SU(2) Yang-Mills Thermodynamics and Radiation in our Universe

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Yang-Mills action

- (thermal) Yang-Mills
  [Pauli, Barker, and Gulmanelli (1953); Yang and Mills (1954)]

\[
S = \frac{\text{tr}}{2} \int_0^\beta d\tau \int d^3x F_{\mu\nu} F_{\mu\nu},
\]

where \( g \) is (dimensionless) coupling, \( \beta \equiv 1/T \),
\( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \), and
\( A_\mu \equiv A^a_\mu t^a \rightarrow \Omega A_\mu \Omega^\dagger + i\Omega \partial_\mu \Omega^\dagger \) \((\Omega(x) \in G)\) is gauge field
such that \( F_{\mu\nu} \rightarrow \Omega F_{\mu\nu} \Omega^\dagger \) and thus \( S \) is gauge invariant.

- at \( T > 0 \): admissible changes of gauge respect periodicity
  of \( A_\mu \)

- in evaluating partition function \( Z \equiv \sum \{ A_\mu \} \ e^{-S} \) in
  fundamental fields: Additional gauge fixing required \( \Rightarrow \)
  1) Faddeev-Popov in PT
  2) restriction to Gribov region (or better) otherwise
Propagating modes

- loop expansion of $N$-point functions in momentum space, propagator $\bar{D}$

$$\bar{D}(\mathbf{p}, \omega_n) \sim \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2},$$

where $\omega_n \equiv 2\pi nT \ (n \in \mathbb{Z})$ $n$th Matsubara frequency.

- re-expressing (but not changing the contour for $\tau$ integration in Euclid. action) summation over $n$ by Cauchy’s theorem $\Rightarrow$

$$- \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2} \rightarrow \frac{i}{\mathbf{p}^2 - m^2} + \delta(p^2 - m^2) \frac{2\pi}{e^{\beta|\mathbf{p}_0|} - 1},$$

where $\sum_n \int d^3p \rightarrow \int d^4p$. 
Real-time interpretation of loop integrals

Remarks:

- A more elaborate $\tau$ integration contour in the action was considered in [Umezawa, Matsumoto, and Tachiki (1982), Niemi and Semenoff (1984)]. This doubles real-time DOE to avoid pinch singularities in PT.

- In Yang-Mills, where selfdual (nonpropagating) field configurations contribute to ground-state physics, such a change of contour for physics of propagating excitations is inconsistent.
Trivial-holonomy calorons

- in singular gauge (winding number $|k| = 1$ is localized in a point) there is a superposition principle of instanton centers in prepotential $\Pi$ ['t Hooft (1976), Jackiw and Rebbi (1976)]:

$$\bar{A}_\mu^+, a(x) = -\bar{\eta}^a_{\mu\nu} \partial_\nu \log \Pi,$$
$$\bar{A}_\mu^-, a(x) = -\eta^a_{\mu\nu} \partial_\nu \log \Pi.$$

- can be used to satisfy at $|k| = 1$ periodic b.c. in strip $(0 \leq \tau \leq \beta) \times \mathbb{R}^3$ [Harrington and Shepard (1978)]:

$$\Pi(\tau, x; \rho, \beta, x_0) = 1 + \sum_{l=-\infty}^{l=\infty} \frac{\rho^2}{(x - x_l)^2}$$

$$= 1 + \frac{\pi \rho^2}{\beta r} \frac{\sinh \left( \frac{2\pi r}{\beta} \right)}{\cosh \left( \frac{2\pi r}{\beta} \right) - \cos \left( \frac{2\pi \tau}{\beta} \right)},$$

where $r \equiv |x|$. 
Trivial-holonomy calorons, cntd.

- holonomy of $\tilde{A}_{\mu}^{\pm,a}(x)$ at $r \to \infty$ trivial:

$$\Pi_{r \to \infty} 1 + \frac{\pi \rho^2}{\beta r} \Rightarrow \lim_{r \to \infty} \tilde{A}_{4}^{\pm} \propto \lim_{r \to \infty} \frac{1}{r^2} = 0 \Rightarrow$$

$$\mathcal{P} \exp \left[ i \int_{0}^{\beta} d\tau \tilde{A}_{4}^{\pm} \right] = 1_2.$$ 

- Gaussian quantum weight [Gross, Pisarski, and Yaffe (1981)]:

$$S_{\text{eff}} = \frac{8\pi^2}{g^2} + \frac{4}{3}\sigma^2 + 16 A(\sigma) \quad (\sigma \equiv \pi \frac{\rho}{\beta}),$$

$$A(\sigma) \to -\frac{1}{6} \log \sigma \quad (\sigma \to \infty) \quad A(\sigma) \to -\frac{\sigma^2}{36} \quad (\sigma \to 0).$$

Conclusion of **semiclassical approx.**:
Trivial-holonomy-caloron weight exponentially suppressed at high $T$. 

Nontrivial holonomy: Magnetic dipoles


- explicitly carried out in [Lee and Lu 1998, Kraan and Van Baal 1998]: $A_4(\tau, r \to \infty) = -iut^3(0 \leq u \leq \frac{2\pi}{\beta})$.

Action density of nontrivial-holonomy caloron with $k = 1$ plotted on 2D spatial slice

Exact cancellation between $A_4$-mediated repulsion and $A_i$-mediated attraction; caloron radius $\rho$ and thus monopole-core separation $D = \frac{\pi}{\beta} \rho^2$ increase from left to right ($T$ and holonomy fixed)
Nontrivial holonomy, cntd.

computation of functional determinant about nontrivial holonomy carried out in [Gross, Pisarski, and Yaffe (1981), Diakonov et al. 2004], in latter paper for (relevant) limit \( \frac{D}{\beta} = \pi \left( \frac{\rho}{\beta} \right)^2 \gg 1 \)

conclusions:

- **total suppression** for nontrivial static holonomy in limit \( V \to \infty \)

- **attraction** of monop. and antimonop. for small holonomy \( (0 \leq u \leq \frac{\pi}{\beta} (1 - \frac{1}{\sqrt{3}}); \frac{\pi}{\beta} (1 + \frac{1}{\sqrt{3}}) \leq u \leq 2 \frac{\pi}{\beta} ) \)

- **repulsion** of monop. and antimonop. for large holonomy \( (\frac{\pi}{\beta} (1 - \frac{1}{\sqrt{3}}) \leq u \leq \frac{\pi}{\beta} (1 + \frac{1}{\sqrt{3}})) \)

- **Instability** of classical configuration under quantum noise \( \Rightarrow \) Nontrivial holonomy does not enter a priori estimate of thermal ground state!
Inert field $\phi$: A priori estimate of thermal ground state

Observations and principles constraining construction of $\phi$:

- $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \Rightarrow$ vanishing energy-momentum:

$$\Theta_{\mu\nu} = -2 \text{tr}\left\{ \delta_{\mu\nu} \left( \mp E \cdot B \pm \frac{1}{4} (2E \cdot B + 2B \cdot E) \right) \right. \left. \mp (\delta_{\mu 4}\delta_{\nu i} + \delta_{\mu i}\delta_{\nu 4})(E \times E)_i \right. \left. \mp \delta_{\mu i}\delta_{\nu(j \neq i)}(E_i B_j - E_j B_i) \pm \delta_{\mu(j \neq i)}\delta_{\nu i}(E_j B_i - E_j B_i) \right\} \equiv 0.$$

- spatial isotropy and homogeneity of effective local field not associated with propagation of energy-momentum by coarse-grained (anti)calorons $\Rightarrow$ inert scalar $\phi$

- modulo admissible gauge transformations $\phi$ does not depend on time

- relevance of $\phi$ (BPS) by gauge-invariant coupling to coarse-grained $k = 0$ sector (perturbative renormalizability) $\Rightarrow$ $\phi$ adjoint scalar
Inert field $\phi$

Observations and principles constraining construction of $\phi$, cntd:

$F_{\mu\nu} \equiv \pm \tilde{F}_{\mu\nu} \Rightarrow$ any local “power” of $F_{\mu\nu}$ with an insertion of $t^a$ vanishes

only trivial holonomy in $F_{\mu\nu} \equiv \pm \tilde{F}_{\mu\nu}$ allowed

$|\phi|$ is spacetime homogeneous $\Rightarrow$ information on $\phi$’s EOM is encoded in phase $\hat{\phi} \equiv \frac{\phi}{|\phi|}$

definition of possible phases $\{\hat{\phi}\}$: due to BPS of $A_{\mu}^{\pm}$ no explicit $T$ dependence, flat measure for admissible integration over moduli (excluding temporal shifts and global gauge rotations), Wilson lines between spatial points along straight lines
Inert field $\phi$

**Unique** definition of $\{\hat{\phi}\}$ [Herbst and RH 2004]:

$$\{\hat{\phi}^a\} \equiv \sum_\pm \text{tr} \int d^3x \int d\rho \ t^a \ F_{\mu\nu}(\tau, 0) \ \{(\tau, 0), (\tau, x)\} \times F_{\mu\nu}(\tau, x) \ \{(\tau, x), (\tau, 0)\},$$

where

$$\{(\tau, 0), (\tau, x)\} \equiv \mathcal{P} \exp \left[ i \int_{(\tau, 0)}^{(\tau, x)} dz_\mu A_\mu(z) \right],$$

$$\{(\tau, x), (\tau, 0)\} \equiv \{(\tau, 0), (\tau, x)\}^\dagger,$$

and sum is over **Harrington-Shepard** (trivial-holonomy) caloron and anticaloron of scale $\rho$.

Higher $n$-point functions, higher topol. charge $k$? **No.**
(Would introduce mass dimension $d = 3 - n - m$ of object, $m > 1$ number of dimension-length caloron moduli at $k > 1$, but $d$ needs to vanish.)
Inert field $\phi$

Some observations, conventions:

- $\hat{\phi}$ indeed transforms as an adjoint scalar:
  
  $\hat{\phi}^a(\tau) \rightarrow R^{ab}(\tau)\hat{\phi}^b(\tau)$,

  where $R^{ab}$ is $\tau$ dependent matrix of adjoint rep.

- What about shift of spatial center $0 \rightarrow z_{\pm}$?

Shift of center amounts to spatially *global* gauge rotation induced by the group element

\[ \Omega^{\pm}_{z} = \{(\tau, 0), (\tau, z_{\pm})\} \].

(a) graphical representation of definition
(b) only possible generalization to $z_{\pm} \neq 0$
Inert field $\phi$

Some observations, conventions, cntd:

- one has

$$\int_{(\tau,0)}^{(\tau,x)} dz_\mu A_\mu(z)|_\pm = \pm \int_0^1 ds \, x_i A_i(\tau, sx)$$

$$= \pm t_b x_b \partial_\tau \int_0^1 ds \, \log \Pi(\tau, sr, \rho) \Rightarrow$$

integrand in the exponent of $\{(\tau, 0), (\tau, x)\}_\pm$ varies along a fixed direction in $\text{su}(2)$ (a hedge hog); **Path-ordering can be ignored.**

- temporal shift freedom in $A_{\mu}^\pm$: set $\tau_\pm = 0$ and re-instate later

- parity: $F_{\mu\nu}(\tau, x)_+ = F_{\mu\nu}(\tau, -x)_-$ and

$$\{(\tau, 0), (\tau, x)\}_+ = \{(\tau, x), (\tau, 0)\}_+^\dagger = \{(\tau, 0), (\tau, -x)\}_-$$

$$= \{(\tau, -x), (\tau, 0)\}_-^\dagger \Rightarrow$$

- contribution to the integrand in **definition** obtained by $x \to -x$ in $+$ contribution
Inert field $\phi$

Some observations, conventions, cntd:

after tedious computation \cite{Herbst and RH 2004}

+ contribution to integrand in **definition** reads:

$$
- i \beta^{-2} \frac{32 \pi^4}{3} \frac{x^a}{r} \frac{\pi^2 \hat{\rho}^4 + \hat{\rho}^2 (2 + \cos(2\pi \hat{\tau}))}{(2\pi^2 \hat{\rho}^2 + 1 - \cos(2\pi \hat{\tau}))^2} \times F[\hat{g}, \Pi],
$$

where $\hat{\rho} \equiv \frac{\rho}{\beta}$, $\hat{r} \equiv \frac{r}{\beta}$, $\hat{\tau} \equiv \frac{\tau}{\beta}$, and functional $F$ is

$$
F[\hat{g}, \Pi] = 2 \cos(2\hat{g}) \left( 2 \frac{[\partial_{\tau} \Pi][\partial_r \Pi]}{\Pi^2} - \frac{\partial_{\tau} \partial_r \Pi}{\Pi} \right) 
+ \sin(2\hat{g}) \left( 2 \frac{[\partial_r \Pi]^2}{\Pi^2} - 2 \frac{[\partial_{\tau} \Pi]^2}{\Pi^2} + \frac{\partial_{\tau}^2 \Pi}{\Pi} - \frac{\partial_r^2 \Pi}{\Pi} \right),
$$

and

$$
\{(\tau, 0), (\tau, x)\}_\pm \equiv \cos \hat{g} \pm 2it_b \frac{x^b}{r} \sin \hat{g}.
$$

One shows that $\hat{g}$ saturates exponentially fast for $\hat{r} > 1$. 
Inert field $\phi$

discussion:

- angular integration would yield $\textbf{zero}$ if radial integration was regular
- **but:** radial integration diverges logarithmically due to term $\frac{\partial^2 \Pi}{\Pi}$; this term arises from the $\textbf{magnetic-magnetic}$ correlation (no convergence in PT due to weakly screened magnetic sector!)
- $\textbf{zero} \times \textbf{infinity}$ yields undetermined, multiplicative, and real constants $\Xi_{\pm}$
- without restriction of generality (global choice of gauge), angular integration regularized by defect azimuthal angle in 1-2 plane of su(2) for both $+$ and $-$ contributions \(\Rightarrow\) **Members of \{\hat{\phi}\} all move in hyperplane of su(2)!**
- re-instate $\tau \rightarrow \tau + \tau_{\pm}$ \(\Rightarrow\)
Inert field $\phi$

discussion, cntd:

result:

$$\{\hat{\phi}^a\} = \left\{\Xi_+ (\delta^a_1 \cos \alpha_+ + \delta^a_2 \sin \alpha_+) \mathcal{A} \left(2\pi(\hat{\tau} + \hat{\tau}_+\right)) \right. + \Xi_- (\delta^a_1 \cos \alpha_- + \delta^a_2 \sin \alpha_-) \mathcal{A} \left(2\pi(\hat{\tau} + \hat{\tau}_-\right))\right\}, \quad \text{where}$$

$\tau$ dependence of function $\mathcal{A}(\frac{2\pi \tau}{\beta});$
saturation property (cutoff independence) for $\hat{\rho}$ integration.
\( \zeta \) dependence of \( \Xi_\pm \)

\[ \rho_{\text{max}} \equiv \zeta \beta : \]

\[ \int d\rho \rightarrow \int_{0}^{\zeta \beta} d\rho, \quad (\zeta > 0). \]

- \( \Xi_\pm = 272 \zeta^3 \times \text{unknown, fixed real}, \quad (\zeta > 5) \)
- integral over \( \rho \) is strongly dominated by contributions just below upper limit
- since upper limit set by \( |\phi|^{-1} \) (yet to be determined), only (anti)calorons with \( \rho \sim |\phi|^{-1} \) contribute to effective theory
- since \( \zeta_\phi \equiv (|\phi|\beta)^{-1} \geq 8.22 \) (later) semiclassical analysis of nontrivial-holonomy calorons in limit

\[ \frac{D}{\beta} = \pi \left( \frac{\rho}{\beta} \right)^2 \geq (8.22)^2 \times \pi \gg 1 \ [\text{Diakonov et al. 2004}] \text{ is justified.} \]
Kernel of a differential operator $D$ and potential for $\phi$

- set $\{\hat{\phi}\}$ contains two real parameters for each “polarization”: $\Xi_{\pm}$ and $\tau_{\pm}$; $\{\hat{\phi}\}$ is annihilated by linear, second-order differential operator $D = \partial_\tau^2 + \left(\frac{2\pi}{\beta}\right)^2$ $\Rightarrow$

  $\{\hat{\phi}\}$ coincides with kernel of $D$ and determines $D$ uniquely

- linearity $\Rightarrow$ also $D\phi = 0$

- **but:** $D$ depends on $\beta$ explicitly, not allowed (BPS, caloron action given by topolog. charge)

- therefore seek potential $V(|\phi|^2)$ such that (Euclidean) action principle applied to

\[
L_\phi = \text{tr} \left( (\partial_\tau \phi)^2 + V(\phi^2) \right).
\]

yields solutions annihilated by $D$, where $L_\phi$ does not depend on $\beta$ explicitly; demand that energy density $\Theta_{44} = 0$ on those solutions
Potential $V(\phi^2)$ and modulus of $\phi$

- pick motion in 1-2 plane of su(2) (gauge invariance $\Rightarrow V$ \textbf{central} potential $\Rightarrow$ cons. angular momentum); ansatz:
  \[ \phi = 2 |\phi| t_1 \exp(\pm \frac{4\pi i}{\beta} t_3 \tau). \]
  (circular motion in 1-2 plane, $|\phi|$ time independent!)
- apply E-L to $\mathcal{L}_\phi$ $\Rightarrow$
  \[ \partial^2_\tau \phi^a = \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \phi^a \text{ (in components)} \iff \]
  \[ \partial^2_\tau \phi = \frac{\partial V(\phi^2)}{\partial \phi^2} \phi \text{ (in matrix form)}. \]
- $\Theta_{44} = 0$ on ansatz $\phi$ $\Rightarrow$ $|\phi|^2 \left(\frac{2\pi}{\beta}\right)^2 - V(|\phi|^2) = 0$ but also:
  \[ \partial^2_\tau \phi + \left(\frac{2\pi}{\beta}\right)^2 \phi = 0 \Rightarrow \]
  \[ \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} = -\frac{V(|\phi|^2)}{|\phi|^2}. \]
Potential $V(\phi^2)$ and modulus of $\phi$, cntd

$\Rightarrow V(|\phi|^2) = \frac{\Lambda^6}{|\phi|^2}$

where $\Lambda$ integration constant of mass dim. unity.

$\Rightarrow |\phi| = \sqrt{\frac{\Lambda^3 \beta}{2\pi}}$ (power-like decay of field $\phi$ with increasing $T$)

The field $\phi$ describes coarse-grained effect of noninteracting trivial-holonomy calorons and anticalorons. It does not propagate, and its modulus $|\phi|$ sets the scale of maximal off-shellness of intermediates in effective theory.

Indeed: cutting off $\rho$ and $r$ integrations at $|\phi|^{-1}$, $\tau$

dependence of $A(\frac{2\pi \tau}{\beta})$ is perfect sine

(Error at level smaller than $10^{-22}$ if knowledge about $T_c = \frac{\lambda_c \Lambda}{2\pi}$ with $\lambda_c = 13.87$ is used, later.)
BPS equation for $\phi$

In addition to E-L equation $\phi$ satisfies first-order, BPS equation:

$$\partial_{\tau}\phi = \pm 2i \Lambda^3 t_3 \phi^{-1} = \pm i V^{1/2}(\phi).$$

Because $\phi$ satisfies both, second-order E-L and first-order BPS equation, usual shift ambiguity in ground-state energy density, as allowed by E-L equation, absent in SU(2) Yang-Mills thermodynamics.
Effective action for deconfining phase

Coupling the coarse-grained $k = 0$ sector to $\phi$, following constraints:

- **perturbative renormalizability**
  [’t Hooft, Veltman, Lee, and Zinn-Justin 1971-1973] ⇒ form invariance of action for effective $k = 0$ gauge field $a_\mu$
  from integrating fundamental $k = 0$ fields only, no higher dim. ops. constr. from $a_\mu$ only

- **no energy-momentum transfer to $\phi$** ⇒ absence of higher dim. ops. involving $a_\mu$ and $\phi$

- **gauge invariance** ⇒ $\partial_\mu \phi \rightarrow D_\mu \phi \equiv \partial_\mu \phi - ie[a_\mu, \phi]$
  (e effective coupling); no momentum transfer to $\phi$ (unitary gauge $\phi = 2|\phi| t_3$), massive 1,2 modes propagate on-shell only

\[ \text{(a)} \]
\[ \text{(b)} \]
Effective action and ground-state estimate

unique effective action density:

\[ \mathcal{L}_{\text{eff}}[a_\mu] = \text{tr} \left( \frac{1}{2} G_{\mu\nu} G_{\mu\nu} + (D_\mu \phi)^2 + \frac{\Lambda^6}{\phi^2} \right), \]

where \( G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - ie[a_\mu, a_\nu] \equiv G_{\mu\nu}^a t_a \)

ground-state estimate:

▷ E-L EOM from \( \mathcal{L}_{\text{eff}}[a_\mu] \)

\[ D_\mu G_{\mu\nu} = ie[\phi, D_\nu \phi]. \]

▷ solved by zero-curvature (pure-gauge) config. \( a_{\mu}^{gs} \):

\[ a_{\mu}^{gs} = \pm \delta_\mu 4 \frac{2\pi}{e\beta} t_3 \quad (D_\nu \phi \equiv G_{\mu\nu} \equiv 0) \quad \Rightarrow \]

\[ \rho^{gs} = -P^{gs} = 4\pi \Lambda^3 T. \]

Unresolvable interaction between \( k = 0 \) to \( |k| = 1 \) sector lifts \( \rho^{gs} \) from zero (BPS). EOS of a cosmological constant; pressure negative.

(Short-lived, attracting magnetic (anti)monopoles by temporary shifts of (anti)caloron holonomies from trivial to small through absorption/emission of unresolved plane-wave fluctuations.)
Winding to unitary gauge: $\mathbb{Z}_2$ degeneracy

- consider gauge rotation $\tilde{\Omega}(\tau) = \Omega_{gl} \cdot Z(\tau) \cdot \Omega(\tau)$ where $\Omega(\tau) \equiv \exp[\pm 2\pi i \frac{\tau}{\beta} t_3]$, $Z(\tau) = \left(2\Theta(\tau - \frac{\beta}{2}) - 1\right)\mathbf{1}_2$, and $\Omega_{gl} = \exp[i \frac{\pi}{2} t_2]$

- $\tilde{\Omega}(\tau)$ transforms $a_{\mu}^{gs}$ to $a_{\mu}^{gs} \equiv 0$ and $\phi$ to $\phi = 2t^3|\phi|

- $\tilde{\Omega}(\tau)$ is admissible because respects periodicity of $\delta a_{\mu}$:

$$a_{\mu} \rightarrow \tilde{\Omega}(a_{\mu}^{gs} + \delta a_{\mu})\tilde{\Omega}^\dagger + \frac{i}{e} \tilde{\Omega} \partial_{\mu} \tilde{\Omega}^\dagger$$

$$= \Omega_{gl} \left(\Omega(a_{\mu}^{gs} + \delta a_{\mu})\Omega^\dagger + \frac{i}{e} \left(\Omega \partial_{\mu} \Omega^\dagger + Z \partial_{\mu} Z\right)\right)\Omega_{gl}^\dagger$$

$$= \Omega_{gl} \left(\Omega \delta a_{\mu} \Omega^\dagger + \frac{2i}{e} \delta(\tau - \frac{\beta}{2})Z\right)\Omega_{gl}^\dagger = \Omega_{gl} \delta a_{\mu} (\Omega_{gl} \Omega)^\dagger.$$ 

- $\tilde{\Omega}(\tau)$ transforms Polyakov loop from $-\mathbf{1}_2$ to $\mathbf{1}_2 \Rightarrow$ ground-state estimate is (electric) $\mathbb{Z}_2$ degenerate $\Rightarrow$ deconfining phase
Mass spectrum; outlook resummed radiative corrections

- Computation in physical and completely fixed unitary, Coulomb gauge ($\phi = 2t^3|\phi|$, $\partial_i a_i^3 = 0$)
- Mass spectrum: $m^2 \equiv m_1^2 = m_2^2 = 4e^2 \frac{\Lambda^3}{2\pi T}$, $m_3 = 0$.
- Resummation of polarization tensor of massless mode as

\[ \begin{array}{c}
\text{\textbullet} \quad = \quad + \quad + \quad + \quad \ldots \\
\end{array} \]

\[ \Rightarrow \] Small linear-in-$T$ correction to tree-level ground-state estimate [Falquez, RH, Baumbach 2010]

\[ \text{Tree-level:} \quad \frac{\rho^{gs}}{T^4} = 3117.09 \lambda^{-3} , \]

\[ \text{One-loop resummed:} \quad \frac{\Delta \rho^{gs}}{T^4} = 3.95 \lambda^{-3} . \]

- Large hierarchy between loop orders (conjecture about termination at finite irreducible order [RH 2006]), one-loop plus two-loop correction practically exact
$T$ dependence of $e$: selfconsistent thermal quasiparticles

$P$ and $\rho$ at one loop:

\[
P(\lambda) = -\Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ 2\bar{P}(0) + 6\bar{P}(2a) \right] + 2\lambda \right\},
\]

\[
\rho(\lambda) = \Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ 2\bar{\rho}(0) + 6\bar{\rho}(2a) \right] + 2\lambda \right\},
\]

where

\[
\bar{P}(y) \equiv \int_0^\infty dx \, x^2 \log \left[ 1 - \exp(-\sqrt{x^2 + y^2}) \right],
\]

\[
\bar{\rho}(y) \equiv \int_0^\infty dx \, x^2 \frac{\sqrt{x^2 + y^2}}{\exp(\sqrt{x^2 + y^2}) - 1},
\]

and $a \equiv \frac{m^2}{2T} = 2\pi e\lambda^{-3/2}$. For later use introduce function $D(2a)$ as

\[
\partial_{y^2} \bar{P} \bigg|_{y=2a} = -\frac{1}{4\pi^2} \int_0^\infty dx \, \frac{x^2}{\sqrt{x^2 + (2a)^2}} \cdot \frac{1}{e^{\sqrt{x^2 + (2a)^2}} - 1} \equiv -\frac{1}{4\pi^2} D(2a).
\]
Legendre transformation and evolution equation

- for $m(T)$ to respect Legendre trafo (fundamental partition function) between $P$ and $\rho \leftrightarrow \partial_m P = 0$
- ⇒ first-order evolution equation

\[
\partial_a \lambda = -\frac{24 \lambda^4 a}{(2\pi)^6} \frac{D(2a)}{1 + \frac{24 \lambda^3 a^2}{(2\pi)^6} D(2a)}.
\]

or

\[
1 = -\frac{24 \lambda^3}{(2\pi)^6} \left( \lambda \frac{da}{d\lambda} + a \right) a D(2a).
\]

- ⇒ $a(\lambda) \propto \lambda^{-\frac{3}{2}}$ for $\lambda \to \infty$
  ⇒ for $\lambda \gg 1$ $a$ must fall below unity

- fixed points of evolution equation:

repulsive at $a = 0$ ($\lambda \to \infty$)

attractive at $a = \infty$ ($\lambda = \lambda_c$)
Solution to evolution equation

- $a \ll 1$ [Dolan, Jackiw 1974] $\Rightarrow 1 = -\frac{\lambda^3}{(2\pi)^4} \left( \lambda \frac{da}{d\lambda} + a \right) a$;
  solution ($a(\lambda_i) = a_i \ll 1$):

$$a(\lambda) = 4\sqrt{2\pi^2} \lambda^{-3/2} \left( 1 - \frac{\lambda}{\lambda_i} \left[ 1 - \frac{a_i^2 \lambda_i^3}{32\pi^4} \right] \right)^{1/2}.$$  

$\Rightarrow$ attractor $a(\lambda) = 4\sqrt{2\pi^2} \lambda^{-3/2}$ as long as $a \ll 1$

$\Rightarrow e = \sqrt{8\pi}$ as long as $a \ll 1$ ($S = \frac{8\pi^2}{e^2} = 1 \Rightarrow$ interpretation of $\hbar$ in terms of caloron winding number, later)

- full solution for $e(\lambda) \Rightarrow \lambda_c = 13.87$: 

![Graph](image-url)
$T$ dependence of $P$ and $\rho$

- notice **negativity** of $P$ shortly above $\lambda_c$
- relative correction to one-loop quasiparticle $P$ and $\rho$ by radiative effects: $< 1\%$
Counting powers of $\hbar$

- re-instating $\hbar$ but keeping $c = k_B = 1$
  $\Rightarrow$ (dimensionless) exponential (fluctuating fields only) in effective partition function

$$- \int_0^\beta d\tau d^3x \frac{\mathcal{L}'_{\text{eff}}[a_\mu]}{\hbar},$$

is re-cast as

$$-\int_0^\beta d\tau d^3x \text{tr} \left( \frac{1}{2} (\partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu - ie\sqrt{\hbar}[\tilde{a}_\mu, \tilde{a}_\nu])^2 - e^2 \hbar[\tilde{a}_\mu, \tilde{\phi}]^2 \right),$$

$\tilde{a}_\mu \equiv a_\mu/\sqrt{\hbar}, \tilde{\phi} \equiv \phi/\sqrt{\hbar}$ assumed not to depend on $\hbar$

(see for example [Brodsky and Hoyer 2011; Iliopoulos, Itzykson, and Martin 1975, Holstein and Donoghue 2004])

- This re-formulation of (effective) action implies that loop expansion is expansion in ascending powers of $\hbar$.

- $[\tilde{a}_\mu]$ is length$^{-1}$ $\Rightarrow$ $[e] = [1/\sqrt{\hbar}]$
Action of just-not-resolved (anti)caloron

- Thus $e = \frac{\sqrt{8\pi}}{\sqrt{\hbar}}$ almost everywhere.

- Since only (anti)calorons of $\rho \sim |\phi|^{-1}$ contribute to $\phi$ in effective theory $\Rightarrow$ effective coupling $e$ admissible in calculation of fundamental (anti)caloron action:

$$S_{C/A} = \frac{8\pi^2}{e^2} = \hbar \quad \text{(almost everywhere)}.$$
Implications: Planck’s quantum=caloron action

1) universality, constancy (quantization) of $\hbar$: no dependence on YM scale $\Lambda$, associated with one unit of topological charge

2) pointlike vertices between effective plane waves induced by just-not-resolved, Euclidean nonpropagating field configuration
   $\Rightarrow$ irreconcilability of Euclidean and Minkowskian signatures as source of indeterminism in scattering event

3) because effective vertices are dominated by (anti)calorons with $\rho \sim |\phi|^{-1}$
   $\Rightarrow$ no interaction between (fundamental) plane waves if potential momentum transfer $\gg |\phi|$
   $\Rightarrow$ absence of plane-wave offshellness $\gg |\phi|$
   $\Rightarrow$ adds justification to renormalization programme of PT
hypothetically resolving an effective vertex:

resolution fixed (here by $T$ through $\phi(T)$):

$\Rightarrow$ no plane-wave interactions beyond that resolution (UV finiteness)

![Diagram](image)

(a) radiative corrections in eff. th.                (b) caloron mediation of vertex

(zero-mode induced fermionic vertex on (anti)instanton: ['t Hooft 1976])
Constraints of momentum transfers in effective 4-vertex

\[ p_3 \]

\[ p_4 = p_1 + p_2 - p_3 \]

\[ p_1 \]

\[ p_2 \]

**sum** over nontrivial s-, t-, and u-channel contributions in physical unitary-Coulomb gauge constrained as

- **s-channel:** \(|(p_1 + p_2)^2| \leq |\phi|^2\)
- **t-channel:** \(|(p_3 - p_1)^2| \leq |\phi|^2\)
- **u-channel:** \(|(p_3 - p_2)^2| \leq |\phi|^2\)
Negligible photon-photon scattering

- diagrams excluded by overall on-shellness:

- coherent channel superposition in remaining diagram:
Details on photon-photon scattering

- Investigate 27 combinations of s, t, u in 3 overall channels S, T, U with 4 energy-sign combinations each subject to:
  - On-shellness constraint on massive modes and
  - 4-vertex constraints

- Distinguish cases for signs of loop energy $\tilde{u}_0$ and $\tilde{v}_0$:

<table>
<thead>
<tr>
<th>$\tilde{u}_0 &gt; 0; \tilde{v}_0 &gt; 0$</th>
<th>$\tilde{u}_0 &gt; 0; \tilde{v}_0 &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{u}_0 &lt; 0; \tilde{v}_0 &gt; 0$</td>
<td>$\tilde{u}_0 &lt; 0; \tilde{v}_0 &lt; 0$</td>
</tr>
</tbody>
</table>

- Example of overall S:
Exclusion of Sss

- from on-shellness and momentum conservation:

\[
(1 - \cos(\angle ab)) \geq \frac{2\tilde{m}^2}{\tilde{a}_0 \tilde{b}_0},
\]

- from momentum transfer constraints:

\[
(1 - \cos(\angle ab)) \leq \frac{1}{2\tilde{a}_0 \tilde{b}_0},
\]

where \( \tilde{m} \equiv \frac{m}{|\phi|} = 2e \geq \sqrt{8\pi} \).

\[ \Rightarrow \text{upper bound smaller than lower bound} \]
\[ \Rightarrow \text{no Sss contribution!} \]
\[ \Rightarrow \text{Stt}^{+-}, \text{Stu}^{+-}, \text{Sut}^{+-}, \text{Suu}^{+-} \text{ remain.} \]
\[ (4 \text{ out of } 36 \text{ combinations}) \]
What about $T$ and $U$?

- for both channels:

<table>
<thead>
<tr>
<th>Vertex 2</th>
<th>Vertex 1</th>
<th>s-ch.</th>
<th>t-ch.</th>
<th>u-ch.</th>
</tr>
</thead>
<tbody>
<tr>
<td>s-ch.</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>t-ch.</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>u-ch.</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

- again: 4 out of 36 combinations remain in each case.
MC sampling of nonexcluded cases: Total hits

2 × 10^{11} test shots into region
\hat{a}_0 = \frac{a_0}{T}, \hat{b}_0 = \frac{b_0}{T} \leq 100, \lambda_c = 13.867 \leq \lambda \leq 100 (noncompact arguments) and nonconstrained ang. domain.

- histogram of hits:

- in Sss analysis of Bose suppression yields factor \leq 10^{-7} for \lambda_c = 13.867 \leq \lambda \leq 30.
MC sampling of nonexcluded cases: Distribution of photon energies

- for $2 \times 10^{11}$ test shots we obtain
  \[
  (\hat{a}_0 \equiv \frac{a_0}{T}, \hat{b}_0 \equiv \frac{b_0}{T})
  \]

- Hard photons do not scatter at all.
- Very feeble participation of soft photons.
Filamented algebraic varieties

Constraints
Vertex 2

[u] is positive and real

Constraints
Vertex 1

S\text{St}_1 \text{ Softening Factor: 150}

S\text{St}_1 \text{ Softening Factor: 30}

S\text{St}_1 \text{ Softening Factor: 5}

S\text{St}_1 \text{ Softening Factor: 1}
Real-world implications

- postulate that photon propagation described by SU(2) rather than U(1) gauge principles:
  [RH 2005; Giacosa and RH 2005] ⇒ blackbody anomaly (transverse polarisations), magnetic charge-density waves (longitudinal polarisations)
  [Schwarz, RH, and Giacosa 2006; Ludescher and RH 2008; Falquez, RH, and Baumbach 2010, 2011]

- in units $c = \epsilon_0 = \mu_0 = k_B = 1$ QED fine-structure constant $\alpha$ is

  \[ \alpha = \frac{Q^2}{4\pi\hbar} \]

  ⇒ to be unitless: $Q \propto 1/e$.

*Is realized if $Q$ taken $\propto$ electric-magnetically dual of $e$:

  \[ Q' = \frac{4\pi}{e} \propto \sqrt{\hbar}, \quad Q' = NQ \quad \text{(mixing of SU(2)'s)} \]
Dual interpretation of charge and flux

⇒ magnetic monopoles of SU(2) are electric monopoles in real world [RH 2005]

⇒ magnetic-monopole condensate of SU(2) is condensate of electric monopoles in real world (no dual Meissner effect) [Giacosa and RH 2005]

⇒ electric charge density waves in SU(2) are longitudinally propagating magnetic field modes in real world [Falquez, RH, and Baumbach 2011]

⇒ magnetic $Z_2$ charge of an SU(2) center-vortex selfintersection is electric charge in real world [Moosmann and RH 2008]
What is $T_c$?

- ARCADE 2 confirms excess of line temperature for CMB frequencies $\nu \leq 3.4$ GHz ($6\sigma$ at $\nu = 3$ GHz) in agreement with earlier radio-frequency observations [Fixsen et al 2009]

- interpreted as onset of evanescence for $\nu \leq m_\gamma \sim 100$ MHz due to (partial) Meissner effect (deconfining-preconfining transition) [RH 2009]

$\Rightarrow T_c = 2.725 \text{ K} \equiv T_0$, fixing $\Lambda \sim 10^{-4}$ eV
Blackbody anomaly

- computation of (only right diagram allowed)
  [Schwarz,RH,Giacosa 2007; Ludescher,RH 2008; Falquez,RH,Baumbach 2010]

\[
\begin{align*}
\nu, a = 3 & \quad \mu, a = 3 \\
p & \quad p \quad k \\
\lambda & \quad \sigma \\
p - k & \quad p \\
f & \quad j \\
\rho & \quad \kappa
\end{align*}
\]

A

\[
\begin{align*}
\nu, a = 3 & \quad \mu, a = 3 \\
p & \quad p \quad k \\
\lambda & \quad \sigma \\
p - k & \quad p \\
f & \quad j \\
\rho & \quad \kappa
\end{align*}
\]

B

to obtain dispersion law for transverse polarisations yields

\[
\frac{\nu^*}{\text{GHz}} = 42.70 \left( \frac{T}{K} \right)^{-0.53} + 0.21 \text{ for } T \geq 4.3 \text{ K}
\]
Blackbody anomaly

- leads to difference $\delta \rho$ between thermal energy density of SU(2) and conventional U(1) photons

$$F(\bar{T}, \delta T) = N \exp \left( -\frac{V \delta \rho}{\bar{T}} \right) \text{ for } \delta T$$

[RH, Nature Physics 2013]
Magnetic charge-density waves

- dispersion law for longitudinal polarisation yields

![Graph showing magnetic modes]

(3 branches of longitudinally propagating magnetic modes)

- assuming energy-free conversion of energy density of these incoherent magnetic modes into magnetic field $B$ of cosmological coherence length:

$$B \sim 10^{-8} \text{ G} \sim \text{upper bound from small-angle TT correlation}$$

[see Widrow 2002; Falquez, RH, Baumbach 2012]
CMB temperature vs. cosmological scale factor

- existence of Yang-Mills scale Λ changes $T \propto a^{-1}$ scaling of CMB temperature at late times
  [RH 2014]

⇒ $T = 0.62 \, a^{-1} \times T_0$, \hspace{1cm} (a ≤ $\frac{1}{10}$)

⇒ repeal of difference between CMB extracted redshift for instantaneous reionisation, $z_{\text{reion}} = 10.8 \pm 3.1$, and that deduced from high-redshift quasar-spectra (Gunn-Peterson trough), $z_{\text{reion}} \sim 6$. 
Redshift of CMB decoupling (recombination)

- due to scaling violation at low $z$ one has
  \[ z_{\text{dec}} = \frac{1}{0.62} \frac{3000}{2.725} - 1 = 1775 \]
  in contrast to $z_{\text{dec}} = 1089$ in conventional U(1) theory

- to keep successful BAO physics, matter density (DM+baryonic) needs to be the same at $z_{\text{dec}} = 1775$ as conventionally was at $z_{\text{dec}} = 1089$; leads to a re-scaling of conventional matter density by
  \[ \left( \frac{1089}{1775} \right)^3 = 0.231 \]
  (today’s 25% matter contribution to $\rho_{\text{crit}}$ conventionally is re-scaled to 5.8% !)

- But what about small-redshift DM contribution, rotation curves of galaxies? Possibly: slow-roll of Planck-scale axion with topologically stabilised U(1) solitons centered at spiral galaxies

  [Giacosa and RH 2005]
Cosmic neutrinos

- redshift dependence of $N_{\text{eff}}$
- $\frac{T_\nu}{T} = \left(\frac{16}{23}\right)^{1/3}$ instead of $\frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3}$
- too low value of $N_{\text{eff}}$ today \(\Rightarrow\)
- way out:
  coupling of CMB and neutrino fluids inducing $T_\nu = T$ and $m_\nu = \xi T$
  (neutrino: single center-vortex loop in confining phase of an SU(2) YM theory)

[Moosmann and RH 2008]
More material

- **Low-order radiative corrections:**
  RH 2006; Schwarz, RH, Giacosa 2007; Ludescher, RH 2008; Falquez, RH, Baumbach 2010, 2011

- **Loop expansions:**
  RH 2006

- **Stable but unresolved monopoles:**
  Keller et al. 2008

- **The two other phases:**
Summary

- mini review on (thermal) Yang-Mills action
- mini review on calorons: trivial vs. nontrivial holonomy for \( |k| = 1 \) plus semiclassical approx.
- construction of thermal ground-state estimate: inert field \( \phi \); BPS and E-L; potential
- discussion of constraints on effective action: pert. renormalizability plus inertness of \( \phi \) \( \Rightarrow \) unique answer
- full ground-state estimate, deconfining nature, tree-level quasiparticles
- evolution of effective coupling
- \( T \) dependence pressure and energy density
- interpretation of \( \hbar \) in terms of caloron action
- photon-photon scattering
Some physics implications:

postulate: SU(2) \((10^{-4} \text{ eV})\) describes photon propagation

⇒ blackbody spectral anomaly at \(T \sim 5 - 20 \text{ K}\)
⇒ low frequency magnetic charge-density waves
⇒ CMB large-angle anomalies
⇒ instantaneous, early reionisation
⇒ cosmic neutrinos
⇒ cold matter and Planck-scale axion

Thank you.