{YM}^2 \\ \dfrac{R}{l_s} & \longleftrightarrow & (g_{YM}^2 N)^{\frac{1}{4}} \end{array}$$

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Yet, Maldacena's duality implies

There is a dual spacetime physics.

What is this dual space-time physics ?
at zero coupling ?
at weak coupling ?

Zero coupling with $g_{YM}^2 = 0$ is

$$\begin{aligned} \frac{R}{l_s} &= (g_{YM}^2 N)^{\frac{1}{4}} &= 0 \\ g_s &= g_{YM}^2 &= 0 \\ \frac{R^4}{g_s l_s} &= N &= \text{finite} \end{aligned}$$

This is a double scaling limit.

Special property of zero coupling : Decoupling of sectors corresponding to different fields.

Simplest sector : One hermitian matrix X from the six matrix scalars.

Correlators of gauge-invariant observable made from X can be computed using only the action for X.

What is the stringy geometry of one hermitian matrix model correlators ?

What is the space-time explanation of the decoupling ?

Knowing about correlators from the gauge theory, how can we construct the dual string theory. This is the philosophy of collective field theory, Gopakumar programme ...

Here we focus on the space-time independent part. Reduce the theory on S^3 as in radial quantization. And further reduce to zero dimensions.

To get simply

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2} trX^2}$$

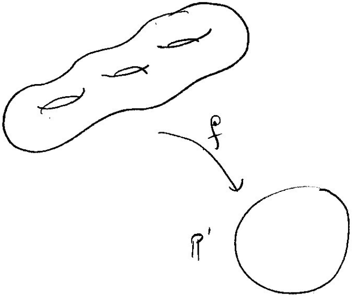
This model was studied, with the possibility of an additional general potential $V(g_i, X)$, around early nineties.

By tuning g_i to different critical points, one had a series of dual string theories $c < 1$ models coupled to 2D gravity (Liouville theory).

Another low-dimensional gauge-string duality from the early-mid nineties : Two dimensional Yang Mills and dual string theory.

Holomorphic maps between Riemann surfaces – Hurwitz space.

We found a simple Hurwitz space description of the correlators in the 1-Hermitian matrix model.



$$\bar{\partial} f = 0$$

What kind of holomorphic maps ?

The curve and map are defined by equations involving coefficients which are algebraic numbers i.e they are in $\bar{\mathbb{Q}}$.

Any curve defined over $\bar{\mathbb{Q}}$ can appear among the worldsheets for appropriate observables.

OUTLINE

- The Matrix model : From correlators to counting triples of permutations.
- Physical Interpretation : Target space of $\mathbb{P}^1$ and strings over algebraic numbers.
- Multi-Matrix Model : New invariants of the absolute Galois group.
- Open problems

PART 1 : One Matrix Model

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2} trX^2}$$

X : $N \times N$ Hermitian matrix

$$dX \equiv \prod_{i<j} dRe(X_{ij}) dIm(X_{ij}) \prod_i dRe(X_{ii})$$

$$\mathcal{Z}(g) = \int dX e^{-\frac{1}{2} trX^2 + V(X,g)} = \int dX e^{-\frac{1}{2} trX^2 + g_3 trX^3 + g_4 trX^4 + \cdots}$$

One Matrix Model : Obserbvables

The Observables of interest : Trace moments of the matrix variables.

$$\langle \mathcal{O}(X) \rangle = \int dX e^{-\frac{1}{2} trX^2} \mathcal{O}(X) \cdots$$

The $\mathcal{O}(X)$ is a function of traces, e.g $\mathcal{O}(X) = (trX)^{p_1} (trX^2)^{p_2} \cdots$.

Fixing the total number of X to be n, the number of these observables is $p(n)$. The number of partitions of n.

$$n = p_1 + 2p_2 + 3p_3 + \cdots$$

Partitions of n correspond to conjugacy classes of the symmetric group S_n, of all permutations of n objects.

It is possible to associate observables to permutations

$$\mathcal{O}_\sigma(X)$$

which only depend of the conjugacy class.

$$\mathcal{O}_{\alpha\sigma\alpha^{-1}}(X) = \mathcal{O}_\sigma(X)$$

$$\mathcal{O}_\sigma(X) = X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_n}_{i_{\sigma(n)}}$$

Conjugacy classes in S_n are characterized by the cycle decomposition of the permutations. e.g a permutation (123)(45) in S_5 cyclically permutes $1, 2, 3$ and swops $4, 5$.

The conjugacy class of such a permutation corresponds to $trX^3 trX^2$, i.e if $\sigma = (123)(45)$,

$$\mathcal{O}_\sigma(X) \quad \sim \quad trX^3 trX^2$$

We will choose a normalization of observables as

$$\begin{aligned}\mathcal{O}_\sigma(X) &= N^{-n+p_1(\sigma)+p_2(\sigma)+\cdots+p_n(\sigma)}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n}\\ &= N^{C_\sigma-n}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n}\end{aligned}$$

We will define a delta function over the symmetric group

$$\begin{aligned}\delta(\sigma) &= 1 \text{ if } \sigma=1\\ &= 0 \text{ otherwise}\end{aligned}$$

Theorem 1 :

$$\langle \mathcal{O}_\sigma \rangle = \frac{1}{(2n)!} \sum_{\sigma \in [\sigma]} \sum_{\gamma \in [2^n]} \sum_{\tau \in S_{2n}} \delta(\sigma\gamma\tau) N^{C_\sigma + C_\tau - n}$$

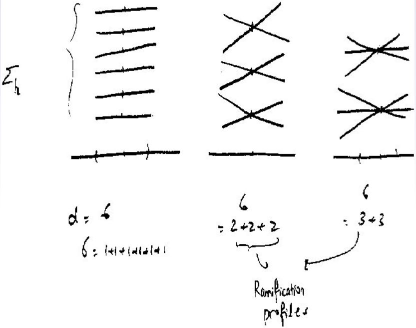
Follows using Wick's theorem. The sum over γ, which is in $[2^n]$ is the sum over Wick contractions.

Equivalently, this is the sum over Feynman diagrams of the Gaussian matrix model.

Use a classic theorem : The Riemann Existence theorem, which relates the counting of such strings of permutations to the counting of equivalence classes of holomorphic maps $f : \Sigma_h \to \mathbb{P}^1$, from Riemann surface Σ_h of genus h to target $\mathbb{P}^1$.

Holomorphic maps between Riemann surfaces are branched covers.

An interval through a generic point on the target Riemann surface : inverse image has d intervals, where d is the degree of the map. A branch point has fewer inverse images.



Σ_h
$d = 6$
$6 = 1+1+1+1+1+1$
$6 = 2+2+2$
$6 = 3+3$
Ramification profiles

Each branch point has a ramification profile which is a partition of the degree d. The ramification data determines the genus h of Σ_h by the Riemann Hurwitz formula.

$$\langle \mathcal{O}_\sigma \rangle = \sum_{f:\Sigma_h \to \mathbb{P}^1} \frac{1}{|Aut\ f|} N^{2-2h}$$

The Gaussian Matrix model correlator is a sum over equivalence classes of holomorphic maps to $\mathbb{P}^1$, branched at 3 points, with ramification profiles $[\sigma]$, $[\gamma] = [2^n]$ and $[\tau]$ which is general.

Weighted by g_{st}^{2h-2} where $g_{st} = \frac{1}{N}$

PART 2 : Physics Interpretation

The Gaussian Matrix model is equivalent to a topological string theory, with target space $\mathbb{P}^1$ which localizes to holomorphic maps with three branch points.

A perturbed Gaussian model also has such an interpretation with e^V treated as an observable.

Implication : Hurwitz counting results from Saddle point

By considering a potential $V = trX^m$ we can get explicit Hurwitz space counting results for maps where $[\sigma] = [m^n]$. The cases $m = 3, 4, 6$ were done in the paper - for genus genus zero worldsheets using saddle point methods.

For the case $m = 6$, $[\sigma] = [6^n]$, $[\gamma] = [2^{3n}]$ and $[\tau]$ is summed over all possible. For such maps,

$$\sum_f \frac{1}{|Autf|} = \frac{1}{2} \frac{(10)^n (3n-1)!}{(2n+1)!(n+1)!}$$

MEANING OF THREE ?

Belyi theorem : A Riemann surface is defined over algebraic numbers iff it admits a map to $\mathbb{P}^1$ with three branch points.

Riemann surface can be described by algebraic equations, e.g an elliptic curve

$$y^2 = x^3 + ax^2 + bx$$

If a, b are algebraic numbers, i.e. solutions to polynomial equations with rational coefficients $\mathbb{Q}$, then the Riemann surface is defined over $\bar{\mathbb{Q}}$, i.e for $x \in \bar{\mathbb{Q}}$, $y \in \bar{\mathbb{Q}}$

The field $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ and contains all solutions of polynomial equations with rational coefficients.

It contains finite extensions of $\mathbb{Q}$ such as $\mathbb{Q}(\sqrt{2})$.

This is numbers of the form $a + b\sqrt{2}$, where a, b are rational. They form a field, closed under addition, multiplication, division.

An important group associated to this extension is the group of automorphisms which preserves the rationals. In this case, the only non-trivial element of the group is $\sqrt{2} \to -\sqrt{2}$. We say $Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \mathbb{Z}_2$

The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ contains as subgroups, all the finite Galois groups of finite dimensional extensions.

It acts on the algebraic numbers coefficients of the defining equations of the curve Σ_h and of the map f.

Hence the Galois group acts on the (equivalence classes of) permutation triples, equivalently the Feynman graphs of the 1-matrix model.

Grothedieck associated Dessins to the permutation triples, which are essentially the Feynman graphs of the 1-matrix model.

The multiplicity of Feynman graphs can be organised into orbits of the Galois group action.

Elements in the same orbit contribute with equal weight, since *Autf* is a Galois invariant.

$$\langle \mathcal{O}_{[\sigma]} \rangle = \underbrace{F_1 + F_2 + F_3}_{\text{Galois orbit}} + \underbrace{F_4 + F_5}_{\text{Galois orbit}}$$

Equivalence class of

σ, γ, τ ; $\dfrac{1}{\mathrm{Aut}\, f} = \underbrace{\dfrac{1}{\mathrm{Aut}(\sigma, \gamma, \tau)}}$

This is a Galois invariant

$\Downarrow$

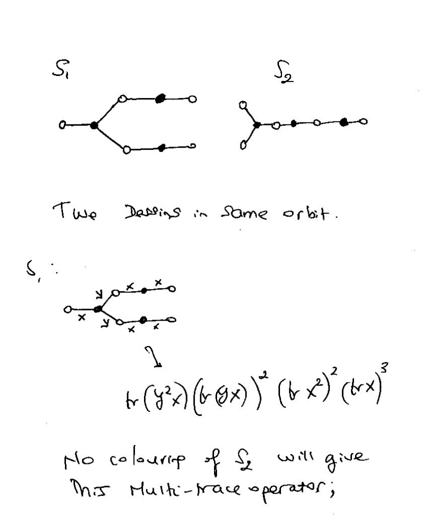
Feynman diagrams in a Galois orbit contribute with same weight

PART 3 : Multi-matrix models

An obvious generalization to consider is multi-matrix models, where we have integrals over multiple matrix variables, e.g X, Y.

The edges of the Feynman graphs, which are propagators are now colored, i.e they can be X or Y propagators. So they correspond to colored-edge versions of Grothendieck Dessins.

A given multi-matrix observable, e.g $trX^2Y^2trXtrY^3$ can receive zero contribution from one Dessin in a Galois orbit and non-zero from another.



- A generic correlator in the multi-matrix model is not a Galois invariant.
- But we can build invariants from multi-matrix model correlators with a few steps.

GAUSSIAN = CLEAN !!

General Grothendieck Dessins have arbitrary $[\sigma], [\gamma], [\tau]$. The branch points can be chosen at $0, 1, \infty$.

The Dessins are bi-partitite graphs drawn the sphere, with black and white vertices. Black vertices have numbers of edges corresponding to the cycle structure of $[\sigma]$. White vertices correspond to $[\gamma]$.

Dessins with $[\gamma] = [2^n]$ are **Clean Dessins**

Any dessin can be mapped to a clean Dessin. Any map with three branch points at $0, 1, \infty$ can be mapped to a new map with only simple branchings using $f \to 4f(1-f)$.

Cleaning is a process used to define Galois invariants in the Dessins literature.

The restriction $[\gamma] = [2^n]$ is not significant from the point of view of Galois invariants.

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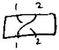
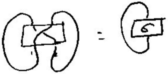
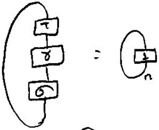




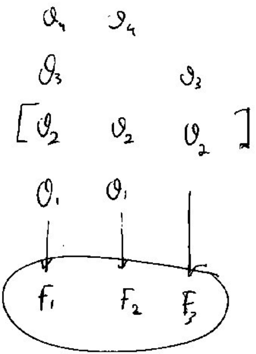


$$\Sigma = (\sigma \circ \gamma)$$

$\rho =$  $= t$

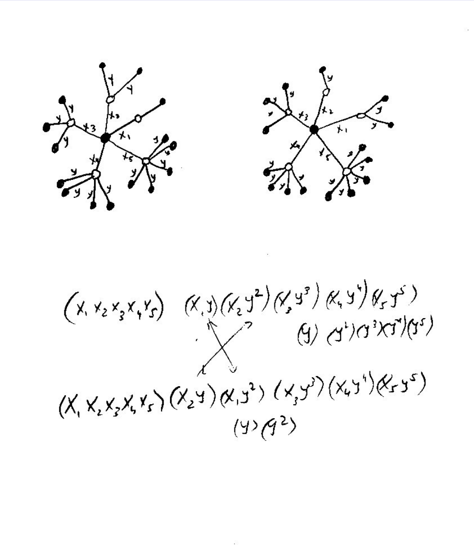
$$\tau = t(\gamma^{-1} \cdot \sigma^{-1})$$

- Colorings of the Dessins allow the definition of new invariants of the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Dessins : constructed from lists of multi-matrix observables which receive contributions from the Dessin.
- Take certain unions and intersections.



Intersection of lists of Multi-Matrix Observables is a Galois Invariant.

Known invariants can be described in terms of these lists, e.g Flower-shaped trees.



Two of Many questions

How are the Physics-inspired invariants constructed from coloured Dessins related to number theoretic invariants ?

Belyi theorem suggests that the string theory of 1-matrix model can be defined over $\bar{\mathbb{Q}}$. Is there an explicit construction of string amplitudes and path integrals over $\bar{\mathbb{Q}}$?