

On One-point Functions and Integrability

Charlotte Kristjansen
Niels Bohr Institute

Based on:

- C. K., G. Semenoff, and D. Young, arXiv: 1210.7015, JHEP 1301 (2013) 117
- M. de Leeuw, C. K., and K. Zarembo , arXiv:1506.06958 [hep-th]
- I. Buhl-Mortensen, M. de Leeuw, C.K., and K. Zarembo, work in progress

Joburg Workshop 2015, September 9th

Plan of the talk

- ◆ The integrable spin chain of AdS/CFT
- ◆ “AdS/dCFT”
- ◆ Fundamentals of the integrable Heisenberg spin chain
- ◆ Calculating one-point functions of the dCFT using integrability
- ◆ Matching with string theory
- ◆ Open problems/Conclusion

AdS/CFT

$\mathcal{N} = 4$ SYM, gauge group $SU(N)$ \longleftrightarrow Type IIB string theory on $AdS_5 \times S^5$

Local gauge invariant conformal operators \longleftrightarrow string states

Conformal dimensions Δ \longleftrightarrow energies of string states

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(0) \rangle = \delta_{\Delta, \Delta'} \frac{1}{|x|^{2\Delta}}, \quad \langle \mathcal{O}_\Delta(x) \rangle = 0$$

Determine Δ 's and \mathcal{O}_Δ 's in the CFT \equiv

Solve the spectral problem \equiv

Diagonalize dilatation operator \hat{D}

In the planar limit, $N \rightarrow \infty$:

$\hat{D} = \hat{H}$ of an integrable spin chain

Beyond the spectral problem

The planar spectral problem solved to all loop orders for all sectors (i.e. conformal dimensions (and thus two-point functions) determined).

Next step:

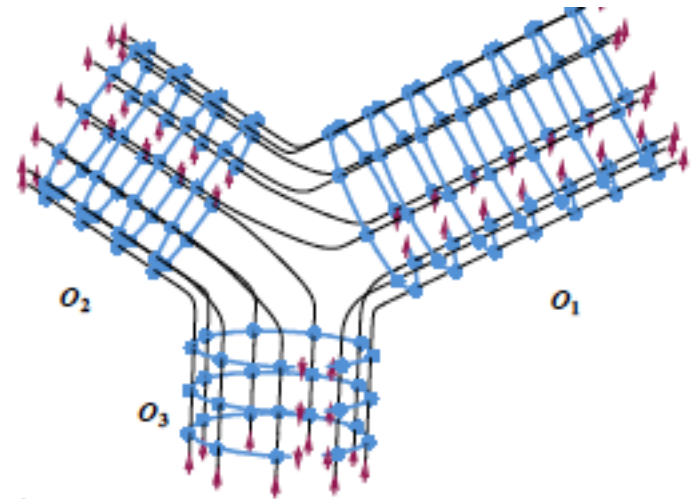
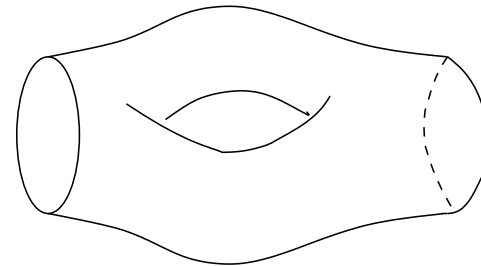
- Move on to the non-planar level

The usual tools of integrability do not work
One and two loop results in certain sub-sectors

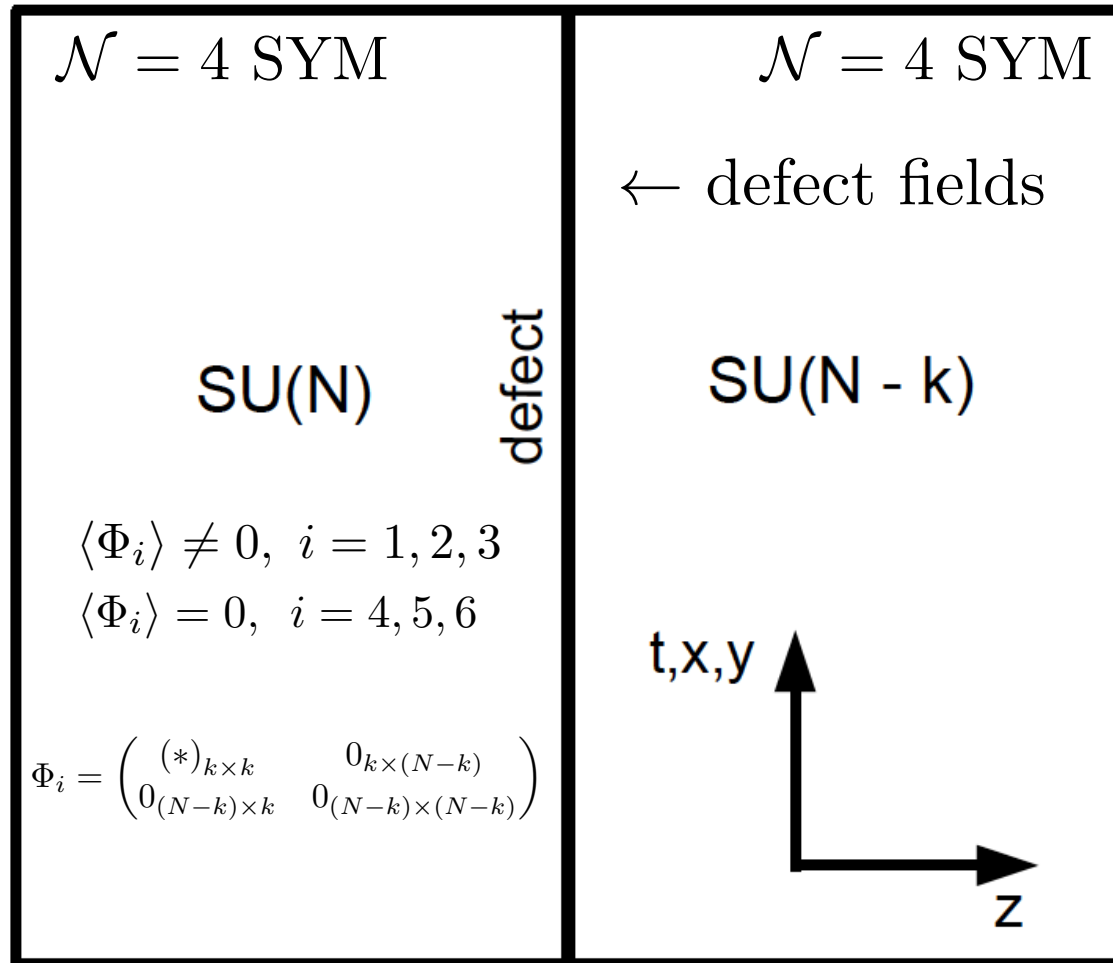
- Move on to three-point functions

Many important works and conjecture for all loop formula in certain sub-sectors,
Basso, Komatsu & Vieira '15

- Consider one-point functions in AdS/dCFT



AdS/dCFT --- The field theory side

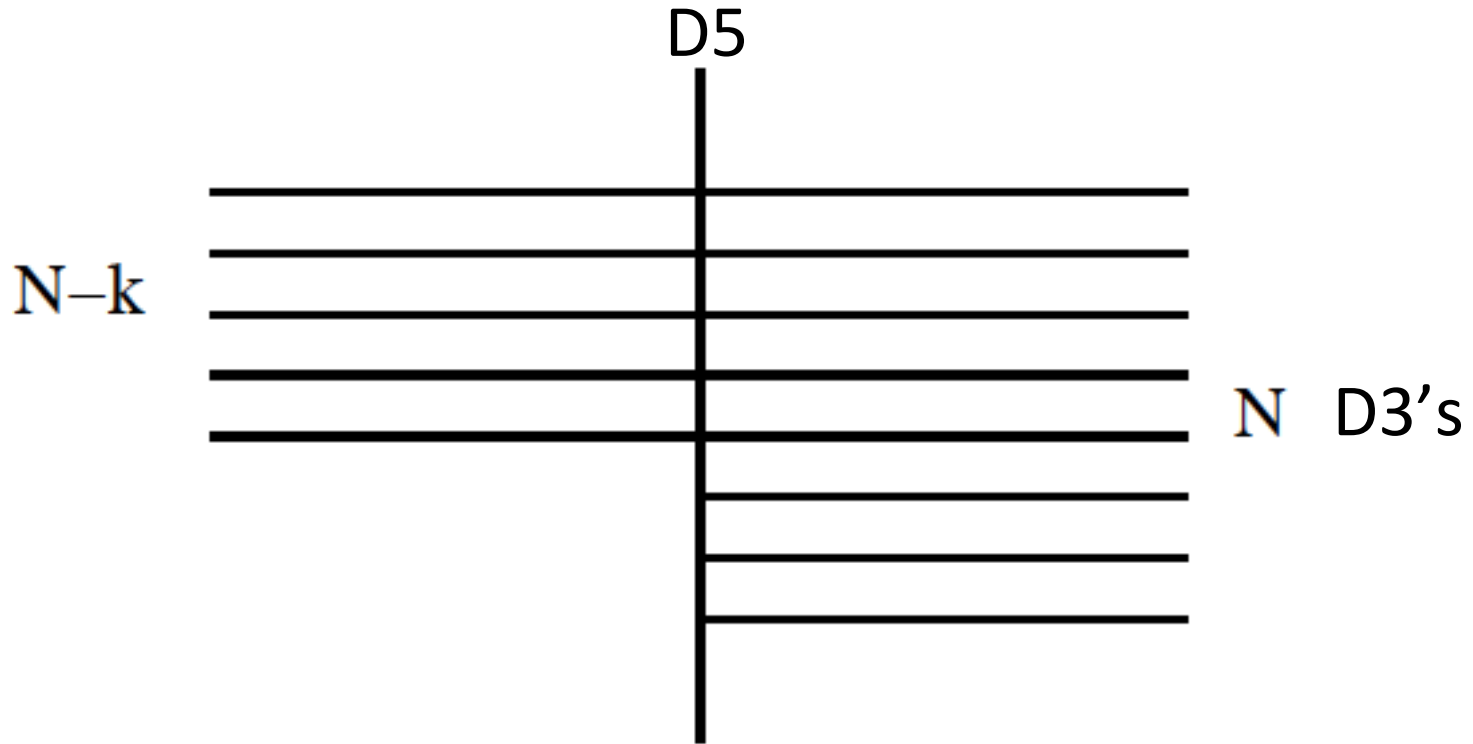


deWolfe, Freedman
& Ooguri '01

Our interest: Tree-level one-point functions of single trace operators built from bulk scalar fields

AdS/dCFT --- The string theory side

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D3$	×	×	×	×						
$D5$	×	×	×		×	×	×			



Geometry of D5 brane: $AdS_4 \times S^2$

Background gauge field: k units of magnetic flux on S^2

Vev's and tree level one-point functions

$$\Phi_i^{\text{cl}} \neq 0, \quad i = 1, 2, 3, \quad \Phi_4^{\text{cl}} = \Phi_5^{\text{cl}} = \Phi_6^{\text{cl}} = 0$$

$$A_\mu = 0, \quad \Psi_A = 0$$

Classical e.o.m.: $\frac{d^2 \Phi_i^{\text{cl}}}{dz^2} = [\Phi_j^{\text{cl}}, [\Phi_j^{\text{cl}}, \Phi_i^{\text{cl}}]]$. Constable, Myers & Tafjord '99
 (z is distance to defect)

Solution: $\Phi_i^{\text{cl}} = \frac{1}{z} \begin{pmatrix} (t_i)_{k \times k} & 0_{k \times (N-k)} \\ 0_{(N-k) \times k} & 0_{(N-k) \times (N-k)} \end{pmatrix}, \quad i = 1, 2, 3$

where t_i constitute a k -dimensional irreducible repr. of $SU(2)$ and where $z < 0$. (Nahm eqns. also fulfilled.)

Op's with tree-level 1-point functions built from $\Phi_i, i = 1, 2, 3$

One-point functions in AdS/dCFT

Wish to calculate: One-point function coefficient, C , at tree level

$$\langle \mathcal{O}_\Delta(z) \rangle = \frac{C}{|z|^\Delta}, \quad z < 0, \quad \langle \mathcal{O}_\Delta(z) \rangle = 0, \quad z > 0$$

Determine Δ 's and \mathcal{O}_Δ 's in the CFT \equiv

Diagonalize dilatation operator \hat{D}

$\hat{D} = \hat{H}_{heisenberg}$ under the following circumstances

- Planar limit
- One-loop level
- Restriction to $SU(2)$ -sector $\subset PSU(2, 2|4)$

$$\mathcal{O} = \text{Tr}(\underbrace{Z Z Z W W Z Z \dots W}_{L \text{ fields}}) \sim | \uparrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \dots \downarrow \rangle$$

Tree level one-point functions

Consider SU(2) subsector: $Z = \Phi_1 + i\Phi_4$, $W = \Phi_2 + i\Phi_5$

$$\Phi_i^{\text{cl}} \neq 0, \quad i = 1, 2, 3, \quad \Phi_4^{\text{cl}} = \Phi_5^{\text{cl}} = \Phi_6^{\text{cl}} = 0$$

$$\Phi_i^{\text{cl}} = \frac{1}{z} \begin{pmatrix} (t_i)_{k \times k} & 0_{k \times (N-k)} \\ 0_{(N-k) \times k} & 0_{(N-k) \times (N-k)} \end{pmatrix}, \quad i = 1, 2, 3$$

$$\begin{aligned} \langle O \rangle &= \langle \text{Tr}(ZZZWZ...W) \rangle \sim | \uparrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \dots \downarrow \rangle \\ &= \frac{1}{z^L} \text{Tr}(t_1 t_1 t_1 t_2 t_2 t_1 t_1 \dots t_2) \end{aligned}$$

Wish: Systematic approach to the computation of 1-pt functions of Bethe eigenstates (i.e. the conf. operators).

Eigenstates of the Heisenberg spin chain

Ground State ($\lambda > 0$): $\hat{H}|\uparrow\uparrow \dots \uparrow\rangle \equiv \hat{H}|0\rangle = 0$

Excited states (with M flipped spins):

$$|\{u_i\}\rangle = \hat{B}(u_M) \dots \hat{B}(u_1)|0\rangle$$

where

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j=1, j \neq k}^M \left(\frac{u_k - u_j + i}{u_k - u_j - i}\right), \quad k = 1, \dots, M$$

with

$$u_i = \frac{1}{2} \cot\left(\frac{p_i}{2}\right) \quad \text{where } p_i \text{ are momenta}$$

Interested in zero-momentum sector (= cyclically inv. states)

$$\sum_{i=1}^M p_i = 0$$

Paired and un-paired solutions

Bethe eqns invariant under $\{u_k\} \rightarrow \{-u_k\}$

Solutions can be split into paired and unpaired:

Paired solutions:

$|\{u_k\}\rangle, | \{-u_k\}\rangle$, where $\{u_k\} \neq \{-u_k\}$

$Q_{2n+1}|\{u_k\}\rangle \neq 0$, $n = 1, 2, \dots$

Unpaired solutions:

$|\{u_k\}\rangle$, where $\{u_k\} = \{-u_k\}$

$Q_{2n+1}|\{u_k\}\rangle = 0$, $n = 1, 2, \dots$

Tree-level one-point functions

$$\langle O \rangle \sim \langle \{u_i\} \rangle$$

Wish: Systematic approach to the computation of 1-pt functions of Bethe eigenstates

Matrix Product State associated with the defect:

$$\langle MPS | = \text{tr}_a \prod_{l=1}^L (\langle \uparrow_l | \otimes t_1 + \langle \downarrow_l | \otimes t_2)$$

$$\text{Object to calculate: } C(\{u_j\}) = \frac{\langle MPS | \{u_j\} \rangle}{\langle \{u_j\} | \{u_j\} \rangle^{\frac{1}{2}}}$$

where $|\{u_j\}\rangle = B(u_1) \dots B(u_M) |0\rangle_L$

is a Bethe eigenstate of length L having M excitations.

Tree-level one-point functions of Bethe eigenstates

Dream scenario:

$C(\{u_j\})$ given by closed expression of determinant type.

Motivation: Several examples of this exist

- Gaudin formula for norm of Bethe eigenstate (i.e. on-shell state).
- Slavnov formula for inner product between an on-shell and an off-shell Bethe state
- Determinant formula for inner product of Bethe on-shell state with the Néel state. [Pozsgay '13, Brockmann et al '14]

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow \dots \uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow \dots \downarrow\uparrow\rangle$$

OBS: The defect state is not a Bethe state! (And not the Néel state)

Strategy of calculation

Matrix Product State indicates the usefulness of the algebraic BA.

$$\langle MPS| = \text{tr}_a \prod_{l=1}^L (\langle \uparrow_l | \otimes t_1 + \langle \downarrow_l | \otimes t_2)$$

$$= \langle \uparrow \dots \uparrow | K, \quad \text{where}$$

$$K = \text{tr}_a \prod_{l=1}^L \{ [s \cdot 1 + (1-s)\sigma_l^3] \otimes t_1 + \sigma_l^+ \otimes t_2 + \sigma_l^- \otimes t \},$$

$$= \text{tr}_a \prod_{l=1}^L (\sigma_l^3 \otimes t_1 + \sigma_l^1 \otimes t_2), \quad \text{for } s=0, t=t_2$$

Reminiscent of the ABA construction

But so far calculations are done by the coordinate space BA

$$|\{p_i\}\rangle = \sum_{\sigma \in S_M} \sum_{1 \leq n_1 < \dots < n_M \leq L} e^{i \sum_m (p_{\sigma_m} n_m + \sum_{j < m} \frac{\theta_{\sigma_j \sigma_m}}{2})} S_{n_1}^- \dots S_{n_M}^- |0\rangle,$$

$$\text{with } u_i = 1/2 \cot(p_i/2)$$

General results

- The overlap vanishes unless M and L are both even
(Easy to see for $k = 2$ where $\{t_i, t_j\} = 0$, but true for any k .)
- The overlap vanishes unless the Bethe eigenstate has $P_{tot} = 0$
Follows from the fact that $|MPS\rangle$ has $P_{tot} = 0$

$$\left(\langle MPS | U \right) | \{u_j\} \rangle = \langle MPS | \{u_j\} \rangle = \langle MPS | (U | \{u_j\} \rangle),$$

where $U = e^{i\hat{P}_{tot}}$

- The overlap vanishes except for unpaired states: $\{u_i\} = \{-u_i\}$
Follows from the fact that $Q_3|MPS\rangle = 0$, paired states have $q_3 \neq 0$ and

$$0 = \langle MPS | Q_3 | \Psi \rangle = q_3 \langle MPS | \Psi \rangle$$

More specific results (can be obtained "by hand")

1. Overlap with the vacuum (any k and L):

$$\langle MPS|0\rangle = \text{tr } t_3^L = \zeta_{-L} \left(\frac{1-k}{2} \right) - \zeta_{-L} \left(\frac{1+k}{2} \right) = \frac{k^{L+1}}{2^L(L+1)} + \mathcal{O}(k^L)$$

2. Two excitations (any k and L)

$$\langle MPS|p, -p\rangle = Lu(u - \frac{i}{2}) \sum_{j=-\frac{k}{2}}^{\frac{k}{2}} \frac{j^2 - \frac{k^2}{4}}{j^2 + u^2} (j - \frac{1}{2})^{L-1}$$

For $k=2$:

$$\langle MPS|p, -p\rangle = 2^{1-L} Lu^{-1}(u - \frac{i}{2})$$

For large k :

$$\langle MPS|p, -p\rangle = \frac{u(u + \frac{i}{2})}{L-3} \frac{k^{L-1}}{2^L} + \frac{u(u + \frac{i}{2})}{(L-1)(L-3)} \frac{k^{L-2}}{2^L} + \mathcal{O}(k^{L-3})$$

Result for $k=2$, any M, L (obtained using Mathematica)

$$|\{u_j\}\rangle = B(u_1)B(-u_1) \dots B(u_{\frac{M}{2}})B(-u_{\frac{M}{2}}) |0\rangle$$

$$C_2(\{u_j\}) = \frac{\langle MPS | \{u_j\} \rangle}{\langle \{u_j\} | \{u_j\} \rangle^{\frac{1}{2}}} = 2^{1-L} \left(\prod_j \frac{u_j^2 + \frac{1}{4}}{u_j^2} \frac{\det G^+}{\det G^-} \right)^{\frac{1}{2}}$$

$$G_{jk}^{\pm} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_n K_{jn}^+ \right) \delta_{jk} + K_{jk}^{\pm}.$$

$$K_{jk}^{\pm} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2},$$

All matrices of size $M/2 \times M/2$

Proof of the formula for $k=2$ ($M=L/2$)

- Proposal based on explicit (Mathematica) calculations up to and including 8 excitations
- Formula can be proved for $M=L/2$

Observation:
$$C(\{u_j\}) = \frac{1}{4^M \left(\frac{i}{2}\right)^{\frac{M}{2}}} \cdot \frac{\langle \text{Néel} | \{u_j\} \rangle}{\langle \{u_j\} | \{u_j\} \rangle^{\frac{1}{2}}},$$

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow \dots \uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow \dots \downarrow\uparrow\rangle$$

$$|MPS\rangle = \frac{1}{4^M \left(\frac{i}{2}\right)^M} |\text{Néel}\rangle + S^- |\dots\rangle, \quad S^+ |\{u_j\}\rangle = 0$$

Result for any k, M, L

$$C_k(\{u_j\}) = k^{L-2M+1} \left[\sum_{j=1-\frac{k}{2}}^{\frac{k}{2}} \frac{1}{2k} \left(\frac{2j-1}{k}\right)^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2 \left(\frac{u_i^2}{k^2} + \frac{1}{4}\right)}{\left(\frac{u_i^2+j^2}{k^2}\right) \left(\frac{u_i^2+(j-1)^2}{k^2}\right)} \right] C_2(\{u_j\}).$$

The limit $k \rightarrow \infty$ is of interest to string theory

BMN-like limit exists: $\lambda \rightarrow \infty, k \rightarrow \infty, \frac{\lambda}{k^2}$ finite and small

Match found to leading order in $\frac{\lambda}{k^2}$ in for chiral primaries

Nagasaki & Yamaguchi '12, C.K, Semenoff & Young '12

Comparison with string theory

The AdS/dCFT set-up: Extra parameter k

Field theory side: dimension of rep. of vev of scalars

String theory side: Number of D3 branes dissolved into D5 brane

First take the planar limit: $N \rightarrow \infty, g_s \rightarrow 0$

Next consider $\lambda \rightarrow \infty, k \rightarrow \infty, \frac{\lambda}{k^2}$ finite (BMN like limit)

Comparisons can be made order by order in $\frac{\lambda}{k^2}$

Comparison with string theory

Agreement found to leading order in $\frac{\lambda}{k^2}$
for operators which are chiral primaries (protected in theory without defects).

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k}{\sqrt{\Delta}} \left(\frac{2\pi^2 k^2}{\lambda} \right)^{\Delta/2} Y_\Delta(0) \frac{1}{|z|^\Delta},$$

Field theory side: Calculated by insertion of vev in spherical harmonics with the appropriate symmetry.

String theory side: Calculated using the supergravity approximation
(Fluctuation of D5 brane action when an appropriate source is inserted on the boundary of AdS)

Open questions

- Proof of the $k=2$ determinant formula for $M \neq L/2$
- Proof of the determinant formula for general k, L, M
- Consider the thermodynamical limit $M, L \rightarrow \infty, M/L$ finite (work in progress)
- Higher loops, other sectors in the dCFT
- Other dCFT's/ other probe brane set-ups such as D3-D7
- More detailed comparisons with string theory:
f.inst. involving spinning strings

Conclusion

- The dream scenario was realized
- The tools of integrability again came in handy
- Many interesting open questions remain