The Thermal Scalar and Random Walks in Curved Spacetime

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Outline

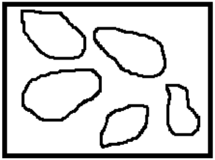
Introduction
- String Thermodynamics in Flat space

Motivation for studying curved space string thermodynamics

Results
- Random walks
- 3d Geometry: AdS_3 WZW model
- Generic Black holes
- Stress Tensor

String Thermodynamics in flat space

Consider a gas of (non-interacting) strings at finite temperature (canonical ensemble):



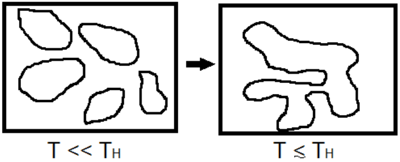
What happens as we heat the gas?

Canonical ensemble of non-interacting strings $Z_\beta = \text{Tr}\left(e^{-\beta H}\right)$ exhibits maximal temperature = Hagedorn temperature T_H

Reason = Exponential increase of $\rho(E) \sim e^{\beta_H E}$ at high E

String Thermodynamics in flat space: Random walks

For $T \lesssim T_H$, no divergence is present yet
Near T_H, string gas recombines into one (or several) highly excited long strings Deo-Jain-Tan '89, Bowick-Giddings '89, Mitchell-Turok '87



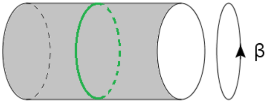
Long string behaves as random walk in space
Hagedorn divergence can be interpreted as instability towards long string formation

Alternative Perspective: Thermal Manifold

Alternatively: Study string theory on the thermal manifold Sathiapalan '87, Kogan '87, O'Brien-Tan '87, McClain-Roth '87, Atick-Witten '88

Thermal manifold = Wick-rotate + periodically identify time direction

Extra feature: winding modes possible



As T (or β) changes $\Rightarrow$ masses of wound strings change

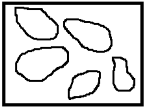
$w = 1$ state is called thermal scalar

- Masslessness of this state determines T_H
- $T > T_H$, state becomes tachyonic $\Rightarrow$ instability of theory
- Dominates TD for $T \lesssim T_H$ (random walk)

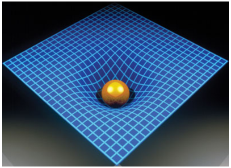
Motivation (1)

GOAL

Study string thermodynamics in the high-temperature regime in curved backgrounds using the thermal manifold approach

Take gas of strings ... 

... and place it on a curved background



- How does the curvature affect the Hagedorn behavior?
- Thermal scalar and its random walk interpretation?

Motivation (2)

In particular: black hole backgrounds

Why study string thermodynamics near black hole horizons?

1. Deeper understanding of black hole horizons (Firewall Almheiri-Marolf-Polchinski-Sully '12)

2. Holography: Black hole horizons $\Leftrightarrow$ deep IR of dual field theory
   Fluid properties of quark-gluon plasma ?

3. Better understanding of string thermodynamics (not completely understood in general)

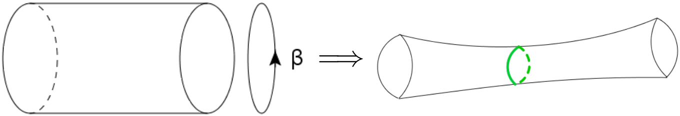
Issues with String Thermodynamics

1. Jeans Instability
   Matter clumps together (Perturbative generation of imaginary mass of gravitons)
   $\Rightarrow$ Restrict amount of thermal matter by using compact space or AdS as a container Barbon-Rabinovici '02
2. Canonical ensemble suffers from large fluctuations near β_H
   $\Rightarrow$ Use the canonical picture as a tool to obtain $\rho(E)$
   Or study this discrepancy further
3. Hagedorn transition could be first order at a lower temperature than T_H (semi-classical tunneling) Atick-Witten '88
   $\Rightarrow$ We will study overheated spaces

Set-up of the geometry

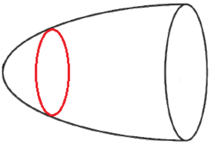
Background metric of the static form
$ds^2 = G_{00}(\mathbf{x})dt^2 + G_{ij}(\mathbf{x})dx^i dx^j$

Thermal manifold
Flat space: Cylinder $\Longrightarrow$ Curved space: varying size



What about black holes?

- Thermal circle pinches off
- Does the thermal scalar exist ?



Random walks in curved backgrounds: Path integral approach (1)

Torus string path integral reduced to particle path integral

Kruczenski-Lawrence '06, TM-Verschelde-Zakharov '13

Starting point = Torus path integral on the strip modular domain:

$$Z_{T_2} = \int_E \frac{d\tau_2 d\tau_1}{2\tau_2} \Delta_{FP} \int [\mathcal{D}X] \sqrt{G} e^{-\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)}$$

Near-Hagedorn approximation:

- We consider strings that are singly wound around the Euclidean time direction: $X^0(\sigma, \tau + 1) = X^0(\sigma, \tau) \pm \beta$
  $\Rightarrow$ Interpretation of a free closed string performing one loop around the thermal dimension
- Small τ_2 region in strip E

Random walks in curved backgrounds: Path integral approach (2)

In the near-Hagedorn limit, it is found that this simplifies to

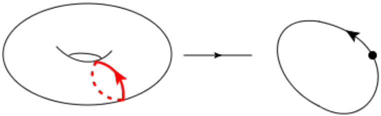
$$Z_p = 2\int_0^\infty \frac{d\tau_2}{2\tau_2}\int_{X^i(\tau_2)=X^i(0)}[\mathcal{D}X]\sqrt{\det G_{ij}}\exp -S_p(X)$$

with particle action S_p:

$$S_p = \frac{1}{4\pi\alpha'}\left[\int_0^{\tau_2} dtG_{ij}\partial_t X^i\partial_t X^j + \beta^2\int_0^{\tau_2} dtG_{00} - \beta_{H,flat}^2\tau_2 + \dots\right]$$

$\Rightarrow$ particle path integral on the spatial submanifold and can be interpreted as a random walk of the long string

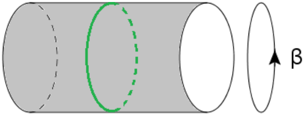
$\Longrightarrow$ Particle Trajectory = Spatial form of long string



Unfortunately, some corrections to the action S_p are missed (naive)

Thermal Scalar action

Alternative perspective: field theory of the thermal scalar (supposed to dominate near T_H)



Thermal scalar field $\phi(\mathbf{x})$ = complex scalar field only depending on the spatial coordinates of the manifold

Its action (non-interacting) can be obtained using T-duality and dimensional reduction:

$$S_{\text{th.sc.}} = \int dV \sqrt{G_{ij}}\sqrt{G_{00}} \left[G^{ij}\partial_i\phi\partial_j\phi^* - \frac{4}{\alpha'}\phi\phi^* + \frac{\beta^2 G_{00}}{4\pi^2\alpha'^2}\phi\phi^* \right]$$

Q: α'-corrections?

A: Possibly, depends on model

Determining Hagedorn Temperature

For $T \lesssim T_H$, partition function is dominated by thermal scalar
$\Longrightarrow Z_\beta = \mathrm{Tr}(e^{-\beta H}) \approx Z_{\text{th.sc.}} = \int [\mathcal{D}\phi]\, e^{-S_{\text{th.sc.}}}$

Determining T_H:

$S_{\text{th.sc.}} \sim \int dV e^{-2\Phi} \sqrt{G} \phi^* \hat{\mathcal{O}} \phi \quad , \hat{\mathcal{O}} \psi_n = \lambda_n \psi_n$
$\Rightarrow Z_{\text{th.sc.}} = \det^{-1} \hat{\mathcal{O}} \quad \Rightarrow \beta F \approx \mathrm{Tr} \ln \hat{\mathcal{O}}$

$\lambda_0 = 0$ determines β_H
ψ_0 determines where random walk is localized

Link with previous random walk picture:
Schwinger proper time representation of logarithm:
$\ln(a) = -\int_0^{+\infty} \frac{d\tau_2}{\tau_2} \left(e^{-a\tau_2} - e^{-\tau_2} \right)$

Predicts corrections to previous random walk

Comparison of two methods

String path integral approach

Advantages

- Intuitive geometric derivation of random walk

Disadvantages

- On-shell backgrounds only
- Corrections to action

Field theory approach

Advantages

- Field Theory $\Rightarrow$ in principle also off-shell
- Direct control on corrections

Disadvantages

- Interpretation on real-time manifold obscured

Both approaches individually not ideal
$\Rightarrow$ Combining them gives full picture

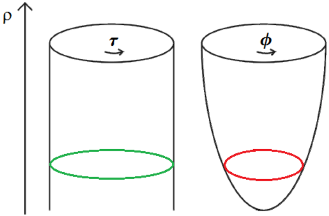
Q: Does thermal scalar exist and predict T_H ? $\Rightarrow$ Thermal Spectrum

Geometry:

AdS_3 WZW model is group manifold $\Rightarrow$ α'-exact background

$ds^2 = \alpha' k \left(\cosh(\rho)^2 d\tau^2 + d\rho^2 + \sinh(\rho)^2 d\phi^2\right)$

$\tau \sim \tau + \beta$ and $\phi \sim \phi + 2\pi$



Additionally $B = -i\alpha' k \sinh(\rho)^2 d\tau \wedge d\phi$

Imaginary in Euclidean signature

The (bosonic) thermal spectrum in AdS_3

Spectrum on non-thermal space is known

Thermal spectrum constructed using: TM-Verschelde-Zakharov '14

- Twisted vertex operator method
- Field theory methods
- Hamiltonian rewriting of the (known) partition function

Maldacena-Ooguri-Son '01 $Z = \mathrm{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right)$

Resulting thermal spectrum:

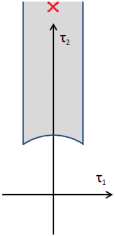
$$h = \frac{s^2 + 1/4}{k-2} - i\frac{qp\beta}{4\pi} + \frac{pn}{2} + \frac{kp^2\beta^2}{4(2\pi)^2} + h_{int}$$

$p =$ Cylinder Winding , $n =$ Cylinder Discrete momentum
$w =$ Cigar Winding , $q =$ Cigar Discrete momentum

Properties

- No Cigar-winding modes
- Only continuous modes (due to B), $s \sim$ radial momentum

The thermal scalar in AdS_3



Defining a tachyon

Genus 1 vacuum amplitude in $\mathcal{F}$ (for large τ_2):

$$Z \approx \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{2\tau_2} \sum_{H_{matter}} q^{h_i - 1} \bar{q}^{\bar{h}_i - 1}$$

where $q^{h-1}\bar{q}^{\bar{h}-1} = e^{2\pi i \tau_1 (h - \bar{h})} e^{-2\pi\tau_2 (h + \bar{h} - 2)}$

$\Rightarrow$ Quantity $\Re(h + \bar{h})$ determines convergence

Thermal tachyon: $p = \pm 1$, $q \in \mathbb{Z}$ are all marginal at

$\beta_H^2 = \frac{4\pi^2}{k}\left(4 - \frac{1}{k-2}\right)$ Berkooz-Komargodski-Reichmann '07, Lin-Matsuo-Tomino '07

- Infinitely many states become marginal at T_H
- Thermal scalar is continuous $\Rightarrow$ Spreads over entire space

Orbifolded thermal spectrum in AdS_3

$\mathbb{Z}_N$ orbifold by modding out a $2\pi/N$ rotation around cigar

Hamiltonian rewriting of $Z \Rightarrow$ Appearance of discrete states
Mathematical Reason = correct analytic continuation of Poisson's summation formula:

$$\sum_{n\in\mathbb{Z}} e^{inz}\hat{f}(n)=2\pi\sum_{k\in\mathbb{Z}}\Big[f(z+2\pi k)+2\pi i\sum_{p_i\in\mathcal{P}}\mathrm{Res}_{p_i}(f)\delta(z-p_i+2\pi k)\Big]$$

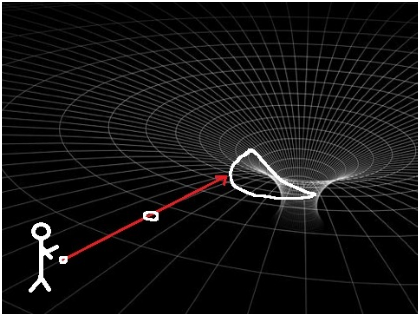
Resulting spectrum:

- Continuous part:
  $$h=\frac{s^2+1/4}{k-2}+\frac{qw}{2}+\frac{i\pi nw}{\beta}+\frac{kw^2}{4}-i\frac{qp\beta}{4\pi}+\frac{pn}{2}+\frac{kp^2\beta^2}{4(2\pi)^2}$$
- Discrete part:
  $$h=-\frac{\tilde{j}(\tilde{j}-1)}{k-2}+\frac{qw}{2}-\frac{\pi iwn}{\beta}+\frac{kw^2}{4}-\frac{i\beta pq}{4\pi}-\frac{pn}{2}+\frac{kp^2\beta^2}{4(2\pi)^2}$$
  where $\tilde{j}=\frac{k|w|}{2}-\frac{|q|}{2}\pm\frac{i\pi n}{\beta}-l$ $,l=0,1,2,\ldots,$ $q\in N\mathbb{Z},$ $n\in\mathbb{Z},$ $p\in\mathbb{Z}$ and $|w|\leq 1/2$

Application: Chemical potential in AdS_3

Application 2: generic black holes

Susskind: an asymptotic observer throws in 1 string



- As it falls, the string is seen to elongate dramatically near the black hole horizon (distance ℓ_s) Susskind '94
- Elongation not seen by free-falling observers, only by fiducial observers
- Two different situations with long strings: high T and near horizon ? $\Rightarrow$ Related ?

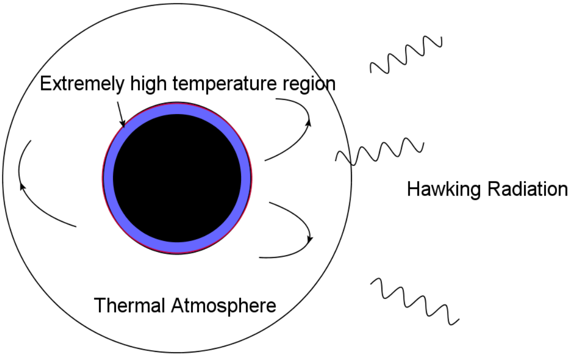
Thermodynamics and Black holes

Susskind's story from canonical point of view
Recall: Unruh effect (intrinsic to black holes)

- Close to the horizon, infalling QFT vacuum around the black hole is viewed by fiducial observers as thermally populated with temperature $T = \frac{1}{2\pi\rho}$ where $\rho \sim \sqrt{r - r_s}$
  For $r \to r_s$, T diverges (QG effects should be important!)

Thermal gas is emergent from the black hole quantum mechanics with $T = T_{\text{Hawking}}$

Thermal structure of black holes



In thermal atmosphere: Rindler approximation of metric:
$ds^2 = -\frac{\rho^2}{(4GM)^2}dt^2 + d\rho^2 + d\mathbf{x}_\perp^2$.
$\Longrightarrow$ Study string theory on thermal manifold = Euclidean Rindler space: $ds^2 = \frac{\rho^2}{(4GM)^2}d\tau^2 + d\rho^2 + d\mathbf{x}_\perp^2$, $\tau \sim \tau + 8\pi GM$

Results from string theory (1)

Approach: Small curvature (large k) limit of $SL(2,\mathbb{R})/U(1)$ cigar
$\Rightarrow$ Euclidean Rindler space Giveon-Itzhaki '13
$\Rightarrow \exists\ w = 1$ state in thermal spectrum
From coset construction $\Rightarrow$ α'-exact thermal scalar field theory for type II superstrings Verlinde-Verlinde-Dijkgraaf '92

From thermal scalar FT perspective:

$$ds^2 = \frac{\rho^2}{(4GM)^2} d\tau^2 + d\rho^2 + d\mathbf{x}_\perp^2$$

Wave operator: $\hat{\mathcal{O}} = -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \frac{2}{\alpha'} + \frac{\beta^2\rho^2}{4\pi^2\alpha'^2(4GM)^2}$

Spectrum (enforcing regularity at the origin and at infinity):

$$\psi_n(\rho) \propto \exp\left(-\frac{\beta\rho^2}{4\pi\alpha'(4GM)}\right) L_n\left(\frac{\beta\rho^2}{2\pi\alpha'(4GM)}\right)$$

$$\lambda_n = \frac{\beta - 8\pi GM + 2\beta n}{\pi\alpha'(4GM)} \qquad , \ n \geq 0$$

With $\beta = \beta_{\text{Hawking}} = 8\pi GM$ and $n = 0$ (lowest mode)

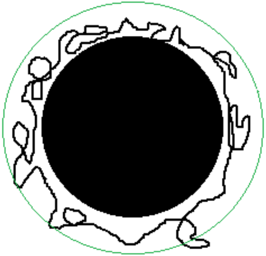
$$\psi_0 \propto \exp\left(-\frac{\rho^2}{2\ell_s^2}\right), \quad \lambda_0 = 0$$

Results from string theory (2)

$$\psi_0 \propto \exp\left(-\frac{\rho^2}{2\ell_s^2}\right), \quad \lambda_0 = 0$$

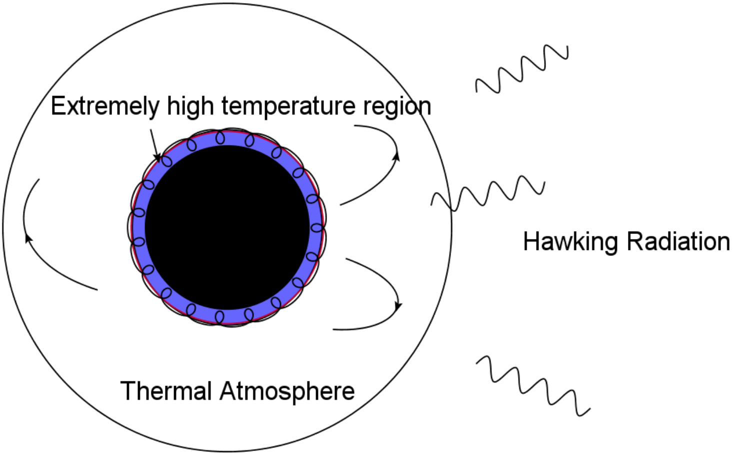
- Bound to black hole, width $\sim \ell_s \Rightarrow$ Random walk near horizon
- Even though no reference to ℓ_s is made in the background, the thermal scalar ψ_0 is sensitive to ℓ_s (reason = T-duality)
- Predicts $T_H = T_{\text{Hawking}}$

Gas around black hole behaves as $T = T_H$ gas of flat space!
Thermal scalar dominates one-loop TD



Susskind's picture of long string surrounding black hole horizon

Resulting thermal structure



Disclaimer: only genus 1 $\Rightarrow$ interactions expected to be important near horizon

Stress Tensor (1)

Energy-momentum tensor of string gas for type II superstrings

Horowitz-Polchinski '98

Microcanonical

High-energy averaged stress tensor

$$\langle T^{\mu\nu}(\mathbf{x})\rangle_E \approx \frac{E}{2N\beta_H^2} T^{\mu\nu}_{\text{th.sc.}}(\mathbf{x})$$

$\Leftrightarrow$

Canonical

Near-Hagedorn stress tensor (compact space):

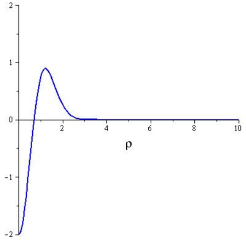
$$\langle T^{\mu\nu}(\mathbf{x})\rangle_\beta \approx \frac{1}{2N\beta_H^2} T^{\mu\nu}_{\text{th.sc.}}(\mathbf{x}) \frac{1}{\beta-\beta_H}$$

- Classical stress tensor evaluated on lowest eigenmode ψ_0 of thermal scalar
- Thermal scalar wavefunction determines where stress-energy is located

Stress Tensor (2)

Example Near-Hagedorn energy density in Rindler space

Rindler space: $\psi_0 \propto \exp\left(-\frac{\rho^2}{2\ell_s^2}\right)$

$\left\langle T_0^0 \right\rangle_{\text{thermal}} =$ 

- Energy density located close to black hole horizon
- Negative energy density zone is present (violates weak-energy condition)

  Generic feature in curved spacetime
- Consistency Check: $E_{\text{tot}} = \partial_\beta(\beta F)$

Conclusions

1. High-temperature string theory contains a divergence, the Hagedorn divergence, related to long string dominance
   Thermal scalar is singly wound state on thermal manifold and encodes the random walk near-Hagedorn thermodynamics of the gas
2. Random walk picture can be generalized to curved spacetimes using 1^{st} quantized string path integral methods or 2^{nd} quantized field theory methods
3. AdS_3 contains infinitely many states becoming marginal at T_H
   Thermal scalar wavefunction spreads over entire space
4. Black hole atmosphere behaves precisely the same as the $T = T_H$ gas of flat space: long strings are present
   The long string configuration is close to the event horizon
5. Other properties such as the stress tensor of the near-Hagedorn gas can also be analyzed

Thank you!