Lecture 1

1. General relativity describes the dynamical response of spacetime to energy and, conversely, the dynamical response of matter to the curvature of spacetime. Einstein proposed our modern theory of classical gravity exactly one hundred years ago. Since then, the gravitational interaction has been studied from sub-millimeter distances to cosmic scales, and so far, no experiment has invalidated general relativity. One of the surprising features of the theory is that it allows for so-called *singular* solutions, which are solutions where the curvature of spacetime becomes very large. Such solutions are realized in our own Universe. *Black holes* are the paradigmatic examples.

   The **Einstein equation** is

$$\boxed{\boxed{G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}}} \ . \tag{1}$$

   We observe the left hand side is purely geometric while the right hand side describes the matter in the theory. Taking the trace of both sides $R = -\frac{8\pi G_N}{c^4}T$. Plugging this in to (1), we can equivalently write the Einstein equation as

$$R_{\mu\nu} = \frac{8\pi G_N}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \ . \tag{2}$$

   When $T_{\mu\nu} = 0$, we are in vacuum. Here, the Einstein equation is simply

$$R_{\mu\nu} = 0 \ . \tag{3}$$

   The **Schwarzschild metric** is the unique spherically symmetric solution to the vacuum Einstein equation:

$$ds^2 = -\left(1 - \frac{2G_NM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2G_NM}{rc^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ . \tag{4}$$

   The *Schwarzschild radius* is

$$R_S = \frac{2G_NM}{c^2} \ . \tag{5}$$

   The parameter M in the metric line element specifies the mass of the black hole. Note that as $M \to 0$, we recover the Minkowski metric. Far away from the mass M, as $r \to \infty$, we also have the flat metric on $\mathbb{R}^{1,3}$. This property is referred to as *asymptotic flatness.*

2. **Exercise:** *Consider the vacuum Einstein equation with non-zero cosmological constant Λ. Guess a spherically symmetric Schwarzschild solution in this spacetime and show that it solves the vacuum Einstein equations with $\Lambda \neq 0$.*

   The vacuum Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\Lambda}{c^2}g_{\mu\nu} = 0 \ . \tag{6}$$

   In our conventions, $[\Lambda] = [\text{time}]^{-2}$. Let us work in d spacetime dimensions. Taking the trace of both sides

$$R = \frac{2d}{d-2}\frac{\Lambda}{c^2} = \text{constant} \ . \tag{7}$$

We know that de sitter (anti-de Sitter) space is the maximally symmetric solution to the Einstein equation with constant positive (respectively, negative) curvature. These are hyperboloids in $\mathbb{R}^{1,d}$ and $\mathbb{R}^{2,d-1}$, respectively:

$$-(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 = L^2 \qquad (\mathrm{dS}) \ , \tag{8}$$

$$(X^0)^2 + (X^d)^2 - \sum_{i=1}^{d-1}(X^i)^2 = L^2 \qquad (\mathrm{AdS}) \ . \tag{9}$$

As these are Einstein spaces,

$$R_{\mu\nu} = \pm\frac{d-1}{L^2}g_{\mu\nu} \qquad \Longrightarrow \qquad R = \pm\frac{d(d-1)}{L^2} \qquad \Longrightarrow \qquad \Lambda = \pm\frac{(d-1)(d-2)}{2L^2} \ . \tag{10}$$

The plus sign is dS, and the minus sign is AdS. We can see that (8) is solved by

$$X^0 = \sqrt{L^2-r^2}\sinh\frac{ct}{L} \ , \quad X^d = \sqrt{L^2-r^2}\cosh\frac{ct}{L} \ , \quad X^i = r\omega_i \ , \tag{11}$$

where $\omega_{i=1,\ldots,d-1}$ are the angular variables on the unit sphere $S^{d-2} \subset \mathbb{R}^{d-1}$. In static coordinates, the metric of de Sitter space is then induced from the metric on $\mathbb{R}^{1,d}$:

$$ds^2 = -\left(1-\frac{r^2}{L^2}\right)c^2dt^2 + \left(1-\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2\,d\Omega_{d-2}^2 \ . \tag{12}$$

Similarly, for AdS, we have

$$X^0 = \sqrt{L^2+r^2}\sin\frac{ct}{L} \ , \quad X^d = \sqrt{L^2+r^2}\cos\frac{ct}{L} \ , \quad X^i = r\omega_i \ , \quad \sum_{i=1}^{d-1}\omega_i^2 = 1 \ , \tag{13}$$

$$\Longrightarrow \qquad ds^2 = -\left(1+\frac{r^2}{L^2}\right)c^2dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2\,d\Omega_{d-2}^2 \ . \tag{14}$$

The spherically symmetric form of the metric is

$$ds^2 = -f(r)c^2dt^2 + f(r)^{-1}dr^2 + r^2\,d\Omega_{d-2}^2 \ . \tag{15}$$

For large values of L, it is difficult to distinguish dS or AdS from flat space. Thus, the natural ansatz is

$$f(r) = 1 \mp \frac{r^2}{L^2} - \frac{C_d G_d M}{r^{d-3}c^2} \ , \tag{16}$$

where C_d is some dimension dependent numerical constant. The powers of r are determined by the gravitational potential in d dimensions. It turns out that

$$C_d = \frac{16\pi}{(d-2)S_{d-2}} \ , \qquad S_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \ . \tag{17}$$

Here, S_n is the area of a unit n-sphere. We can check that $C_4 = 2$ as required. Verifying that this guess is correct by plugging in to the vacuum Einstein equation is left to you.

3. We notice that two potential **singularities** occur in the Schwarzschild metric. These are at

$$r = R_S \ , \qquad r = 0 \ , \tag{18}$$

corresponding to where the metric functions diverge. We determine whether these are coordinate singularities or physical singularities by probing the behavior of curvature invariants, in which all of the indices are contracted so that we have a basis independent scalar quantity. Since $R_{\mu\nu} = 0$, it follows the Ricci scalar $R = 0$, so this is not a good invariant to test. A non-vanishing curvature invariant turns out to be

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12 R_S^2}{r^6} \ . \tag{19}$$

At $r = R_S$, this is $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 12/R_S^4$, which is finite, whereas $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ blows up at $r = 0$. All of the other non-trivial curvature invariants that we can write share this property. A clever choice of coordinates removes the singularity at $r = R_S$, whereas this is not possible for $r = 0$. This is problematic because at the singularity one of the central assumptions of general relativity, that geometry is smooth, breaks down. General relativity is a classical theory and is not the end of physics. We have so far nowhere invoked quantum mechanics. Because general relativity and quantum mechanics are incompatible — the former is non-renormalizable because the coupling $G_N \sim L^3 M^{-1} T^{-2}$ has negative mass dimension — we need to extend both theories. It is expected that a fully quantum mechanical description of gravitation explains the physics at the spacetime singularities such as those we encounter in black holes and cosmology. We will offer a scenario in string theory wherein the classical singularity of certain supersymmetric black holes is explained as an artifact of gravitational thermodynamics.

4. Consider the equation for escape velocity in Newtonian theory:

$$\frac{1}{2} m v_e^2 - \frac{G_N M m}{r} = 0 \quad \Longrightarrow \quad v_e = \sqrt{\frac{2 G_N M}{r}} \ . \tag{20}$$

Now suppose that the escape velocity was the speed of light c. We have $r = R_S$. For distances smaller than R_S even light cannot escape the attractive force of gravity.

The **event horizon** represents the boundary beyond which events in the interior spacetime cannot affect an observer in the exterior spacetime. It is the point of no return. The horizon is a *null surface*, meaning that normal vectors to the surface are null (they have length zero with respect to the local Lorentzian metric). For our purposes, we define $g_{tt} = 0$ as specifying the *static limit* and $g^{rr} = 0$ as defining the *horizon*. These are coincident for the Schwarzschild solution. From the metric (4), we see that

$$g^{rr} = \left(1 - \frac{R_S}{r}\right) \quad \Longrightarrow \quad g^{rr} = 0 \quad \text{when} \quad r = R_S \ . \tag{21}$$

In the classical theory, the horizon at $r = R_S$ shields us from the singularity. This observation is in accord with the *cosmic censorship conjecture*, which argues that naked singularities cannot form in gravitational collapse from generic states under the dominant energy condition.

5. *Birkhoff's theorem* states that (4) is the unique, spherically symmetric solution to the vacuum Einstein equation. The metric (4) is well behaved for $R > R_S$ (exterior region) and also for $0 < r < R_S$ (interior region). The locus $R = R_S$ is the **event horizon** of the Schwarzschild black hole. It is a null surface that separates the exterior region from the interior region. At $R = R_S$, there is a coordinate singularity. The geometry has a genuine **singularity** at $r = 0$.

This metric is both **static** and **stationary**. The former is a subset of the latter. For a stationary solution, there is a Killing vector that is timelike near infinity. This Killing vector is ∂_t, and we can choose coordinates so that the metric is

$$ds^2 = g_{00}(\{x^k\})dt^2 + 2g_{0i}(\{x^k\})dt\,dx^i + g_{ij}(\{x^k\})dx^i dx^j \ . \tag{22}$$

In particular, the metric is independent of the time coordinate. It can, however, include functions of the spatial coordinates, $\{x^k\} = x^1, \dots, x^{d-1}$.

For a static solution, the timelike Killing vector is orthogonal to a family of hypersurfaces. The hypersurfaces to which the timelike Killing vector is orthogonal are defined by $t =$ constant. We can write the metric so that there are no off diagonal components in the metric linking space with time:

$$ds^2 = g_{00}(\{x^k\})dt^2 + g_{ij}(\{x^k\})dx^i dx^j \ . \tag{23}$$

Static spacetimes are stationary and also invariant under time reversal, which is the map $t \to -t$. While the stationary metric does the same thing at every time, the static metric does nothing at all at any time. The *Kerr metric* is stationary but not static:

$$\begin{aligned} ds^2 \ &= \ -\left(1 - \frac{R_S r}{\rho^2}\right)c^2 dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + \alpha^2 + \frac{R_S r \alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2 \\ &\quad - \frac{2R_S r\alpha\sin^2\theta}{\rho^2}c\,dt\,d\phi \ , \end{aligned} \tag{24}$$

where

$$\alpha = \frac{J}{Mc} \ , \quad \rho^2 = r^2 + \alpha^2\cos^2\theta \ , \quad \Delta = r^2 - R_S r + \alpha^2 \ . \tag{25}$$

The parameter J describes the angular momentum of a spinning black hole. There is a $dt\,d\phi$ cross term in the metric. These are the astrophysical black holes that astronomers observe in the sky. There are other solutions similar to Schwarzschild and Kerr with electric charges, for example the Reissner–Nordström black hole and the Kerr–Newman black hole.

There is a *no hair theorem* which in four dimensions states that once the mass, electric charge, and angular momentum are fixed, the metric of the black hole is determined uniquely in Einstein–Maxwell theory. The horizon always has a spherical topology. In higher dimensions, there are always black holes with horizon topology S^{d-2}. These are the Myers–Perry solutions, which are the analog of the Kerr black hole. In fact, in five dimensions, these black holes can have two independent angular momenta, J_1 and J_2. This is because there are $\lfloor \frac{d-1}{2} \rfloor$ orthogonal planes of rotation in d spacetime dimensions. The no hair theorem fails in higher dimensions, however. The gravitational self-attraction of a *black ring* can exactly balance the centrifugal force so that a solution with horizon topology $S^2 \times S^1$ is possible for $d = 5$:

$$U_{\rm grav} \sim -\frac{G_d M}{r^{d-3}} \ , \qquad U_{\rm cent} \sim \frac{J^2}{M^2 r^2} \ . \tag{26}$$

The black ring was discovered in 2001 by Emparan and Reall.

Lecture 2

6. Black holes were studied in semiclassical gravity in the 1970s. By *semiclassical* we mean that we treat the matter sector as a quantum field theory on a curved spacetime, which is a solution to general relativity. We treat the gravitational field classically, however. In this context, it

was realized that black holes are thermodynamic objects. The laws of black hole mechanics are stated in analogy to the laws of ordinary thermodynamics.

The zeroth law of thermodynamics states that the temperature T at equilibrium is constant. For a black hole, there is a quantity called *surface gravity* that describes the gravitational acceleration at the surface. On Earth, this is $g \approx 9.81$ m/s^2. Depending upon where we are on the Earth, which has an equatorial bulge, this figure changes slightly. Remarkably, the surface gravity κ of a stationary black hole is constant everywhere on the horizon. This is the **zeroth law of black hole mechanics**. When $\kappa = 0$, the black hole is said to be *extremal*; it is *non-extremal* otherwise. For the Kerr solution,

$$J \leq \frac{G_N M^2}{c} \ . \tag{27}$$

This inequality is saturated at extremality. When we say that the surface gravity is zero, we mean that rotation balances gravitational attraction.

The first law of thermodynamics expresses the conservation of energy. We have a state function, the entropy, which is a function of energy, volume, and particle number. Inverting this to write $E(S, V, N)$, we then have

$$dE = \left.\frac{\partial E}{\partial S}\right|_{V,N} dS + \left.\frac{\partial E}{\partial V}\right|_{S,N} dV + \left.\frac{\partial E}{\partial N}\right|_{S,V} dN = T\,dS - P\,dV + \mu, dN \ , \tag{28}$$

where E is energy, T is temperature, S is entropy, P is pressure, V is volume, μ is chemical potential, and N is particle number.

The no hair theorem for black holes explains that in Einstein–Maxwell theory, a solution with spherical horizon topology is specified by knowing the mass M, angular momentum J, and electric charge Q. These quantities give us the area of the event horizon:

$$A = 4\pi \left[\frac{2G_N^2 M^2}{c^4} - \frac{G_N Q^2}{4\pi\epsilon_0 c^4} + \frac{2G_N M}{c^2}\sqrt{\frac{G_N^2 M^2}{c^4} - \frac{G_N Q^2}{4\pi\epsilon_0 c^4} - \frac{J^2}{M^2 c^2}} \right] \ . \tag{29}$$

Inverting this, we can express the mass of a black hole as a function of area, angular momentum, and electric charge:

$$M = \sqrt{\frac{\pi}{A}} \left[\frac{c^4}{G_N^2} \left(\frac{A}{4\pi} + \frac{G_N Q^2}{4\pi\epsilon_0 c^4} \right)^2 + 4\frac{J^2}{c^2} \right]^{\frac{1}{2}} \ . \tag{30}$$

Differentiating, we find the **first law of black hole mechanics**:

$$dE = d(Mc^2) = \frac{\kappa c^2}{8\pi G_N} dA + \Omega\, dJ + \Phi\, dQ \ , \tag{31}$$

where the left hand side describes the differential change in the energy. On the right hand side, Ω is the angular velocity and Φ is the electric potential. In analogy to the first law of thermodynamics, the **Hawking temperature** and **Bekenstein–Hawking entropy** are identified as

$$T_H = \frac{\hbar\kappa}{2\pi k_B c} \ , \quad S_{BH} = \frac{A k_B c^3}{4\hbar G_N} \ . \tag{32}$$

This means extremal black holes have zero temperature. Sending $\hbar \to 0$, the temperature vanishes. All classical black holes are extremal. Temperature is a semiclassical property that we have invoked quantum mechanics to define.

This is the first appearance of k_B and $\hbar$ in our story. The k_B is not mysterious — it simply sets the units of entropy and temperature. The $\hbar$ indicates that the entropy of a black hole is an intrinsically quantum effect. It tells us that entropy counts the quantum states of a system. We expect the number of states is

$$\Gamma = \exp\left(\frac{S_{BH}}{k_B}\right) \quad \Longleftrightarrow \quad S_{BH} = k_B \log \Gamma \ . \tag{33}$$

In the general setting, we do not know what black hole statistical physics is, so the identification of the quantum states of a black hole is still mysterious. We will explore progress on this problem in highly symmetric settings.

The second law of thermodynamics states that entropy is a non-decreasing function in time: $\Delta S \geq 0$. We can imagine taking a highly entropic object of low energy and throwing it into a black hole. What happens to the entropy of the Universe then? The *weak energy condition* demands that for every timelike vector t^μ,

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \ . \tag{34}$$

This implies that the density $\rho \geq 0$ and the combination of density and pressure $\rho c^2 + P \geq 0$. As a consequence of the weak energy condition

$$\frac{dA}{dt} \geq 0 \ . \tag{35}$$

It is tempting then to take seriously the identification of entropy with horizon area. If we take an ordinary thermodynamic system and throw it into a black hole, the entropy of the Universe seemingly decreases because an external observer cannot probe behind the horizon. In order to account for this, we must write a **generalized second law of black hole mechanics** so that in any physical process

$$\Delta S_{BH} + \Delta S_X \geq 0 \ . \tag{36}$$

Here, ΔS_X is the change in the entropy of the rest of the Universe.

In ordinary thermodynamics, the entropy is an *extensive* state variable. It adds for independent, non-interacting systems. If we increase the size of the system by a factor λ, the entropy $S \to \lambda S$ in response. The intensive quantities, temperature, pressure, etc., are independent of the system size. According to (32), the entropy of a black hole (the logarithm of the number of states) scales with the area of the event horizon rather than the volume contained within the event horizon. This is a dramatic revision of the expected behavior of thermodynamic systems.

The third law of thermodynamics states that we cannot reduce the entropy of a system to its absolute zero value at which the entropy enumerates the degeneracy of the ground state. The analogous statement in gravity, **third law of black hole mechanics** is that it is not possible to form a black hole with vanishing surface gravity in a finite number of steps without violating energy conditions. Note that black holes can have finite area and therefore finite entropy at zero temperature.

7. Let us contemplate the collapse process in the context of the following *gedankenexperiment*. In quantum mechanics, we can always prepare a pure state. Suppose two degenerate pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ have the same mass M, angular momentum J, and electric charge Q. The two states may differ in their other properties, for instance their shape and size. Let us dial the

Newton constant to make gravity stronger. Most objects (for example, the solar system) would become smaller as the strength of the gravitational interaction, the coupling G_N, is increased. When the mass occupies a volume smaller than $\frac{4}{3}\pi R_S^3$, it collapses to a black hole. Note that once the black hole forms, as $R_S \propto G_N$, it becomes larger as we make gravity stronger. Again, we see that black holes behave in a counterintuitive manner.

Both $|\psi_1\rangle$ and $|\psi_2\rangle$ collapse to a geometry with the same metric. According to the no hair theorem, any black hole with the quantum numbers (M, J, Q) specifies the same spacetime. A naïve guess is that the entropy of a black hole enumerates all the possible initial states with the quantum numbers (M, J, Q). Entropy therefore parametrizes our ignorance about the true initial quantum state of the system that formed the black hole in the first place.

There is a problem, however. Quantum mechanics is a unitary theory, so it must preserve *information* about the initial pure state. We know that according to the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \ . \tag{37}$$

For a time independent Hamiltonian,

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle \ . \tag{38}$$

The data about the initial state cannot therefore be lost. How do we recover this information from a black hole? For an external observer, it seems that data about the initial state has disappeared once the black hole forms. This is the **information paradox** in a nutshell.

8. The vacuum in a quantum theory is dynamical. The Heisenberg uncertainty principle tells us that

$$\Delta E\, \Delta t \geq \frac{\hbar}{2} \ . \tag{39}$$

(Aharonov and Bohm tell us that we need to be careful with energy/time uncertainty relations, but let us accept this as more or less true.) In a short time, Δt, we can therefore have large uncertainties in the vacuum energy. This is sufficient to pair create a particle and an antiparticle. Let us imagine this process happens near the horizon of a black hole. Suppose the particle, say the electron, crosses the horizon. By momentum conservation, the positron travels in the opposite direction, toward the spatial asymptopia. From the perspective of an observer at infinity, the black hole seems to be emitting particles and thereby shedding mass. We can interpret this as a particle escaping quantum mechanically by tunneling through an infinite potential barrier at the horizon. This is **Hawking radiation**. We obtain this by applying quantum mechanical reasoning in a semiclassical description of the black hole geometry.

Note that the process is only sensitive to the geometry near the horizon, which is the same for all black holes with given (M, J, Q) quantum numbers. Thus, the Hawking radiation process does not know about any microstructure that may exist near the singularity. The temperature of the radiation is T_H, which means that extremal black holes do not radiate. The surface gravity of the Schwarzschild solution is

$$\kappa = \frac{c^4}{4G_N M} \ . \tag{40}$$

Plugging this expression into (32), we determine that

$$T_H = \frac{\hbar c^3}{8\pi G_N M k_B} \ . \tag{41}$$

For the Schwarzschild solution Hawking radiation is perfect *blackbody radiation* emitted isotropically at this characteristic temperature (41). These numbers are typically tiny: a solar mass black hole has a Hawking temperature of 10^{-8} K. This is comparable to the lowest temperatures achieved in a laboratory.

Because the black hole is not in thermal equilibrium with its surroundings, it loses mass and decreases in size. As the mass appears in the denominator of (41), the temperature increases as the black hole evaporates. Light black holes are hotter than more massive black holes. A black hole has negative specific heat. This evaporation process is depicted in Figure 1.

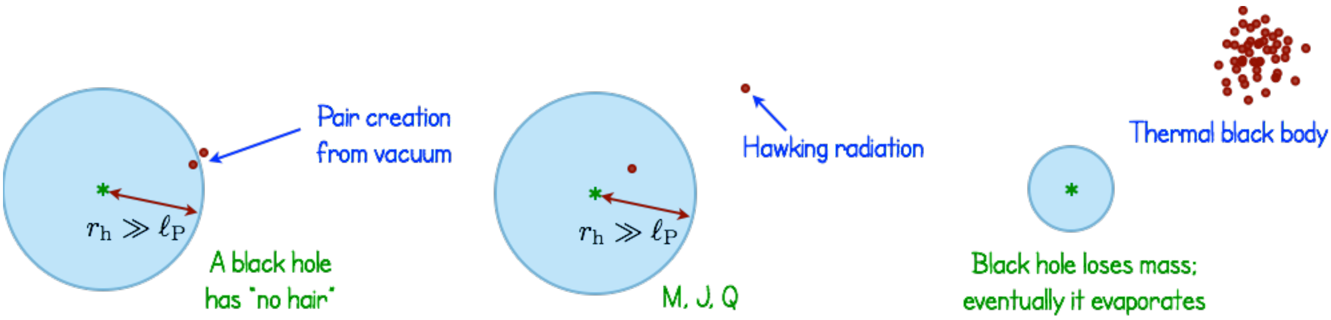

Figure 1: A black hole evaporates through *Hawking radiation*.

9. **Exercise:** *Calculate how long it takes for a solar mass black hole to evaporate.*

The surface area of the Schwarzschild black hole is

$$A = 4\pi R_S^2 = \frac{16\pi G_N^2 M^2}{c^4} \ . \tag{42}$$

According to the Stefan–Boltzmann law, the power emitted by the black hole is $\mathcal{P} = A\sigma T_H^4$:

$$\mathcal{P} = \frac{16\pi G_N M^2}{c^4} \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \left(\frac{\hbar c^3}{8\pi G_N M k_B}\right)^4 = \frac{\hbar c^6}{15360\pi G_N^2 M^2} \ . \tag{43}$$

Suppose no matter falls into the black hole during the evaporation process. Then the power is

$$\mathcal{P} = -\frac{d(Mc^2)}{dt} = \frac{\hbar c^6}{15360\pi G_N^2 M^2} \quad \Longrightarrow \quad dt = -\frac{15360\pi G_N^2}{\hbar c^4} M^2\, dM \ . \tag{44}$$

Integrating this expression, we find that the *evaporation time* for a black hole is

$$\begin{aligned} \int_0^{t_{\rm ev}} dt &= \frac{15360\pi G_N^2}{\hbar c^4} \int_0^{M_0} dM\ M^2 \\ t_{\rm ev} &= \frac{15360\pi G_N^2}{\hbar c^4} \frac{M_0^3}{3} = \frac{5120\pi G_N^2 M_0^3}{\hbar c^4} \ , \end{aligned} \tag{45}$$

where M_0 is the mass of the black hole at the time it was formed. These time scales are typically huge: a solar mass black hole takes 10^{57} Gyr to evaporate. By comparison, the Universe is only 13.8 Gyr old. [*N.B.* We cheated. The cosmic microwave background (CMB) has a temperature $T = 2.725$ K. Only black holes with a present day temperature greater than this can evaporate. Black holes at a lower temperature (such as solar mass black holes) will eventually come into thermal equilibrium with the CMB, which acts as a heat bath, and never fully evaporate.]

Suppose an observer at infinity very carefully collects all the Hawking radiation the black hole emits. How does this recover the data about the initial state that formed the black hole in the first place? This is a restatement of the information paradox. To date, there isn't an exact answer to this puzzle. Because we believe in the primacy of quantum mechanics, the puzzle must have a resolution in quantum gravity. In string theory, certain *dualities* ensure unitarity, but the recipe whereby the information is recovered is still to be determined. The hope is that investigations in this direction will lead us to an understanding of the quantum states of a black hole that explain the entropy. This is what we turn to in the next lecture.

10. The expressions above included $\hbar$ to emphasize the quantum nature of entropy. For the sake of my sanity, we will now switch to natural units, where $\hbar = c = k_B = \frac{1}{4\pi\epsilon_0} = 1$. In these units [length] = [time] = $[\text{energy}]^{-1}$ = $[\text{mass}]^{-1}$ = $[\text{temperature}]^{-1}$, [angular momentum] = [charge] = $[\text{length}]^0$, and $[G_N] = [\text{length}]^2$. The four dimensional reduced Planck mass is $M_P = 2.435 \times 10^{18}$ GeV. (By comparison, the beam energy at LHC is 13 TeV.) This is the ultraviolet scale at which quantum gravity effects manifest. We will assume that the string scale is near the Planck scale.

11. Recall that the partition function in the canonical ensemble is the sum over Boltzmann factors for each of the states:

$$Z = \sum_j e^{-\beta E_j} = \sum_j \langle j|e^{-\beta H}|j\rangle = \operatorname{tr} e^{-\beta H} \ . \tag{46}$$

Here, E_j is the energy of a state $|j\rangle$ and β is the inverse temperature, which is held fixed. Let us compare this expression to the propagator at time $\tau = -i\beta$:

$$K(q', -i\beta; q, 0) = \langle q'|e^{-iH(-i\beta)}|q\rangle = \langle q'|e^{-\beta H}\sum_j |j\rangle\langle j|q\rangle = \sum_j e^{-\beta E_j}\langle q'|j\rangle\langle j|q\rangle \ . \tag{47}$$

The time is imaginary due to *analytic continuation*. If we put $q' = q$ and integrate over the position,

$$\int dq\ K(q, -i\beta; q, 0) = \sum_j e^{-\beta E_j}\langle j| \int dq\ |q\rangle\langle q|j\rangle = Z \ . \tag{48}$$

The partition function of a statistical system is the integral of the propagator evaluated at imaginary time. The time evolution operator in quantum mechanics maps to the basic object in statistical physics.

In quantum field theory, we can recast the previous expression as

$$\operatorname{tr} e^{-\beta H} = \int dq \int [D\Phi]\ e^{-S_E[\Phi]} \ , \tag{49}$$

where $\Phi(\beta) = \Phi(0) = q$ is the boundary condition on the field Φ. The periodicity in Euclidean time t_E is identified with the inverse temperature.

Remember that the Schwarzschild metric is

$$ds^2 = \left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} + r^2\, d\Omega_2^2 \ , \qquad R_S = 2G_N M \ . \tag{50}$$

Let us perform a change of coordinates $z = r - R_S$. Assuming that $z \ll R_S$ yields the metric in the *near-horizon limit*. In this limit, the metric at a fixed angle on S^2 becomes

$$ds^2 \approx -\frac{z}{R_S} dt^2 + \frac{R_S}{z} dz^2 \tag{51}$$

up to corrections that are $O(R_S^{-1})$. For convenience, let us define $\rho^2 = 4R_S z$ and rewrite the metric in terms of the surface gravity κ:

$$ds^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2 \ , \qquad \kappa = \frac{1}{2R_S} \ . \tag{52}$$

Analytically continuing time: $t \mapsto -it_E$, we find

$$ds^2 = \kappa^2 \rho^2 dt_E^2 + d\rho^2 \ . \tag{53}$$

Notice that $\kappa\, t_E$ is just an angle θ. This is the metric in polar coordinates so long as $\theta \in [0, 2\pi)$. The Euclidean time then has periodicity

$$\beta = \frac{2\pi}{\kappa} = 8\pi G_N M \qquad \Longrightarrow \qquad T_H = \frac{1}{8\pi G_N M} \ . \tag{54}$$

Lecture 3

12. The remainder of the notes are based on Mathur, hep-th/0502050.

We will consider BPS states in string theory. These saturate some bound relating charge to mass; supersymmetric states of this type typically fall in short multiplets of the algebra. For definiteness, let us start in critical type IIA string theory. This is a theory of closed and oriented superstrings in ten dimensions in which left and right movers transform under separate spacetime supersymmetries with opposite chiralities. We have F1-strings, NS5-branes, and Dp-branes, for p even. Suppose we have a compact circle $y \sim y + 2\pi R$. A one charge black hole is obtained from wrapping a fundamental string about this circle. Suppose such a string has winding number n_1. The metric is

$$ds^2 = H_1(r)^{-1}(-dt^2 + dy^2) + \sum_{i=1}^{8} (dx^i)^2 \ , \qquad H_1(r) = \left(1 + \frac{Q_1}{r^6}\right) = e^{-2\phi} \ , \tag{55}$$

where $r = \delta_{ij} x^i x^j$. In the limit $r \to 0$, the dilaton behaves as follows:

$$\phi = -\frac{1}{2} \log \left(1 + \frac{Q_1}{r^6}\right) \longrightarrow -\infty \ . \tag{56}$$

The length of the y circle goes to zero, so the horizon vanishes. From the M-theory perspective, we interpret the fundamental string as an M2-brane wrapping y and R_{10}. The tension in the brane collapses the R_{10} cycle, and thus $g_s \sim e^{\phi}$ goes to zero. As the F1-string has 2^8 oscillator ground states, the entropy is $S = \log(256)$, which is independent of n_1.

To construct a two charge black hole, let us consider the space $S^1 \times T^4 \times \mathbb{R}^5$. We wrap the F1-strings on the S^1 and NS5-branes on $S^1 \times T^4$. The metric is

$$ds^2 = H_1(r)^{-1}(-dt^2 + dy^2) + \sum_{a=1}^{4} (dz^a)^2 + H_5(r) \sum_{i=1}^{4} (dx^i)^2 \ , \qquad H_i(r) = 1 + \frac{Q_i}{r^2} \ , \qquad e^{2\phi} = \frac{H_5}{H_1} \ , \tag{57}$$

where the z^a parameterize the T^4. As $r \to 0$, the dilaton becomes

$$\phi = \frac{1}{2} \log \frac{Q_5}{Q_1} \ . \tag{58}$$

The y circle still shrinks to zero size, so again the horizon vanishes.

Let us a third charge, representing momentum along the y circle. The contribution to the energy of n_p units of momentum on the compact direction is n_p/R, which counteracts the linear increase in energy with the winding number. Here, the metric is

$$ds^2 = H_1(r)^{-1}[-dt^2 + dy^2 + K(dt+dy)^2] + \sum_{a=1}^{4}(dz^a)^2 + H_5(r)\sum_{i=1}^{4}(dx^i)^2 \ , \tag{59}$$

$$H_i(r) = 1 + \frac{Q_i}{r^2} \ , \qquad e^{2\phi} = \frac{H_5}{H_1} \ , \qquad K = \frac{Q_p}{r^2} \ , \qquad e^{2\phi} = \frac{H_5}{H_1} \ . \tag{60}$$

We can now compute the length of the string wrapping the y circle at $r = 0$:

$$L = (2\pi R)\sqrt{\frac{K}{H_1}} \approx 2\pi R\sqrt{\frac{Q_p}{Q_1}} \ . \tag{61}$$

In the transverse directions, the metric is

$$H_5(r)\sum_{i=1}^{4}(dx^i)^2 \approx Q_5(\frac{dr^2}{r^2} + d\Omega_3^2) \ , \tag{62}$$

which means that the area of the sphere as $r \to 0$ is $A = Q_5^{3/2}\, S_3$, where $S_3 = 2\pi^2$. The area of the horizon in string frame is then

$$A_S = \left[2\pi R\left(\frac{Q_p}{Q_1}\right)^{\frac{1}{2}}\right]\left[2\pi^2 Q_5^{\frac{3}{2}}\right]\left[(2\pi)^4 V\right] \ , \tag{63}$$

where the last factor is the volume of T^4. The Einstein metric and the string sigma model metric differ by a dilaton dependent Weyl transformation: $g^E_{\mu\nu} = e^{-\phi/2} g^S_{\mu\nu}$. Thus, in Einstein frame, the area is

$$A = A_S\frac{Q_1}{Q_5} = (2\pi^2)(2\pi R)((2\pi)^4 V)\sqrt{Q_1 Q_5 Q_p} \ . \tag{64}$$

Noting that

$$G_5 = \frac{G_{10}}{(2\pi R)((2\pi)^4 V)} \ , \tag{65}$$

we have

$$S_{BH} = \frac{A}{4G_{10}} = \frac{(2\pi^2)(Q_1 Q_5 Q_p)^{\frac{1}{2}}}{4G_5} \ . \tag{66}$$

The ten dimensional Newton constant is $G_{10} = 8\pi^6 g_s^2 \alpha'^4$. On dimensional grounds

$$Q_1 = \frac{g_s^2\alpha'^3}{V} n_1 \ , \qquad Q_5 = \alpha' n_5 \ , \qquad Q_p = \frac{g_s^2\alpha'^4}{VR^2} n_p \ . \tag{67}$$

Then

$$S_{BH} = 2\pi\sqrt{n_1 n_5 n_p} \ . \tag{68}$$

13. Let the y direction be x^5 and the T^4 be parameterized by $x^6, \ldots, x^9$. A T-duality on the x^6 direction gives us a system of F1/NS5/P in type IIB. Now, performing an S-duality, we have D1/D5/P. Through dualities, we can permute these charges.

Exercise: *Using T-duality and S-duality, map D1/D5/P to P/F1/NS5.*

$$\begin{aligned} S: &\quad \text{F1/NS5/P} \\ T_{56}: &\quad \text{P/NS5/F1} \\ S: &\quad \text{P/D5/D1} \\ T_{6789}: &\quad \text{P/D1/D5} \\ S: &\quad \text{P/F1/NS5} \end{aligned}$$

14. In calculating the entropy, we will need to enumerate quantum states. We can do this in two dimensional CFTs by examining the leading term in the high temperature expansion of the partition function. This enables us to solve certain counting problems in a large-N limit.

    Let us take a step backward and examine the partitions of positive integers m. A **partition** is a collection of positive integers $\{m_1, \ldots, m_k\}$ such that $\sum_{i=1}^k m_i = m$. For example, $\{4, 7\}$ is one possible way of partitioning 11. The expression $p(m)$ counts the number of partitions of m. Let us suppose that there is a function $f(q)$ such that the coefficients in its Taylor expansion are $p(m)$. What does this function look like? We write

$$\begin{aligned} \sum_{m=0}^{\infty} p(m) q^m &= (1 + q^1 + q^{1+1} + q^{1+1+1} + \ldots)(1 + q^2 + q^{2+2} + q^{2+2+2} + \ldots) \ldots \\ &= \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \ldots) \qquad (69) \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \ . \end{aligned}$$

    Suppose we wish to extract the q^5 term from the right hand side of the first expression. We have

$$q^{1+1+1+1+1} \ , \quad q^{1+1+1} q^2 \ , \quad q^{1+1} q^3 \ , \quad q^1 q^{2+2} \ , \quad q^1 q^4 \ , \quad q^2 q^3 \ , \quad q^5 \ . \qquad (70)$$

    These are the seven integer partitions of 5. Thus, $p(5) = 7$. The final expression in (69) is the generating function for the partitions of the integers. It was first written down by Euler, who also proved the *pentagonal number theorem*:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{g_k} \ , \qquad g_k = \frac{k(3k-1)}{2} = 0, 1, 2, 5, 7, 12, 15, \ldots \ , \qquad (71)$$

    where g_k is the k-th generalized pentagonal number. The expression is convergent for $|q| < 1$, but we can equally think of it is a formal power series. It is a special case of Jacobi's *triple product identity*. We can apply (71) to recast (69) as

$$\left(\sum_{k=-\infty}^{\infty} (-1)^k q^{g_k} \right) \left(\sum_{m=0}^{\infty} p(m) q^m \right) = 1 \ . \qquad (72)$$

    We can then use this result to derive a recursive expression for the number of partitions of n:

$$p(n) = \sum_{k=-\infty}^{\infty} (-1)^{k-1} p(n - g_k) \ . \qquad (73)$$

    This remains the most efficient method for computing the integer partitions.

Hardy and Ramanujan derived an asymptotic expression for $p(n)$:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \,, \qquad \text{as } n \to \infty \,. \tag{74}$$

(This was later refined by Rademacher, who found a convergent infinite series for $p(n)$.) The logarithm of the asymptotic expression (74) is

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log(4\sqrt{3}n) \sim 2\pi\sqrt{\frac{n}{6}} \,. \tag{75}$$

Physicists call this the *Cardy formula*.

15. To compute the entropy of the F1/P bound state, we have

$$m^2 = \left(2\pi RTn_1 + \frac{n_p}{R}\right)^2 + 8\pi TN_R = \left(2\pi RTn_1 - \frac{n_p}{R}\right)^2 + 8\pi TN_L \,. \tag{76}$$

In order to preserve supersymmetry, only the left moving modes are turned on, so $N_R = 0$. Then,

$$m = 2\pi RTn_1 + \frac{n_p}{R} \,, \qquad 8\pi T(N_L - n_1 n_p) = 0 \,. \tag{77}$$

The oscillator level $N_L = n_1 n_p$ is partitioned among 8 bosonic and 8 fermionic oscillators corresponding to the transverse directions. The central charge is then $c = 8 + \frac{1}{2}\cdot 8 = 12$. The number of states is then given to us by the Cardy formula:

$$\mathcal{N} \sim \exp\left(2\pi\sqrt{\frac{c}{6}N_L}\right) = e^{2\sqrt{2}\pi\sqrt{n_1 n_p}} \,. \tag{78}$$

To compute the entropy of the D1/D5/P bound state, let us work in the frame where we have P/F1/NS5. Suppose there is only one NS5 brane. As the F1 string lies along the NS5, it can only vibrate in this surface. There are four transverse oscillatory modes. The central charge is $c = 4 + \frac{1}{2}\cdot 4 = 6$, so the Cardy formula computes the entropy as

$$S = 2\pi\sqrt{n_1 n_p} \,. \tag{79}$$

Because of duality, we know that the answer must be symmetric under interchange of n_1, n_5, and n_p, which means

$$S = 2\pi\sqrt{n_1 n_5 n_p} \,. \tag{80}$$

This matches S_{BH} from before.

16. Lunin and Mathur then construct explicit microstate geometries. We need these geometries in a strong coupling regime in which the black hole exists, so the previous weak coupling expressions we have written are not good enough. Indeed, the enumeration of states we have seen earlier is at zero coupling, but the number is protected by the BPS condition. This is essentially the Strominger–Vafa computation of the entropy.

For the D1/D5 system, the solutions are

$$ds^2 = \sqrt{\frac{H}{1+K}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + \sqrt{\frac{1+K}{H}} dx^i dx^i + \sqrt{H(1+K)} dz^a dz^a \,, \tag{81}$$

where

$$
\begin{aligned}
H^{-1} &= 1 + \frac{\mu^2 Q_1}{\mu L_T}\int_0^{\mu L_T} \frac{dv}{|\vec{x}-\mu\vec{F}(v)|^2}\,, && (82)\\
K &= \frac{\mu^2 Q_1}{\mu L_T}\int_0^{\mu L_T} \frac{dv\ (\mu^2 \dot{F}(v))^2}{|\vec{x}-\mu\vec{F}(v)|^2}\,, && (83)\\
A_i &= -\frac{\mu^2 Q_1}{\mu L_T}\int_0^{\mu L_T} \frac{dv\ \mu\dot{F}_i(v)}{|\vec{x}-\mu\vec{F}(v)|^2}\,, && (84)\\
dB &= -\ast_4\, dA\,. && (85)
\end{aligned}
$$

I won't explain this in detail, but it suffices to say that the functions F in the previous expressions describe the profile of the effective string. Quantizing the vibrational modes should recover the Bekenstein–Hawking entropy that we noted previously. The near horizon geometry of these solutions is $\mathrm{AdS}_3 \times S^3$. By the gauge/gravity correspondence, we should be able to construct $e^{S_{BH}}$ states in CFT_2 with the same macroscopic charges as the black hole. This has been done for the D1/D5 system. The D1/D5/P system, which we have seen has a non-zero horizon area in supergravity, is more difficult. The state of the art up to a few years ago was that the number of explicit solutions scaled like $Q^{\frac{5}{4}} \ll Q^{\frac{3}{2}}$. In the past year, work by Bena, Giusto, Russo, Shigemori, and Warner suggest that there may be superstrata solutions whose number scale like $Q^{\frac{3}{2}}$.

17. The heuristic picture is depicted in Figure 2.

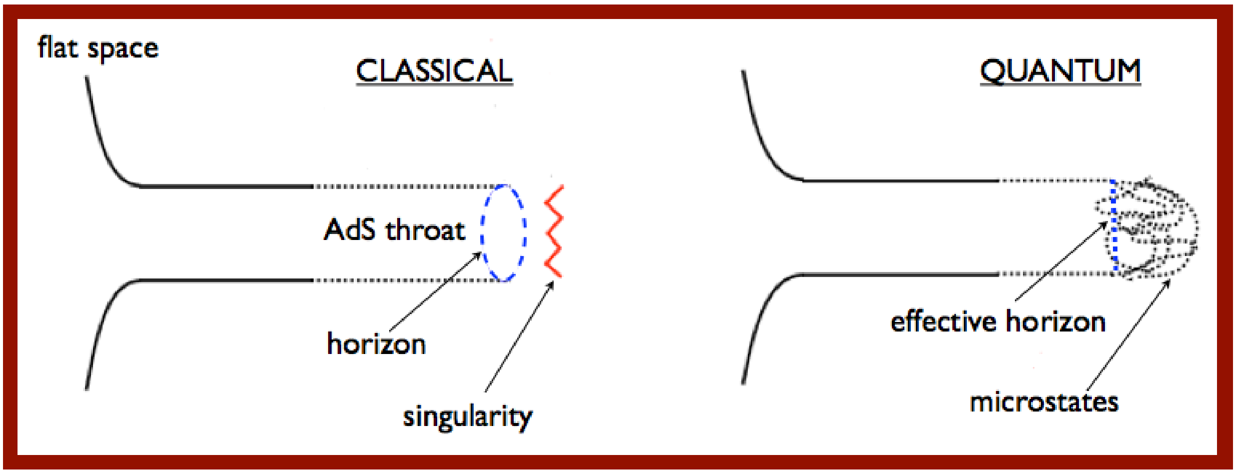

Figure 2: The *fuzzball* picture of a black hole.

In the classical solution, the near horizon geometry is $\mathrm{AdS}_3 \times S^3$. There is a horizon and the inner region contains a singularity. The solutions (81) in string theory are qualitatively different.

The lessons from this investigation are as follows.

- There are many horizonless configurations with the same global charges as the black hole. Their metrics are the same up to where the horizon appears in the supergravity theory, but differ from each other in the capped region.
- The individual microstates for which supergravity is sensible are regular solutions. They have no horizons or singularities.

- The generic state is intrinsically quantum. It doesn't make sense to speak of geometry in the interior region. Thus, a typical state is characteristically stringy all the way to the effective horizon and well approximated by the black hole metric in the exterior.
- Horizons and singularities in the classical picture are *effective* notions in gravity that arise from a thermodynamic averaging or *coarse-graining* over the individual microstates.
- There is a new scale in quantum gravity. Stringy physics manifest not at ℓ_P but at $N^\alpha \ell_P \sim R_h$, where N is a large number.
- Like the black hole, as the gravitational constant G_N is increased, the fuzzball solutions grow. Only string theory has geometries that behave in this manner (*cf.* Horowitz).
- The origin of entropy lies in the inability of a semiclassical observer to distinguish the different stringy microstates.
- We can quantize the moduli space to find $e^{S_{BH}}$ solutions.
- We can as well construct explicit states in CFT with the same charges as the black hole.
- These ideas do not depend on supersymmetry or extremality (*cf.* JMaRT solutions).
- Many of these notions can be made precise by studying superstars (bound states of N giant gravitons) in $\mathrm{AdS}_5 \times S^5$. Here, the gauge theory dual is $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group $SU(N)$, a well studied and well understood superconformal field theory.
- The AMPS firewall paradox is another argument that supports stringy physics at the horizon scale.
- Much work remains to be done. Please join the adventure.