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Representation basis of gauge invariant operators by Brauer algebra

Yusuke Kimura (U. Oviedo)

Based on works with S. Ramgoolam and D. Turton

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Outline

- a basis of two-matrix models
- Brauer algebra
- natural classification based on an integer k which expresses the mixing of two matrices
- one-loop operator mixing
- some protected operators

The problem of AdS/CFT

$$\lambda / N (= g_{YM}^2) = g_s$$

$$\sqrt{\lambda} = R^2 / \alpha'$$

Map between string states and gauge invariant operators

$$\left\langle O_i(x)^\dagger O_j(y) \right\rangle = \frac{c(1/N)\delta_{ij}}{(x-y)^{2d_i(g,1/N)}}$$

4D *N*=*4* SYM (CFT)

Energy (in global time) of a string state = scaling dimension of a local operator

Computation of two-pt functions of the CFT is particularly important because they tell us the correspondence to string theory.

$$\left\langle O_\alpha(x)^\dagger O_\beta(y) \right\rangle = \frac{1}{(x-y)^{2d_0}} \left(S_{\alpha\beta} + T_{\alpha\beta} \log|\Lambda x| \right)$$

Operator mixing

$$\hat{D} O_\alpha = d_{\beta\alpha} O_\beta$$

The definite scaling dimension is related to the eigenstate of the dilatation operator.

Example − operator mixing

holomorphic gauge invariant ops constructed from two complex matrices, *X*, *Y*,

(This sector is closed in all order perturbation theory.)

$$X = \Phi_1 + i\Phi_2$$
$$Y = \Phi_3 + i\Phi_4$$
$$Z = \Phi_5 + i\Phi_6$$

$$D_2 = -2tr\left(\left[X, Y\right]\left[\partial_X, \partial_Y\right]\right) \equiv -2H$$

$$HO_1' = -6NO_1'$$
$$HO_a' = 0 \qquad (a = 2,3,4)$$

$$O_1' = O_1 - O_2 \qquad O_3' = O_2 - NO_4$$
$$O_2' = O_1 + \frac{N}{2}O_4 \qquad O_4' = O_3 + 2O_4$$

$$O_1 = tr(XXYY)$$
$$O_2 = tr(XYXY)$$
$$O_3 = tr(XX)tr(YY)$$
$$O_4 = tr(XY)tr(XY)$$

They all have classical dim. 4, but they mix in a nontrivial way.

c.f.) D-branes are relevant to the case of *O(N)* fields. If we do not have a good way to organise operators, it seems to be almost impossible to study the non-planar corrections.

Recall some important facts of the 1/2 BPS primary.

$$\left\langle X^i_j[x] X^{\dagger k}_l[y] \right\rangle_0 = \delta^i_l \delta^k_j \frac{1}{(x-y)^2}$$

$$O_R(X) = tr_n\left(p_R X^{\otimes n}\right) \qquad p_R = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma$$

Schur polynomial $\left\langle O_R(X)^\dagger O_S(X)\right\rangle \propto \delta_{RS}$ [Corley, Jevicki, Ramgoolam 01]

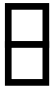
$$X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_n}_{j_n}$$

the fundamental rep of *GL(N)*

The upper indices transform as the reducible representation.

The irreducible rep *R* is projected out by acting with the projector.

$$p_{[2]} = \frac{1}{2}(1+s) \qquad \frac{1}{2}\left(X^{i_1}_{j_1} X^{i_2}_{j_2} + X^{i_2}_{j_1} X^{i_1}_{j_2}\right) \Rightarrow \frac{1}{2}\left(trXtrX + tr\left(XX\right)\right) = O_{[2]}$$



$$p_{[1,1]} = \frac{1}{2}(1-s) \qquad \frac{1}{2}\left(X^{i_1}_{j_1} X^{i_2}_{j_2} - X^{i_2}_{j_1} X^{i_1}_{j_2}\right) \Rightarrow \frac{1}{2}\left(trXtrX - tr\left(XX\right)\right) = O_{[1.1]}$$

Let me summarise the importance of the basis

(1) The multi-trace structure is conveniently summarised in a Young diagram.

You can systematically list all operators for a given number of fields. (It is easy to understand the finite N cut off.)

We can associate this with string state directly.

(2) The symmetric group can be a very efficient tool to calculate correlation functions.

See the talk by [Brown, De Mello Koch, Ramgoolam, Turton]

It will be a good idea to take the same strategy even for the multi-matrix case.

Extension to two-matrix system

[YK-Ramgoolam 0709.2158]

(1) Non-holomorphic operators of X

(2) Holomorphic operators of X, Y [su(2) sector]

$$tr(XX^\dagger), \quad tr(XX)tr(X^\dagger X)$$

$$tr(XXY), \quad tr(XX)tr(YY)$$

They have the same kinematical problem (e.g counting operators, classical correlators).

$$X^* \leftrightarrow Y^T$$

$$X = \Phi_1 + i\Phi_2$$

$$Y = \Phi_3 + i\Phi_4$$

In general, there are some ways to introduce a representation basis for two-matrix system. e.g. respecting the global symmetry, introducing the second matrix as the restriction...

[Brown, Heslop, Ramgoolam; Bhattacharyya, Collons, de Mello Koch].

Here we will introduce a basis in another way by the following idea.

Suppose the following operators are given.

$O_R(X)$, e.g. $R=[m]$ ($m \sim N$), a giant graviton with $J_1=m$

$O_S(Y)$,

Then it is very natural to think that one wants an operator dual to a composite system of these two objects. A very natural composite operator to consider will be something like

$$O_R(X)O_S(Y)+\cdots$$

Let me say that this operator is in $k=0$ (where k is introduced as a quantity to express the mixing between X and Y).

If this problem (situation) is given, it will be more natural to construct a set of ops which are classified by k.

Example

R=[1], S=[1] $N\otimes\bar{N}=\left(N^2-1\right)\oplus 1$

$$X^{i_1}\left(Y^T\right)^{j_1}=\underline{X^{i_1}\left(Y^T\right)^{j_1}-\frac{1}{N}\delta^{i_1j_1}X^k\left(Y^T\right)^k}+\underline{\frac{1}{N}\delta^{i_1j_1}X^k\left(Y^T\right)^k}$$

$$trXtrY-\frac{1}{N}tr\left(XY\right)$$ k=0

$$\frac{1}{N}tr\left(XY\right)$$ k=1

For the purpose, it seems to be convenient to use this method (i.e. transposing Y before considering the irreducible decomposition).

$$tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

Projector associated with an irreducible rep. γ of $GL(N)$

$$X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_m}_{j_m} Y^{T\,k_1}_{l_1} Y^{T\,k_2}_{l_2} \cdots Y^{T\,k_n}_{l_n} \rightarrow \gamma$$

The irreducible representation of $GL(N)$:

$$\gamma = (\gamma_+, \gamma_-, k),$$

$$0 \le k \le \min(m,n), \quad \gamma_+ \mapsto (m-k), \ \gamma_- \mapsto (n-k),$$

This k is nothing but the k we introduced to measure the mixing between X and Y.

3Xs 2Ys

$(3,2)$	γ_+	γ_-
$k=0$	$[3]$	$[2]$
	$[2,1]$	$[2]$
	$[1,1,1]$	$[2]$
	$[3]$	$[1,1]$
	$[2,1]$	$[1,1]$
	$[1,1,1]$	$[1,1]$
$k=1$	$[2]$	$[1]$
	$[1,1]$	$[1]$
$k=2$	$[1]$	$\emptyset$

Examples

1 X 1 Y

$$O^{\gamma(k=0)} = trXtrY - \frac{1}{N} tr(XY) \qquad O^{\gamma(k=1)} = \frac{1}{N} tr(XY)$$

2 Xs 1 Y

$$O^{\gamma(k=0,[2],[1])} = \frac{1}{2}\left((trX)^2 + tr(X)^2\right)trY - \frac{1}{N+1}\left(trXtr(XY) + tr(X^2Y)\right)$$

$$O^{\gamma(k=0,[1,1],[1])} = \frac{1}{2}\left((trX)^2 - tr(X)^2\right)trY - \frac{1}{N+1}\left(trXtr(XY) - tr(X^2Y)\right)$$

$$O^{\gamma(k=1,[1],[0])} = \frac{2}{N^2-1}\left(NtrXtr(XY) - tr(X^2Y)\right)$$

$(2,1)$	γ_+	γ_-
$k=0$	[2]	[1]
	[1, 1]	[1]
$k=1$	[1]	$\emptyset$

The projection operator can be expressed by elements of the *Brauer algebra*.

$$X^{i_1}\left(Y^T\right)^{j_1} = X^{i_1}\left(Y^T\right)^{j_1} - \frac{1}{N}\delta^{i_1 j_1} X^k\left(Y^T\right)^k + \frac{1}{N}\delta^{i_1 j_1} X^k\left(Y^T\right)^k$$

$$C^2 = NC$$

Contraction C $\qquad C \cdot X^{i_1}\left(Y^T\right)^{j_1} = \delta^{i_1 j_1} X^k\left(Y^T\right)^k$

Projectors at $(m,n)=(1,1)$

$$(p^{(k=0)})^2 = p^{(k=0)},\ (p^{(k=1)})^2 = p^{(k=1)},$$
$$p^{(k=0)} p^{(k=1)} = 0$$

$$p^{(k=0)} = 1 - \frac{C}{N},\quad p^{(k=1)} = \frac{C}{N}$$

$$tr(X)tr(Y) - \frac{1}{N}tr(XY),\quad \frac{1}{N}tr(XY)$$

N-dependence (difference from the symmetric group)

Brauer algebra $B_N(m,n)$ = the group algebra of Symmetric group $S_m \times S_n$ + contractions

S_m acts on indices of *X*s, S_n on indices of *Y*s. A contraction can acts on a upper index of *X* and a lower index of *Y*.

Schur-Weyl duality

The action of *GL(N)* commutes with the action of the Brauer algebra. (The Brauer algebra is the commutant of *GL(N)*.)

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} \left(V_{\gamma}^{GL(N)} \otimes V_{\gamma}^{B_N(m,n)} \right)$$

Irreps of *GL(N)* = irreps of Brauer algebra

(In this sense, we may say that *GL(N)* is dual to the Brauer algebra on the space.)

Comment on the complete set

[YK Ramgoolam 0709.2158, 0807.3696]

$$tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

The number of the operators is not enough to provide a complete set. A complete basis is given by

$$O^{\gamma}_{A,ij}(X,Y) \equiv tr_{m,n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes (Y^T)^{\otimes n}\right)$$

$$P^{\gamma} = \sum_{A,i} Q^{\gamma}_{A,ii}$$

A is an irreducible representation of $S_m \times S_n$, and *i,j* are indices associated with the embedding of *A* in γ. (*Q* takes out *A* inside γ.)

$$\left\langle O^{\gamma_1}_{A_1,i_1 j_1}(X,Y)^{\dagger} O^{\gamma_2}_{A_2,i_2 j_2}(X,Y)\right\rangle_0 \propto \delta^{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}\delta_{j_1j_2}$$

But when $k=0$, $i,j=1$ and $\gamma=A=(R,S)$.

$$Q^{\gamma(k=0)}_{A,ij} = P^{\gamma(k=0,\gamma_+,\gamma_-)} = P_{RS}$$

R : *Young diagram with m boxes.*

S : *Young diagram with n boxes.*

The k=0 Sector

For a physics, [talk by D.Turton].

$$\gamma=(R,S),\quad R\mapsto m,\ S\mapsto n,$$

[YK-Ramgoolam 07]

$$tr_{m,n}\left(P_{RS}X^{\otimes m}\otimes Y^{T\otimes n}\right)\propto tr_{m+n}\left(\Omega_{m+n}{}^{-1}p_R p_S X^{\otimes m}\otimes Y^{\otimes n}\right)$$

The omega factor is a specific sum of central elements of the symmetric group S_{m+n}. (Recall the importance in the large N expansion of 2DYM.)

$$\Omega_{m+n}{}^{-1}\propto 1+O\left(1/N\right)\cdots$$

$$\Omega_{m+n}=\sum_{\sigma\in S_{m+n}}N^{C_\sigma-(m+n)}\sigma$$

$$tr_{m.n}\left(P^\gamma X^{\otimes m}\otimes Y^{T\otimes n}\right)=tr_m\left(p_R X^{\otimes m}\right)tr_n\left(p_S Y^{\otimes m}\right)+\cdots$$

(I will see this operator is protected against quantum corrections.)

$$CP^{\gamma(k=0)} = 0$$

$$P^{\gamma(k\neq 0)}P^{\gamma(k=0)} = 0$$

(Definition of irreducible representation.)

The projector of the irreducible representations in $k \neq 0$ is proportional to the contraction.

$$X^{i_1}\left(Y^T\right)^{j_1} = \underline{X^{i_1}\left(Y^T\right)^{j_1} - \frac{1}{N}\delta^{i_1 j_1}X^k\left(Y^T\right)^k} + \underline{\frac{1}{N}\delta^{i_1 j_1}X^k\left(Y^T\right)^k}$$

$$p^{(k=0)} = 1 - \frac{C}{N}, \quad p^{(k=1)} = \frac{C}{N}$$

You will see this equation is very powerful and interesting.

$tr_{m.n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes X^{*\otimes n}\right)$

When we discuss a composite operator containing the complex conjugate, we have to regularise it properly by point-spilitting e.g.

$$\left\langle X^{\dagger i}{}_{j}(x) X_{l}^{k}(0) \right\rangle \propto \delta_{l}^{i} \delta_{j}^{k} \frac{1}{x^{2}}$$

The *k=0* operators are special in the sense that they are well-defined without regularisation because of the property *CP=0*.

The contraction can be considered as a linear map acting on $V \otimes \overline{V}$

$$(C)^{il}_{jk} = \delta^{il} \delta_{jk} \qquad \left\langle X^{*} \otimes X \right\rangle \propto C$$

$$\left\langle tr_{m.n}\left(P^{\gamma(k=0)} X^{\otimes m} \otimes X^{*\otimes n}\right)\right\rangle = 0 \quad \Leftarrow \quad \boxed{CP^{\gamma(k=0)} = 0}$$

For the other sector, we need a regularisation.

The *k*=*m*=*n* Sector

$$\gamma = (\gamma_+, \gamma_-, k = m), \quad \gamma_+ \mapsto 0, \ \gamma_- \mapsto 0,$$

[YK-Ramgoolam-Turton 09][YK 10]

$$tr_{m.n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes Y^{T\otimes n}\right) \propto tr_k\left(p_\alpha S^{\otimes k}\right)$$

$$S = XY$$

X and Y are completely combined.

α : *Young diagram with k boxes.*

Brief summary (before going to the next story)

Representation basis

$$\langle O_s(X[x],Y^T[x])^\dagger O_t(X[0],Y^T[0])\rangle_0 = \delta_{s,t}\frac{C(N)}{|x-y|^{2(m+n)}}$$

s, t are sets of labels characterised by the Brauer algebra.

$$s,t=(\gamma,A,i,j)$$

$$\gamma=(\gamma_+,\gamma_-,k),$$

$$0\le k\le \min(m,n),\quad \gamma_+\mapsto(m-k),\ \gamma_-\mapsto(n-k),$$

$$A=(\alpha,\beta),$$

$$\alpha\mapsto m,\ \beta\mapsto n,$$

The basis managed by the Brauer algebra are naturally classified by the integer k, which (roughly speaking) expresses how much two matrices are mixed.

One-loop analisis

[YK 1002.2424]

In the remaining part of this talk, I will discuss the one-loop physics of the su(2) sector.

$$\widehat{D}_2 = -2tr\left([X,Y][\partial_X,\partial_Y]\right) \equiv -2\hat{H}$$

The problem is to find eigenstates of *D*. Lets see how this acts on the basis given by the Brauer algebra.

$$(\partial_X \partial_Y)_{pq} X^i_j Y^{Tk}_{\ \ l} = \underline{\delta_{ik}} \delta_{pj} \delta_{ql} \qquad CX^i_j Y^{Tk}_{\ \ l} = \underline{\delta_{ik} X^s_j} Y^{Ts}_{\ \ l}$$

$$(\partial_Y \partial_X)_{pq} X^i_j Y^{Tk}_{\ \ k} = \underline{\delta_{jl}} \delta_{pk} \delta_{qi} \qquad X^i_j Y^{Tk}_{\ \ l} \overleftarrow{C} = \underline{\delta_{jl} X^i_s} Y^{Tk}_{\ \ s}$$

$$\hat{H}\ tr_{m.n}\left(P^{\gamma(k=0)} X^{\otimes m} \otimes Y^{T\otimes n}\right) = 0 \quad \Leftarrow \quad CP^{\gamma(k=0)} = 0$$

$$\hat{H}O = \sum_{r,s} tr_{m,n}(bC_{r,s}\mathrm{P}_{r,s}X^{\otimes m}\otimes Y^{T\otimes n}) - \sum_{r,s} tr_{m,n}(bC_{r,s}X^{\otimes m}\otimes Y^{T\otimes n}) - \sum_{r,s} tr_{m,n}(C_{r,s}bX^{\otimes m}\otimes Y^{T\otimes n}) + \sum_{r,s} tr_{m,n}(C_{r,s}b\mathrm{P}_{r,s}X^{\otimes m}\otimes Y^{T\otimes n})$$

$$O = tr_{m.n}\left(bX^{\otimes m}\otimes Y^{T\otimes n}\right)$$

(The action of *H* is expressed by the element of Brauer algebra.)

where we have introduced the operation $\mathbf{P}_{r,s}$ to exchange the r-th X with the s-th Y

$$\mathbf{P}_{r,s}X^{\otimes m}\otimes(Y^T)^{\otimes n} = X^{\otimes r-1}\otimes Y\otimes X^{m-r}\otimes(Y^T)^{\otimes s-1}\otimes X^T\otimes(Y^T)^{n-s}.$$

$$\hat{H}O = \sum_{r,s} tr_{m,n}\left(\left[b, C_{r,s}\right]X^{\otimes r-1}[X,Y]\otimes X^{\otimes m-r}\otimes Y^{T\otimes s-1}\otimes 1\otimes Y^{T\otimes n-s}\right)$$

Similar expression for the other basis [de Mello Koch, Meshile, Park]

$$\gamma = (\gamma_+, \gamma_-, k),$$
$$0 \le k \le \min(m,n), \quad \gamma_+ \mapsto (m-k), \; \gamma_- \mapsto (n-k)$$

The projector associated with an irreducible rep. of the Brauer algebra commutes with any element in the algebra (Schur' s lemma).

$$C_{r,s}P^{\gamma} = P^{\gamma}C_{r,s} \qquad C_{r,s}P^{\gamma(k=0)} = 0$$

$$\hat{H}\ tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right) = 0$$

$$P^{\gamma} = \sum_{A,i} Q^{\gamma}_{A,ii}$$

This is valid for any *m*, *n*, *N*.

Number of protected operators we found here

$$\sum_{k=0}^{min(m,n)} p(m-k)p(n-k)$$

p(l) is the partition of *l*.

$3Xs \quad 2Ys$

$(3,2)$	γ_+	γ_-
$k=0$	[3]	[2]
	[2, 1]	[2]
	[1, 1, 1]	[2]
	[3]	[1, 1]
	[2, 1]	[1, 1]
	[1, 1, 1]	[1, 1]
$k=1$	[2]	[1]
	[1, 1]	[1]
$k=2$	[1]	$\emptyset$

X and *Y* are always combined after the action of the dilatation operator.

This means the (leading term of) *k=0* operators can not appear as the image of the dilatation operator.

In this sense, the *k=0* operators are not mixed with the other sectors (*k≠0*).

$$\left\langle O^{\gamma(k=0)}(X,Y)^{\dagger} O^{\gamma_2 (k\neq 0)}_{A_2, i_2 j_2}(X,Y)\right\rangle_1 = 0$$

$$\left\langle O^{\gamma_1(k=0)}(X,Y)^{\dagger} O^{\gamma_2 (k=0)}(X,Y)\right\rangle_1 \propto \delta^{\gamma_1\gamma_2}$$

Comment on the classification

BPS operators are classified by the global symmetry. (descendants of 1/2 BPS, descendants of 1/4 BPS, primary of 1/4 BPS...)

In general, our operators are linear combinations of them. (But the global symmetry is not so manifest in this method, so it is not so easy to identify which with which..)

In other words,

This means some BPS operators are combined to be classified by the representation of GL(N).

Summary

$$O^{\gamma}_{A,ij}(X,Y) \equiv tr_{m,n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

- *Q* is a linear combination of elements in the Brauer algebra
- Orthogonal at classical level

$$\left\langle O_s^{\dagger} O_t \right\rangle_0 \propto \delta_{st} \qquad s,t=(\gamma, A, i, j)$$

$$O^{(k=0)}_{RS}(X,Y) \equiv tr_{m,n}\left(P_{RS} X^{\otimes m} \otimes Y^{T\otimes n}\right) \qquad \gamma=(k,\gamma_+,\gamma_-)$$

$$= tr_m\left(p_R X^{\otimes m}\right) tr_n\left(p_S Y^{\otimes m}\right) + \cdots$$

- Leading term is the product of Schur polynomials
- Protected against quantum correction
- Not mixed with the other class *(k≠0)*

(Positive) my impressions I get from this workshop

- Using the dual algebra or group (in this talk, the Brauer algebra, in the sense of Schur-Weyl duality), seems to be a promising way to understand the non-planar physics.
- Young diagrams would know about how to summarise the non-planar corrections in an efficient way.