Cut-and-join operators and $\mathcal{N} = 4$ SYM

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based on 1002.2099 [hep-th] and forthcoming

Two for one

Actually I want to give TWO talks.

Cut-and-join operators and $\mathcal{N} = 4$ SYM

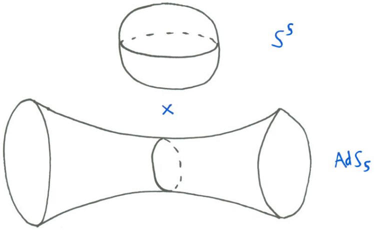
$\sim$ 40 min

and

Ideas for the string dual of the free theory

$\sim$ 20 min

IIB superstrings on $AdS_5 \times S^5$



$$\frac{1}{\alpha'} \int d\tau d\sigma \, [\text{ non-linear } \sigma\text{-model }]$$

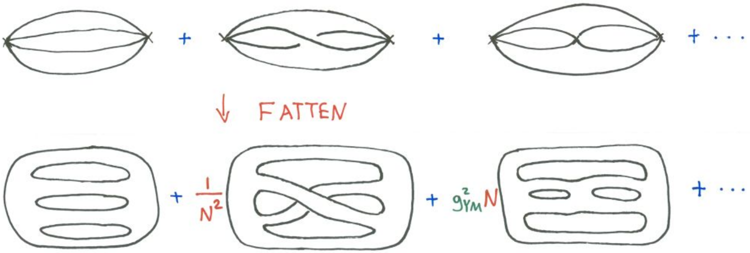
Perturbative expansion in the string coupling g_s

$\mathcal{N} = 4$ SUSY Yang-Mills: a Conformal Field Theory

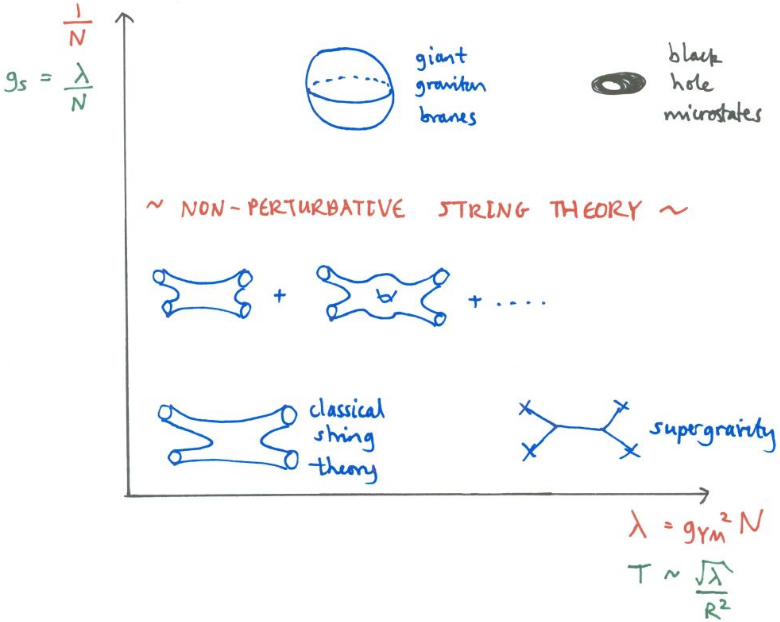
$$\frac{N}{\lambda} \int d^4x \, \mathrm{tr} \Big[F_{\mu\nu} F^{\mu\nu} + D^\mu \phi_i D_\mu \phi_i - [\phi_i, \phi_j][\phi_i, \phi_j]$$

$$+ \psi \sigma^\mu D_\mu \psi - \psi \phi \psi \Big] \;+\; \theta \int d^4x \, \mathrm{tr}[F_{\mu\nu} \tilde{F}^{\mu\nu}]$$

Gauge group $U(N)$; fields in adjoint. Compute correlation functions of gauge-invariant local operators.



Parameter space



General programme for cut-and-join operators in SYM

Study non-planar corrections to $\mathcal{N} = 4, d = 4$ super Yang-Mills.

Two different attitudes to $\frac{1}{N}$ corrections, depending on coupling.

- For free theory, $\lambda = 0$, treat $\frac{1}{N}$ as a string coupling ordering the non-planar expansion of correlation functions. Multi-trace operators identified with multi-string states.
- For $\lambda > 0$ the correct string expansion is in $g_s = \frac{\lambda}{N}$. Treat $\frac{1}{N}$ corrections as a modification to the gauge theory/string theory state identification.

Recast splitting and joining of traces using symmetric group.

Review of half-BPS sector

Based on [Vaman and Verline 0209215]; [Corley, Jevicki and Ramgoolam 0111222].

Trace structures of operators map to conjugacy classes of S_n.
E.g. for $\alpha = (123)(45) \in S_5$

$$\begin{aligned} \mathrm{tr}(X^3)\,\mathrm{tr}(X^2) &= X^{i_1}_{i_2} X^{i_2}_{i_3} X^{i_3}_{i_1} X^{i_4}_{i_5} X^{i_5}_{i_4} \\ &= X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}} X^{i_3}_{i_{\alpha(3)}} X^{i_4}_{i_{\alpha(4)}} X^{i_5}_{i_{\alpha(5)}} \\ &:= \mathrm{tr}(\alpha\, X^5) \end{aligned}$$

Conjugacy classes labelled by partitions of n, e.g. $[3, 2]$ here.

Two-point function given by cut-and-join operators

$$\left\langle \mathrm{tr}(\alpha'\, X^{\dagger n})\ \mathrm{tr}(\alpha\, X^n) \right\rangle_{\text{non-planar}} = N^n \left\langle \alpha' \right|\ \Omega_n \left(\tfrac{1}{N}\right)\ \left|\alpha\right\rangle$$

(We're dropping the spacetime dependence here and onwards.)

Cut-and-join operators

Basic cut-and-join operator is a sum over the transpositions in S_n

$$\Sigma_{[2]} = \sum_{i<j} (ij)$$

It cuts a single trace/cycle $[n] = (123 \cdots n)$ into two

$$\Sigma_{[2]} \, |n\rangle \sim |n_1, n_2\rangle$$

It both joins a double trace and cuts it into three

$$\Sigma_{[2]} \, |n_1, n_2\rangle \sim |n\rangle + |n_1, n_2, n_3\rangle$$

Tree-level mixing given by

$$\begin{aligned} \Omega_n &= \sum_{\sigma \in S_n} \frac{1}{N^{T(\sigma)}} \, \sigma \\ &= 1 + \frac{1}{N} \Sigma_{[2]} + \frac{1}{N^2} \left(\Sigma_{[3]} + \Sigma_{[2,2]} \right) + \mathcal{O}\left(\frac{1}{N^3} \right) \end{aligned}$$

Inner product and full non-planar correlation function

The inner product is given by the leading *planar* two-point function

$$\langle \alpha' | \alpha \rangle \sim \delta_{\alpha' \in [\alpha]}$$

The leading term of the (extremal) three-point function

$$\langle n_1, n_2 | \left(\frac{1}{N} \Sigma_{[2]} \right) | n \rangle = \frac{n n_1 n_2}{N}$$

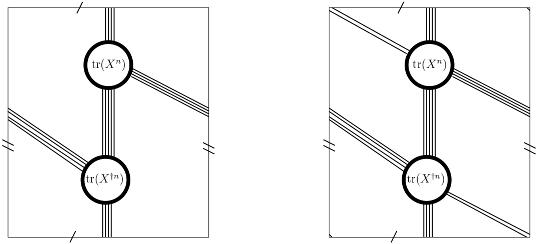
The first correction to the single-trace 2-p't f'n from the torus

$$\langle n | \left(\frac{1}{N^2} \left[\Sigma_{[3]} + \Sigma_{[2,2]} \right] \right) | n \rangle = \frac{n}{N^2} \left[\binom{n}{3} + \binom{n}{4} \right]$$

What do these numbers mean in a putative worldsheet theory?

Bunching of homotopic propagators

The $\Sigma_{[3]}$ term gives propagators on the torus bunched into 3 groups; $\Sigma_{[2,2]}$ gives propagators bunched into 4 groups.



In Gopakumar's model, each Σ_C gives a different skeleton graph of homotopically-bunched propagators for the relevant genus g.

Relation to Hurwitz theory

Suggestively, these are Hurwitz numbers counting n-branched covers of $\mathbb{C}P^1$ by surfaces of genus g with three branch points, two labelled by the operators and the third by the cut-and-join Σ_C. For example for

$$\begin{aligned}\left\langle C' \middle| \Sigma_C \middle| C'' \right\rangle &= \frac{1}{n!}\delta(\Sigma_{C'}\Sigma_C\Sigma_{C''}) \\ &= \frac{1}{(n!)^2}\sum_{|R|=n} d_R^{2-3}\chi_R(\Sigma_C)\chi_R(\Sigma_{C'})\chi_R(\Sigma_{C''}) \\ &= \sum_{f(C,C',C''):\Sigma_g\to\mathbb{C}P^1}\frac{1}{|\mathrm{Aut}(f)|}\end{aligned}$$

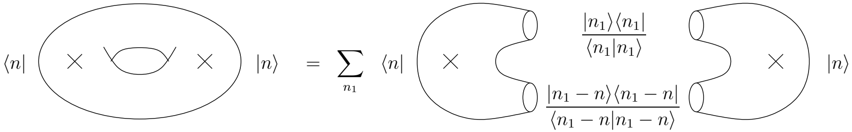
Where Σ_g is the appropriate genus surface from the field theory, e.g. for extremal k-point function $T(C) = k + 2g - 2$.

cf. [de Mello Koch, Ramgoolam 1002.1634] on Hermitian matrix model

Two-dimensional factorisation of correlation functions

Another feature is that for large n the higher genus correlation functions factorise into planar 3-point functions, e.g. for torus

$$\frac{1}{N^2}\left(\Sigma_{[3]}+\Sigma_{[2,2]}\right)\rightarrow\frac{1}{2}\left(\frac{1}{N}\Sigma_{[2]}\right)^2$$



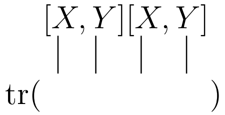
This is the result of the exponentiation of the tree-level mixer

$$\Omega_n=\exp\left(\frac{1}{N}\Sigma_{[2]}-\frac{1}{2N^2}\left[\binom{n}{2}+\Sigma_{[3]}\right]+\mathcal{O}\left(\frac{1}{N^3}\right)\right)$$
$$\rightarrow\exp\left(\frac{1}{N}\Sigma_{[2]}\right)$$

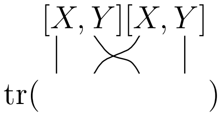
NB: additional terms subleading in $\frac{n^2}{N}$.

Multiple fields: a few simple examples I

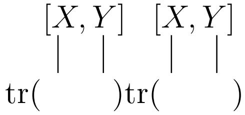
Tracing the same field content for $U(2) \subset SU(4)_R$ rep $\Lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$ we sometimes have to 'twist' the trace to get a non-vanishing operator



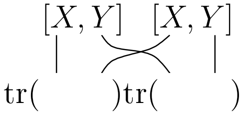
$$= \mathrm{tr}([X,Y][X,Y])$$



$$= 0$$



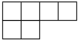
$$= 0$$



$$= \mathrm{tr}(\Phi^r \Phi^s)\mathrm{tr}(\Phi_r \Phi_s)$$

where $\Phi^p \Phi_p = \epsilon^{pq} \Phi_p \Phi_q = [X, Y]$.

Multiple fields: a few simple examples II

Things also get complicated when for a given representation and trace structure there is more than one operator, e.g. for the $U(2)$ rep  $\sim [X, Y][X, Y]\, XX$ with trace structure [4, 2]

$$\mathrm{tr}([X, Y][X, Y])\, \mathrm{tr}(XX)$$

$$\mathrm{tr}(XX\Phi^r\Phi^s)\, \mathrm{tr}(\Phi_r\Phi_s)$$

(remembering that $\Phi^p\Phi_p = \epsilon^{pq}\Phi_p\Phi_q = [X, Y]$).

So our goal is to organise multi-trace, multi-field operators into:

- Representations of the global symmetry group;
- Operators with fixed trace structure, e.g. single/double trace;
- And to describe any attendant multiplicity.

Solution for multiple fields

For $U(2)$ sector organise n copies of fields $\{X, Y\}$ into reps

$$V_{\mathbf{2}}^{\otimes n} = \bigoplus_{|\Lambda|=n} V_\Lambda^{U(2)} \otimes V_\Lambda^{S_n}$$

Can then write all multitrace operators as

$$|\Lambda, M; \alpha, \gamma\rangle \equiv \frac{1}{n!} \sum_{\sigma \in S_n} S_{a\gamma}^{\Lambda,\alpha}\ B_{b\beta}^{\Lambda,\vec{\mu}}\ D_{ab}^{\Lambda}(\sigma)\ \mathrm{tr}(\sigma^{-1}\alpha\sigma\ \overbrace{X\cdots X}^{\mu_1}\ \overbrace{Y\cdots Y}^{\mu_2})$$

- Λ tells us the rep. of $U(2)$ (a two-row n-box Young diagram)
- M tells us the state within that rep.
- α is a partition of n giving the trace structure
- γ labels the multiplicity for this Λ and α; no. of values is

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} \chi_\Lambda(\rho)$$

Example operators

$$\left|\Lambda = \yng(2,2), M = HWS; \alpha = [4], \gamma = 1\right\rangle = \mathrm{tr}([X,Y][X,Y])$$

$$\left|\Lambda = \yng(2,2), M = HWS; \alpha = [2,2], \gamma = 1\right\rangle = \mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)$$

$$\left|\yng(4,2), HWS; [4,2], 1\right\rangle = \mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX)$$

$$\left|\yng(4,2), HWS; [4,2], 2\right\rangle = \mathrm{tr}(XX\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s) + \frac{1}{6}\,\mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX)$$

Inner product and non-planar 2-point function

The inner product (i.e. *planar* two-point function) is diagonal

$$\langle \Lambda', M'; \alpha', \gamma' | \Lambda, M; \alpha, \gamma \rangle \propto \delta^{\Lambda\Lambda'} \delta^{MM'} \delta^{\alpha\alpha'} \delta^{\gamma\gamma'}$$

As for the half-BPS sector, the cut-and-join operators give the full non-planar free two-point function

$$\left\langle \mathcal{O}^\dagger[\Lambda', M'; \alpha', \gamma'] \;\; \mathcal{O}[\Lambda, M; \alpha, \gamma] \right\rangle_{\text{non-planar}} = \delta^{\Lambda\Lambda'} \delta^{MM'} \; N^n \; \langle \Lambda, M; \alpha', \gamma' | \; \Omega_n \; | \Lambda, M; \alpha, \gamma \rangle$$

From $U(2)$ to $PSU(2,2|4)$

This works automatically for $U(2) \to U(K_1|K_2)$. To extend these results for the free theory to the other fields of $\mathcal{N} = 4$ SYM treat the infinite-dimensional singleton rep. of $PSU(2,2|4)$ as the fundamental of $U(\infty|\infty)$. (The Λ are now unrestricted S_n reps, also known as the higher spin YT-pletons.)

However as soon as we turn on the coupling the $PSU(2,2|4)$ group structure asserts itself. Each rep Λ breaks down into an infinite number of $PSU(2,2|4)$ reps. This decomposition is tricky and not known in general. Using the technology of Schur-Weyl duality we *can* do this for e.g. $SO(6)$ and $SO(2,4)$.

One-loop

Analyse mixing with one-loop dilatation operator, e.g. $U(2)$ sector

$$: \mathrm{tr}([X, Y][\tfrac{\partial}{\partial X}, \tfrac{\partial}{\partial Y}]) :$$

Operators with anomalous dimensions have commutators $[X, Y]$ within a trace. Label them $|\Lambda, M; \alpha^a, \gamma^a\rangle$, e.g.

$$\left| \yng(2,2), HWS; [4]^a, 1^a \right\rangle = \mathrm{tr}([X, Y][X, Y])$$

$$\left| \yng(4,2), HWS; [4, 2]^a, 1^a \right\rangle = \mathrm{tr}([X, Y][X, Y])\, \mathrm{tr}(XX)$$

How do we find the quarter-BPS operators?

On general grounds the protected BPS operators must be orthogonal to those operators with anomalous dimensions in the full non-planar two-point function. So choose α^q, γ^q such that these operators are orthogonal in the planar inner product

$$\langle \Lambda, M; \alpha^a, \gamma^a | \Lambda, M; \alpha^q, \gamma^q \rangle = 0 \qquad \forall a, q$$

The $\frac{1}{4}$-BPS ops. are defined with the inverse of the tree-level mixer

$$\boxed{\tfrac{1}{4}\text{-BPS} \;= \Omega_n^{-1} \; |\Lambda, M; \alpha^q, \gamma^q\rangle}$$

$$\text{for} \qquad \Omega_n^{-1} = 1 - \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}\left[\frac{n(n-1)}{2} + 2\Sigma_{[3]} + \Sigma_{[2,2]}\right] + \mathcal{O}\left(\frac{1}{N^3}\right)$$

Now they are guaranteed to be orthogonal in the non-planar two-point function because $\Omega_n \Omega_n^{-1} = 1$.

Quarter-BPS examples

$$
\begin{aligned}
\Omega_n^{-1} \left| \yng(2,2) , HWS; [2,2]^q, 1^q \right\rangle = &\operatorname{tr}(\Phi^r\Phi^s)\operatorname{tr}(\Phi_r\Phi_s) \\
&+ \frac{2}{N}\operatorname{tr}([X,Y][X,Y]) \\
&- \frac{2}{N^2}\operatorname{tr}(\Phi^r\Phi^s)\operatorname{tr}(\Phi_r)\operatorname{tr}(\Phi_s)
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_n^{-1} \left| \yng(4,2) , HWS; [4,2]^q, 2^q \right\rangle \\
&= \operatorname{tr}(XX\Phi^r\Phi^s)\operatorname{tr}(\Phi_r\Phi_s) + \frac{1}{6}\operatorname{tr}([X,Y][X,Y])\operatorname{tr}(XX) \\
&\quad + \frac{8}{3N}\operatorname{tr}(\Phi^r\Phi_r\Phi^s\Phi_s XX) - \frac{16}{3N}\operatorname{tr}(\Phi^r\Phi^s\Phi_r\Phi_s XX) \\
&\quad - \frac{4}{3N}\operatorname{tr}(\Phi^r\Phi^s)\operatorname{tr}(\Phi_r\Phi_s)\operatorname{tr}(XX) \\
&\quad - \frac{1}{N}\operatorname{tr}(\Phi^r\Phi^s XX)\operatorname{tr}(\Phi_r)\operatorname{tr}(\Phi_s) - \frac{1}{6N}\operatorname{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\operatorname{tr}(X)\operatorname{tr}(X) \\
&\quad - \frac{4}{N}\operatorname{tr}(\Phi^r\Phi^s X)\operatorname{tr}(\Phi_r\Phi_s)\operatorname{tr}(X) + \frac{2}{N}\operatorname{tr}(\Phi^r\Phi^s X)\operatorname{tr}(\Phi_r X)\operatorname{tr}(\Phi_s) + \mathcal{O}\left(\frac{1}{N^2}\right)
\end{aligned}
$$

INTERMISSION

General programme for the string dual of the free theory

Want to understand exactly how the Maldacena conjecture works.

1. Find a string theory that reproduces free $\mathcal{N} = 4$ SYM, i.e. that has both the same spectrum and correlation functions.
2. Find a deformation that pushes out to weak coupling $\lambda > 0$.
3. Describe the strong/weak worldsheet duality that relates this theory to the standard large radius string on $AdS_5 \times S^5$.

Revision of non-critical $c = 1$ string

This is a string theory with a two-dimensional target space.

$c = 1$ free boson X	$\oplus$	$c = 25$ Liouville field ϕ	$\oplus$	$c = -26$ b, c ghost system

In 2d the 'tachyons' are massless

$$T_n = c\tilde{c}\ e^{inX} e^{(2-|n|)\phi}$$

Calculate tachyon scattering amplitudes using matrix quantum mechanics ($X \cong t$, Liouville direction emerges), $g_s \sim \frac{1}{\mu}$.

MQM has free fermion description very similar to MQM for half-BPS sector, except harmonic oscillator potential is inverted.

Exhibit A: $c = 1$ two-point S-matrix

For the two-point S-matrix up to genus two for the $c = 1$ string

$$\sum_{g=0} \mu^{2-2g} \left\langle T_{-n} T_n \right\rangle_g$$

$$\begin{aligned}
&\propto 1 \\
&\quad - \frac{1}{\mu^2} \left[\frac{1}{24} n(n-1)(n^2 - n - 1) \right] \\
&\quad + \frac{1}{\mu^4} \left[\frac{1}{5760} n \prod_{r=1}^{3} (n-r)(3n^4 - 10n^3 - 5n^2 + 12n + 7) \right] \\
&\quad + \mathcal{O}\left(\frac{1}{\mu^6} \right)
\end{aligned}$$

[Moore, Plesser, Ramgoolam 9111035]

Exhibit B: half-BPS two-point function

Compare this to the half-BPS result with $N = -i\mu$

$$\left\langle \mathrm{tr}(Z^{\dagger n})[x] \ \mathrm{tr}(Z^n)[0] \right\rangle$$

$$\propto 1 + \frac{1}{N^2}\left[\frac{1}{24} n(n-1)(n^2-n-2)\right] + \frac{1}{N^4}\left[\frac{1}{5760} n \prod_{r=0}^{3}(n-r)(3n^4 - 10n^3 - 15n^2 + 22n + 24)\right] + \mathcal{O}\left(\frac{1}{N^6}\right)$$

[Corley, Jevicki, Ramgoolam 0111222]

Exhibit C: compactified $c = 1$ two-point function

But for $c = 1$ string compactified at radius R

$$
\begin{aligned}
\sum_{g=0} \mu^{2-2g} \left\langle T_{-n} T_n \right\rangle_g \\
\propto 1 \\
- \frac{1}{\mu^2} \left[\frac{1}{24} n(n-1) \times \left\{ (n^2 - n - 1) - \frac{1}{R^2} \right\} \right] \\
+ \frac{1}{\mu^4} \left[\frac{1}{5760} n \prod_{r=1}^{3} (n-r) \times \left\{ (3n^4 - 10n^3 - 5n^2 + 12n + 7) \right. \right. \\
\left. \left. - \frac{10}{R^2} (n^2 - n - 1) + \frac{7}{R^4} \right\} \right] \\
+ \mathcal{O}\left(\frac{1}{\mu^6} \right)
\end{aligned}
$$

[Klebanov, Lowe (1991)]

Mapping $c = 1, R = 1$ string to half-BPS sector

$c = 1$ string at self-dual radius	half-BPS sector of $\mathcal{N} = 4$ SYM
T_n	$\mathrm{tr}(Z^n)$
T_{-n}	$\mathrm{tr}(Z^{\dagger n})$
$g_s = \frac{1}{\mu}$	$\frac{i}{N}$
k-point S-matrix	k-point function

[Jevicki, Yoneya 0612262; Tai 0701086]

Can show that correspondence holds exactly using Normal Matrix Model for $c = 1$ string of [Alexandrov, Kazakov, Kostov 0302106].

Topological cousins of $c = 1, R = 1$ string

The $c = 1$ string at self-dual radius is very special. It has many alternative topological descriptions.

- Various topological matrix models
- Topological coset $SL(2, \mathbb{R})/U(1)$ coset at $k = 3$ [Mukhi, Vafa 9301083]
- Topological Landau-Ginzburg with $W(X) = X^{-1}$ ($T_n \sim X^{n-1}$) [Ghoshal, Mukhi 9312189]
- topological B-model on deformed conifold ($T_n \sim$ deformations of geometry (like LLM?)) [Ghoshal, Vafa 9506122]
- $t \to 0$ limit of topological A-model on resolved conifold [Ooguri, Vafa 0205297]

Which do we choose to extend the map to the rest of the spectrum of $\mathcal{N} = 4$ SYM?

A Berkovits-inspired conjecture

[Berkovits 0806.1960] suggested a string dual for the free theory: a G/G principal chiral model for $G = PSU(2,2|4)$.

By gauge-fixing a subgroup of the G symmetry, the associated ghosts combine with the left-over coset to form a topological A-twisted $\mathcal{N} = 2$ theory on the coset.

Possible suggestion for the correct gauging: A-model on 'super' resolved conifold

$$\frac{U(2,2|4)}{U(1) \times U(1,2|4)}$$

Has correct symmetry, bosonic part is resolved conifold, part of cohomology is identifiable with the half-BPS operators. Need to check: rest of spectrum + correlation functions.

Conclusions

- Full non-planar free theory has a universal structure given by cut-and-join operators, with many stringy features.
  - Can we turn this into a concrete description of the dual string?
  - Can we generalise the $c = 1$ string description of the half-BPS spectrum and correlation functions to the rest of free $\mathcal{N} = 4$?
- Some features also appear in the weak coupling regime, at least in identifying the quarter-BPS operators.
  - Does any of this apply to ops with anomalous dimensions?