The Thermal Scalar and Random Walks in Curved Spacetime

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Outline

Introduction

String Thermodynamics in Flat space

Motivation for studying curved space string thermodynamics

Results

Random walks
3d Geometry: $AdS_3$ WZW model
Generic Black holes
Stress Tensor
Consider a gas of (non-interacting) strings at finite temperature (canonical ensemble):

What happens as we heat the gas?

Canonical ensemble of non-interacting strings $Z_\beta = \text{Tr} \left( e^{-\beta H} \right)$ exhibits maximal temperature = Hagedorn temperature $T_H$

Reason = Exponential increase of $\rho(E) \sim e^{\beta H E}$ at high $E$
For $T \lesssim T_H$, no divergence is present yet
Near $T_H$, string gas recombines into one (or several) highly excited long strings

Deo-Jain-Tan '89, Bowick-Giddings '89, Mitchell-Turok '87

Long string behaves as random walk in space
Hagedorn divergence can be interpreted as instability towards long string formation
Alternative Perspective: Thermal Manifold

Alternatively: Study string theory on the thermal manifold Sathiapalan '87, Kogan '87, O’Brien-Tan '87, McClain-Roth '87, Atick-Witten '88

**Thermal manifold** = Wick-rotate + periodically identify time direction

Extra feature: **winding modes** possible

As \( T \) (or \( \beta \)) changes \( \Rightarrow \) masses of wound strings change

\( w = 1 \) state is called **thermal scalar**

- Masslessness of this state determines \( T_H \)
- \( T > T_H \), state becomes tachyonic \( \Rightarrow \) instability of theory
- Dominates TD for \( T \lesssim T_H \) (random walk)
**GOAL**

Study string thermodynamics in the high-temperature regime in curved backgrounds using the thermal manifold approach.

Take gas of strings ...

... and place it on a curved background

- How does the curvature affect the Hagedorn behavior?
- Thermal scalar and its random walk interpretation?
In particular: black hole backgrounds

Why study string thermodynamics near black hole horizons?

1. Deeper understanding of black hole horizons (Firewall Almheiri-Marolf-Polchinski-Sully '12)

2. Holography: Black hole horizons $\Leftrightarrow$ deep IR of dual field theory
   Fluid properties of quark-gluon plasma?

3. Better understanding of string thermodynamics (not completely understood in general)
1. **Jeans Instability**
   Matter clumps together (Perturbative generation of imaginary mass of gravitons)
   ⇒ Restrict amount of thermal matter by using compact space or $AdS$ as a container Barbon-Rabinovici ’02

2. Canonical ensemble suffers from large fluctuations near $\beta_H$
   ⇒ Use the canonical picture as a tool to obtain $\rho(E)$
   Or study this discrepancy further

3. **Hagedorn transition** could be first order at a lower temperature than $T_H$ (semi-classical tunneling) Atick-Witten ’88
   ⇒ We will study overheated spaces
Set-up of the geometry

Background metric of the static form
\[ ds^2 = G_{00}(x)dt^2 + G_{ij}(x)dx^i dx^j \]

Thermal manifold

**Flat space:** Cylinder \(\rightarrow\) **Curved space:** varying size

What about black holes?

- **Thermal circle pinches off**
- **Does the thermal scalar exist?**
Random walks in curved backgrounds: Path integral approach (1)

Torus string path integral reduced to particle path integral

Kruczenski-Lawrence '06, TM-Verschelde-Zakharov '13

Starting point = Torus path integral on the strip modular domain:

\[ Z_{T^2} = \int [E] \Delta_{FP} \int \mathcal{D}X \sqrt{G} e^{-\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu G_{\mu\nu}(X)} \]

Near-Hagedorn approximation:

- We consider strings that are singly wound around the Euclidean time direction: \( X^0(\sigma, \tau + 1) = X^0(\sigma, \tau) \pm \beta \)
  \( \Rightarrow \) Interpretation of a free closed string performing one loop around the thermal dimension

- Small \( \tau_2 \) region in strip \( E \)
Random walks in curved backgrounds: Path integral approach (2)

In the near-Hagedorn limit, it is found that this simplifies to

\[ Z_p = 2 \int_0^\infty \frac{d\tau_2}{2\tau_2} \int_{X_i(\tau_2) = X_i(0)} [\mathcal{D}X] \sqrt{\det G_{ij}} \exp -S_p(X) \]

with particle action \( S_p \):

\[ S_p = \frac{1}{4\pi\alpha'} \left[ \int_0^{\tau_2} dt G_{ij} \partial_t X^i \partial_t X^j + \beta^2 \int_0^{\tau_2} dt G_{00} - \beta_{H,flat}^2 \tau_2 + \ldots \right] \]

⇒ particle path integral on the spatial submanifold and can be interpreted as a random walk of the long string

⇒⇒ Particle Trajectory = Spatial form of long string

Unfortunately, some corrections to the action \( S_p \) are missed (naive)
Thermal Scalar action

Alternative perspective: field theory of the thermal scalar (supposed to dominate near $T_H$)

Thermal scalar field $\phi(x) = \text{complex scalar field}$ only depending on the spatial coordinates of the manifold

Its action (non-interacting) can be obtained using T-duality and dimensional reduction:

$$S_{\text{th.sc.}} = \int dV \sqrt{G_{ij}} \sqrt{G_{00}} \left[ G^{ij} \partial_i \phi \partial_j \phi^* - \frac{4}{\alpha'} \phi \phi^* + \frac{\beta^2 G_{00}}{4\pi^2 \alpha'^2} \phi \phi^* \right]$$

Q: $\alpha'$-corrections?
A: Possibly, depends on model
Determining Hagedorn Temperature

For $T \lesssim T_H$, partition function is dominated by thermal scalar

$$Z_\beta = \text{Tr}(e^{-\beta H}) \approx Z_{\text{th.sc.}} = \int [\mathcal{D}\phi] \ e^{-S_{\text{th.sc.}}}$$

**Determining $T_H$:**

$$S_{\text{th.sc.}} \sim \int dV e^{-2\Phi} \sqrt{G} \phi^* \hat{O} \phi \ , \ \hat{O} \psi_n = \lambda_n \psi_n$$

$$\Rightarrow Z_{\text{th.sc.}} = \text{det}^{-1} \hat{O} \ \Rightarrow \beta F \approx \text{Tr} \ln \hat{O}$$

$\lambda_0 = 0$ determines $\beta_H$

$\psi_0$ determines where random walk is localized

**Link with previous random walk picture:**

Schwinger proper time representation of logarithm:

$$\ln(a) = - \int_0^{+\infty} \frac{d\tau_2}{\tau_2} \ (e^{-a\tau_2} - e^{-\tau_2})$$

Predicts corrections to previous random walk
Comparison of two methods

String path integral approach

Advantages
- Intuitive geometric derivation of random walk

Disadvantages
- On-shell backgrounds only
- Corrections to action

Field theory approach

Advantages
- Field Theory $\Rightarrow$ in principle also off-shell
- Direct control on corrections

Disadvantages
- Interpretation on real-time manifold obscured

Both approaches individually not ideal
$\Rightarrow$ Combining them gives full picture
Application 1: thermal $AdS_3$

Q: Does thermal scalar exist and predict $T_H$? $\Rightarrow$ Thermal Spectrum

Geometry:

$AdS_3$ WZW model is group manifold $\Rightarrow \alpha'$-exact background

$$ds^2 = \alpha' k \left( \cosh(\rho)^2 d\tau^2 + d\rho^2 + \sinh(\rho)^2 d\phi^2 \right)$$

$\tau \sim \tau + \beta$ and $\phi \sim \phi + 2\pi$

Additionally $B = -i\alpha' k \sinh(\rho)^2 d\tau \wedge d\phi$

Imaginary in Euclidean signature
The (bosonic) thermal spectrum in $AdS_3$

Spectrum on non-thermal space is known

Thermal spectrum constructed using: TM-Verschelde-Zakharov '14

- Twisted vertex operator method
- Field theory methods
- Hamiltonian rewriting of the (known) partition function

Maldacena-Ooguri-Son '01 $Z = \text{Tr} \left( q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} \right)$

Resulting thermal spectrum:

$$h = \frac{s^2 + 1/4}{k - 2} - i \frac{qp\beta}{4\pi} + \frac{pn}{2} + \frac{kp^2\beta^2}{4(2\pi)^2} + h_{int}$$

$p = \text{Cylinder Winding}$, $n = \text{Cylinder Discrete momentum}$

$q = \text{Cigar Winding}$, $w = \text{Cigar Discrete momentum}$

Properties

- No Cigar-winding modes
- Only continuous modes (due to $B$), $s \sim \text{radial momentum}$
The thermal scalar in $AdS_3$

Defining a tachyon
Genus 1 vacuum amplitude in $\mathcal{F}$ (for large $\tau_2$):

$$Z \approx \int_\mathcal{F} \frac{d\tau_1 d\tau_2}{2\tau_2} \sum_{H_{\text{matter}}} q^{h_i-1} \bar{q}^{\bar{h}_i-1}$$

where $q^{h-1} \bar{q}^{\bar{h}-1} = e^{2\pi i \tau_1 (h-\bar{h})} e^{-2\pi \tau_2 (h+\bar{h}-2)}$

$\Rightarrow$ Quantity $\Re(h + \bar{h})$ determines convergence

Thermal tachyon: $p = \pm 1$, $q \in \mathbb{Z}$ are all marginal at

$$\beta_H^2 = \frac{4\pi^2}{k} \left( 4 - \frac{1}{k-2} \right)$$

Berkooz-Komargodski-Reichmann '07, Lin-Matsuo-Tomino '07

- Infinitely many states become marginal at $T_H$
- Thermal scalar is continuous $\Rightarrow$ Spreads over entire space
\( \mathbb{Z}_N \) orbifold by modding out a \( 2\pi/N \) rotation around cigar

Hamiltonian rewriting of \( Z \Rightarrow \text{Appearance of discrete states} \)

Mathematical Reason = correct analytic continuation of Poisson’s summation formula:

\[
\sum_{n \in \mathbb{Z}} e^{inz} \hat{f}(n) = 2\pi \sum_{k \in \mathbb{Z}} \left[ f(z+2\pi k) + 2\pi i \sum_{p_i \in \mathcal{P}} \text{Res}_{p_i}(f) \delta(z-p_i+2\pi k) \right]
\]

Resulting spectrum:

- **Continuous part:**
  \[
h = \frac{s^2 + 1/4}{k-2} + \frac{qw}{2} + \frac{i\pi nw}{\beta} + \frac{kw^2}{4} - \frac{iqp\beta}{4\pi} + \frac{pn}{2} + \frac{kp^2\beta^2}{4(2\pi)^2}
\]

- **Discrete part:**
  \[
h = -\frac{j(j-1)}{k-2} + \frac{qw}{2} - \frac{i\pi inw}{\beta} + \frac{kw^2}{4} - \frac{i\beta pq}{4\pi} - \frac{pn}{2} + \frac{kp^2\beta^2}{4(2\pi)^2}
\]

where \( j = \frac{k|w|}{2} - \frac{|q|}{2} \pm \frac{i\pi n}{\beta} - l, l=0,1,2,..., q \in \mathbb{N}\mathbb{Z}, n \in \mathbb{Z}, p \in \mathbb{Z} \) and \(|w| \leq 1/2\)

**Application:** Chemical potential in \( AdS_3 \)
Application 2: generic black holes

Susskind: an asymptotic observer throws in 1 string

- As it falls, the string is seen to elongate dramatically near the black hole horizon (distance $\ell_s$) \textsuperscript{Susskind ’94}
- Elongation not seen by free-falling observers, only by fiducial observers
- Two different situations with long strings: high $T$ and near horizon? \Rightarrow Related?
Susskind’s story from canonical point of view
Recall: Unruh effect (intrinsic to black holes)

- Close to the horizon, infalling QFT vacuum around the black hole is viewed by fiducial observers as thermally populated with temperature $T = \frac{1}{2\pi \rho}$ where $\rho \sim \sqrt{r - r_s}$
For $r \to r_s$, $T$ diverges (QG effects should be important!)

Thermal gas is emergent from the black hole quantum mechanics with $T = T_{\text{Hawking}}$
Thermal structure of black holes

In thermal atmosphere: Rindler approximation of metric:
\[ ds^2 = -\frac{\rho^2}{(4GM)^2} dt^2 + d\rho^2 + dx_{\perp}^2. \]

\[ \implies \text{Study string theory on thermal manifold = Euclidean Rindler space: } \]
\[ ds^2 = \frac{\rho^2}{(4GM)^2} d\tau^2 + d\rho^2 + dx_{\perp}^2, \quad \tau \sim \tau + 8\pi GM \]
**Results from string theory (1)**

**Approach:** Small curvature (large \(k\)) limit of \(SL(2, \mathbb{R})/U(1)\) cigar

⇒ Euclidean Rindler space \(^{Giveon-Itzhaki '13}\)

⇒ \(\exists\ w = 1\) state in thermal spectrum

From coset construction ⇒ \(\alpha'\)-exact thermal scalar field theory for type II superstrings \(^{Verlinde-Verlinde-Dijkgraaf '92}\)

From thermal scalar FT perspective:

\[
ds^2 = \frac{\rho^2}{(4GM)^2} d\tau^2 + d\rho^2 + dx_{\perp}^2
\]

Wave operator: \(\hat{\mathcal{O}} = -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho - \frac{2}{\alpha'} + \frac{\beta^2 \rho^2}{4\pi^2 \alpha'^2 (4GM)^2}\)

Spectrum (enforcing regularity at the origin and at infinity):

\[
\psi_n(\rho) \propto \exp \left( -\frac{\beta \rho^2}{4\pi \alpha' (4GM)} \right) L_n \left( \frac{\beta \rho^2}{2\pi \alpha' (4GM)} \right)
\]

\[
\lambda_n = \frac{\beta - 8\pi GM + 2\beta n}{\pi \alpha' (4GM)}, \quad n \geq 0
\]

With \(\beta = \beta_{\text{Hawking}} = 8\pi GM\) and \(n = 0\) (lowest mode)

\[
\psi_0 \propto \exp \left( -\frac{\rho^2}{2\ell_s^2} \right), \quad \lambda_0 = 0
\]
\[ \psi_0 \propto \exp \left( -\frac{\rho^2}{2\ell_s^2} \right), \quad \lambda_0 = 0 \]

- Bound to black hole, width \( \sim \ell_s \) \( \Rightarrow \) Random walk near horizon
- Even though no reference to \( \ell_s \) is made in the background, the thermal scalar \( \psi_0 \) is sensitive to \( \ell_s \) (reason = T-duality)
- Predicts \( T_H = T_{\text{Hawking}} \)

Gas around black hole behaves as \( T = T_H \) gas of flat space!
Thermal scalar \textit{dominates} one-loop TD

Susskind’s picture of long string surrounding black hole horizon
Resulting thermal structure

Disclaimer: only genus 1 $\Rightarrow$ interactions expected to be important near horizon
Energy-momentum tensor of string gas for type II superstrings

Horowitz-Polchinski '98

**Microcanonical**
High-energy averaged stress tensor

\[ \langle T_{\mu\nu}(x) \rangle_E \approx \frac{E}{2N\beta_H^2} T_{\text{th.sc.}}^{\mu\nu}(x) \]

**Canonical**
Near-Hagedorn stress tensor (compact space):

\[ \langle T_{\mu\nu}(x) \rangle_\beta \approx \frac{1}{2N\beta_H^2} T_{\text{th.sc.}}^{\mu\nu}(x) \frac{1}{\beta - \beta_H} \]

- Classical stress tensor evaluated on lowest eigenmode \( \psi_0 \) of thermal scalar
- Thermal scalar wavefunction determines where stress-energy is located
Example Near-Hagedorn energy density in Rindler space

Rindler space: \( \psi_0 \propto \exp \left( -\frac{\rho^2}{2\ell_s^2} \right) \)

\[ \langle T^0_0 \rangle_{\text{thermal}} = \]

- Energy density located close to black hole horizon
- Negative energy density zone is present (violates weak-energy condition)

Generic feature in curved spacetime

- Consistency Check: \( E_{\text{tot}} = \partial_\beta (\beta F) \)
Conclusions

1. High-temperature string theory contains a divergence, the Hagedorn divergence, related to long string dominance. Thermal scalar is singly wound state on thermal manifold and encodes the random walk near-Hagedorn thermodynamics of the gas.

2. Random walk picture can be generalized to curved spacetimes using $1^{st}$ quantized string path integral methods or $2^{nd}$ quantized field theory methods.

3. $AdS_3$ contains infinitely many states becoming marginal at $T_H$. Thermal scalar wavefunction spreads over entire space.

4. Black hole atmosphere behaves precisely the same as the $T = T_H$ gas of flat space: long strings are present. The long string configuration is close to the event horizon.

5. Other properties such as the stress tensor of the near-Hagedorn gas can also be analyzed.
Thank you!