De Alfaro, Fubini and Furlan from multi Matrix Systems

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Recent Trends in CFT and the Gauge / Gravity Correspondence - Sept 2015

Plan of the talk

- Multi matrix hamiltonians and matrix valued radial coordinate
- "Fermionic" picture and dAFF
- AdS_2
- Density description of the underlying Calogero model

Motivation

- $D0$
- Higgs sector of compactified $\mathcal{N} = 4$ SYM on $S^3 \times S^1$
- Single matrix - Giant gravitons - LLM liquid droplet
- Emergent geometry
- AdS_2/CFT_1

Hamiltonian and radial sector

- Hamiltonian of $d = 2m$ hermitean matrices X_a , $a = 1, ..., 2m$:

$$\hat{H} = -\frac{1}{2}\mathrm{Tr}\sum_{a=1}^{2m}\frac{\partial}{\partial M_a}\frac{\partial}{\partial M_a} + V$$
$$= -\frac{1}{2}\nabla^2 + V.$$

- Complexify by introducing complex matrices:

  $Z_1 = X_1 + iX_2$, $Z_2 = X_3 + iX_4$, etc. Label them Z_A , $A = 1, ..., m$.

- Consider the matrix

$$\sum_{A=1}^{m} Z_A^\dagger Z_A \ .$$

  Matrix is hermitean and positive definite, and its eigenvalues

$$\rho_i = r_i^2 \ , \ i = 1, ..., N \ , \ \rho_i \geq 0,$$

  have a natural interpretation as the eigenvalues of a matrix valued radial coordinate.

Radial sector

- This sector dependa only only on the eigenvalues ρ_i. It has an enhanced $U(N)^{m+1}$ symmetry

$$Z_A \to V_A Z_A V^\dagger \ , \ A = 1, ..., m.$$

- Note:

$$Z^\dagger Z = X_1^2 + X_2^2 + i[X_1, X_2]$$

- Measure in the innner product of two such wave functions is:

$$\int \prod_A \prod_{ij} dZ_A{}^\dagger{}_{ij} dZ_{Aij} = \int \prod_i d\rho_i \mathcal{J}(\rho_i) d[\text{Angular}],$$

with the "angular" degrees of freedom being integrated out.

Volume element in the radial sector

$\mathcal{J}(\rho_i)$ has been obtained in closed form:

$$\begin{aligned}\mathcal{J}(\rho_i) &= C_m \prod_i d\rho_i \rho_i^{m-1} \prod_{i>j} \rho_i^{m-1} \rho_j^{m-1} (\rho_i - \rho_j)^2 \\ &= D_m \prod_i dr_i r_i^{2m-1} \prod_{i>j} r_i^{2m-2} r_j^{2m-2} (r_i^2 - r_j^2)^2 \\ &= C_m \prod_i d\rho_i \rho_i^{m-1} \Delta_{RM}^2(\rho_i) = D_m \prod_i dr_i r_i^{2m-1} \Delta_{RM}^2(r_i^2),\end{aligned}$$

The antisymmetric product

$$\Delta_{RM}(\rho_i) \equiv \prod_{i>j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} (\rho_i - \rho_j)$$

generalizes the well known Van der Monde determinant $\Delta = \prod_{i>j}(\rho_i - \rho_j)$, and C_m and D_m are numerical constants.

Radial Laplacian

- In the radial sector the Laplacian then takes the form:

$$
\begin{aligned}
-\frac{1}{2}\nabla^2_{Radial} &= -\frac{1}{2}\sum_i \frac{1}{\Delta^2_{RM}(r_i^2)}\frac{1}{r_i^{2m-1}}\frac{\partial}{\partial r_i} r_i^{2m-1}\Delta^2_{RM}(r_i^2)\frac{\partial}{\partial r_i} \\
&= -2\sum_i \frac{1}{\Delta^2_{RM}(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial \rho_i}\rho_i^m \Delta^2_{RM}(\rho_i)\frac{\partial}{\partial \rho_i},
\end{aligned}
$$

- The Hamiltonian acts on symmetric wavefunctions $\Phi(\rho_i)$ of the eigenvalues:

$$
\begin{aligned}
\hat{H}\Phi(\rho_i) &= E\Phi(\rho_i) \\
\left(-\frac{1}{2}\nabla^2_{Radial} + V(\rho_i)\right)\Phi(\rho_i) &= E\Phi(\rho_i)
\end{aligned}
$$

"Radial" fermions

Define the anti-symmetric wavefunction $\Psi(\rho_i)$ as follows:

$$\Psi(\rho_i) \equiv \Delta_{RM}(\rho_i)\Phi(\rho_i)$$

The Laplacian operator ∇^2_{Radial} now acts on $\Psi(\rho_i)$ as:

$$4\sum_i \left(\frac{1}{\rho_i^{m-1}}\frac{1}{\Delta_{RM}(\rho_i)}\frac{\partial}{\partial\rho_i}\Delta_{RM}(\rho_i)\right)\rho_i^m\left(\Delta_{RM}(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta_{RM}(\rho_i)}\right)\Psi(\rho_i)$$

However, one has the identity:

$$\begin{aligned} 4 \quad & \sum_i \left(\frac{1}{\rho_i^{m-1}}\frac{1}{\Delta_{RM}(\rho_i)}\frac{\partial}{\partial\rho_i}\Delta_{RM}(\rho_i)\right)\rho_i^m\left(\Delta_{RM}(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta_{RM}(\rho_i)}\right) \\ = \quad & \left(\sum_i \frac{4}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i} - \frac{(N^2-1)(m-1)^2}{\rho_i}\right) \end{aligned}$$

Radial fermions - ctd

- The Hamiltonian acting on $\Psi(\rho_i)$ now takes the form:

$$\left[-2\sum_i \frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}+\frac{(N^2-1)(m-1)^2}{2\rho_i}+V(\rho_i)\right],$$

$$\left[-\frac{1}{2}\sum_i \frac{1}{r_i^{2m-1}}\frac{\partial}{\partial r_i}r_i^{2m-1}\frac{\partial}{\partial r_i}+\frac{(N^2-1)(m-1)^2}{2r_i^2}+V(r_i)\right]$$

- This is the sum of single particle $d+1=2m+1$ dimensional s-state hamiltonians, with an additional radial dAFF potential. The coefficient is uniquely determined.
- This first quantized hamiltonian acts on wavefunctions which are antisymmetric under the exchange of radial coordinates only, hence their referral to as radial fermions.
- Generalizes single hermitean matrix. Absent when $m=1$.

Conformal quantum mechanics

- The conformal quantum mechanical hamiltonian

$$h = \frac{1}{2}p^2 + \frac{q^2}{2x^2}$$

has a conformal symmetry generated by h and

$$k = \frac{x^2}{2} \qquad d = \frac{1}{2}(xp + px),$$

with algebra

$$[d, h] = 2ih \qquad [d, k] = -2ik \qquad [h, k] = -id$$

- This is mapped to $SO(2,1)$ generators:

$$L_0 = \frac{1}{2}(H + K) \quad L_{\pm 1} = \frac{1}{2}(H - K \mp iD)$$

with algebra

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1}; \quad [L_{-1}, L_1] = 2L_0$$

Second quantized fermionic picture

- In the single matrix hamiltonian fermionic picture, in terms of second quantized fields,

$$\begin{aligned} H &= \int dx \Psi^\dagger(x) \left(\frac{1}{2} p^2 + \frac{q^2}{2x^2} \right) \Psi(x); \\ K &= \int dx \Psi^\dagger(x) \frac{x^2}{2} \Psi(x); \\ D &= \frac{1}{2} \int dx \Psi^\dagger(x)(xp + px)\Psi(x), \end{aligned}$$

- with

$$\left\{ \Psi(x), \Psi^\dagger(x') \right\} = \delta(x - x').$$

AdS_2

For the higher dimensional case, at the 1st quantized level ($d = 2m \; p_r = -i\partial_r$)

$$\begin{aligned} \hat{h}(p_r, r) &= \frac{1}{2}\frac{1}{r^{d-1}} p_r \, r^{d-1} \, p_r + \frac{(N^2-1)(d-2)^2}{8r^2} \\ \hat{d}(p_r, r) &= r p_r - i\frac{d}{2}; \qquad \hat{k}(p_r, r) = \frac{r^2}{2} \end{aligned}$$

The conformal algebra is satisfied. With second quantized operators

$$\left\{\Psi(r), \Psi^\dagger(r')\right\} = \frac{\delta(r-r')}{r^{d-1}}, \tag{1}$$

one has as generators

$$\begin{aligned} H &= \int r^{d-1} dr \Psi^\dagger(r) \hat{h}(p_r, r) \Psi(r) \\ K &= \int r^{d-1} dr \Psi^\dagger(r) \hat{k}(p_r, r) \Psi(r) \\ D &= \int r^{d-1} dr \Psi^\dagger(r) \hat{h}(p_r, r) \Psi(r) \end{aligned} \tag{2}$$

More on AdS_2

- Simplest way to verify this is to redefine

$$\tilde{\Psi}(r) \equiv r^{\frac{d-1}{2}} \Psi(r), \quad \tilde{\Psi}^\dagger(r') \equiv r^{\frac{d-1}{2}} \Psi^\dagger(r')$$

This is also the redefinition of the fields in terms of which p_r becomes explicitly hermitean. One finds:

$$\begin{aligned} K &= \int dr \tilde{\Psi}^\dagger(r) \frac{r^2}{2} \tilde{\Psi}(r) \\ D &= \int dr \tilde{\Psi}^\dagger(r) \frac{1}{2}(r p_r + p_r r) \tilde{\Psi}(r) \\ H &= \int dr \tilde{\Psi}^\dagger(r) (\frac{p_r^2}{2} + \frac{N^2(d-2)^2 - 1}{8r^2}) \tilde{\Psi}(r) \end{aligned}$$

- The higher dimensional case has been mapped to a one-dimensional quantum mechanical conformal hamiltionian with

$$q^2 = \frac{1}{4}(N^2(d-2)^2 - 1)$$

which has the required symmetry.

Density description - Collective Field Theory

- One changes variables from the original variables to the invariant variables: $X_\alpha \to \phi(C)$.
- There is a reduction in the number of degrees of freedom. In the large N limit, the invariant variables are independent.
- There is jacobian J associated with this change of variables which satisfies

$$\boxed{-(\partial_{C'} \ln J)\Omega(C', C) = \omega(C) + \partial_{C'}\Omega(C', C)}$$

- The operator $\Omega(C, C')$ "joins" loops, or words. One may then write schematically $\Omega(C, C') = \sum \phi_{C+C'}$, with $C + C'$ obtained by adding the two words C and C'. Similarly, ω "splits" loops. Schematically again, $\omega(C) = \sum \phi_{C'}\phi_{C''}$

Collective Field Theory Hamiltonian

- For an Hamiltonian

$$H = -\frac{1}{2}(\sum_{\alpha=1}^{M} \frac{\partial}{\partial X_A}\frac{\partial}{\partial X_A}) + V(\phi_C)$$

- the Collective Field Hamiltonian takes the form

$$H = \frac{1}{2}(\frac{\partial}{\partial\phi(C)} + \frac{1}{2}\frac{\partial \ln J}{\partial\phi_C})\Omega(C, C')(-\frac{\partial}{\partial\phi(C')} + \frac{1}{2}\frac{\partial \ln J}{\partial\phi_{C'}}) + V$$

- The leading contribution is

$$\boxed{H' = \frac{1}{2}\Pi(C)\Omega(C, C')\Pi(C') + (\frac{1}{4}\omega(C)\Omega^{-1}(C, C')\omega(C') + V)}$$

where

$$\Pi(C) = -i\frac{\partial}{\partial\phi_C}$$

Density description

- Our collective field variables are defined as

$$\phi_k = \mathrm{Tr} e^{ik\sum_A Z_A^\dagger Z_A} = \sum_i e^{ikr_i^2} = \sum_i e^{ik\rho_i};$$
$$\phi(\rho) = \int dk e^{-ik\rho}\phi_k = \sum_i \delta(\rho - r_i^2) = \sum_i \delta(\rho - \rho_i).$$

- The collective field construction is based on two operators ("joining"-$\Omega(C, C')$ and "splitting"- $\omega(C)$):
- The leading (in N) form of the collective field hamiltonian is

$$\boxed{H' = 2\Pi(C)\Omega(C, C')\Pi(C') + (\frac{1}{2}\omega(C)\Omega^{-1}(C, C')\omega(C') + V)}$$

where

$$\Pi(C) = -i\frac{\partial}{\partial\phi_C}$$

Some technical details

- $$\begin{aligned}\Omega(\rho,\rho';[\phi]) &= \int \frac{dk'}{2\pi}\int \frac{dk}{2\pi} e^{-ik\rho}e^{-ik'\rho'}\sum_A \frac{\partial \phi_k}{\partial Z_A^\dagger}\frac{\partial \phi_{k'}}{\partial Z_A} \\ &= \partial_\rho \partial_{\rho'}\left[\rho\phi(\rho)\delta(\rho-\rho')\right],\end{aligned}$$

- $$\begin{aligned}\omega(\rho;[\phi]) &= \int \frac{dk}{2\pi} e^{-ik\rho}\frac{\partial^2 \phi_k}{\partial Z_A^\dagger \partial Z_A} \\ &= \partial_\rho\left(\rho\phi(\rho)\left[2⨍ \frac{d\rho'\phi(\rho')}{(\rho-\rho')}+\frac{N(m-1)}{\rho}\right]\right).\end{aligned}$$

- $$\partial_\rho \frac{\partial}{\partial \Phi(\rho)}\ \ln J = 2⨍ \frac{d\rho'\Phi(\rho')}{\rho-\rho'}+\frac{N(m-1)}{\rho}$$

Hamiltonian

- $$-\frac{1}{2}\nabla^2_{Radial} \rightarrow 2\int d\rho\,(\partial_\rho \Pi(\rho))\,[\rho\phi(\rho)]\,(\partial_\rho \Pi(\rho))$$
$$+ \frac{1}{2}\int d\rho\,(\rho\phi(\rho))\left[2\fint \frac{d\rho'\phi(\rho')}{(\rho-\rho')} + \frac{N(m-1)}{\rho}\right]^2$$

- $$\Delta V = 2\int_0^\infty d\rho\rho\phi(\rho)\left[\fint_0^\infty \frac{d\rho'\phi(\rho')}{(\rho-\rho')}\right]^2 + \frac{N^2(m-1)^2}{2}\int_0^\infty d\rho\frac{\phi(\rho)}{\rho}$$

- $$\Delta V = 2\sum_i \rho_i\left[\sum_{j\neq i}\frac{1}{(\rho_i-\rho_j)}\right]^2 + \frac{N^2(m-1)^2}{2}\sum_i\frac{1}{\rho_i}$$

Contributions to effective potential

Extend to the whole line: $\Phi(r) \equiv 2r\phi(r^2) = \Phi(-r)$. One can use the identity

$$\int_{-\infty}^{\infty} dr\phi(r) \left(⨍_{-\infty}^{\infty} \frac{dr'\phi(r')}{(r-r')} \right)^2 = \frac{\pi^2}{3} \int_{-\infty}^{\infty} dr\phi^3(r)$$

Then

$$\begin{aligned} H &= \frac{1}{2} \int_0^{\infty} dr \partial_r \Pi(r)\, \phi(r)\, \partial_r \Pi(r) \\ &+ \frac{\pi^2}{6} \int_0^{\infty} dr \phi^3(r) + \frac{N^2(m-1)^2}{2} \int_0^{\infty} dr \left[\frac{\phi(r)}{r^2} \right] \end{aligned}$$

Harmonic potential - L_0

$$\begin{aligned} V_{eff} &= \frac{\pi^2}{6}\int_0^\infty dr\Phi^3(r) + \frac{N^2(d-2)^2}{8}\int_0^\infty dr\left[\frac{\Phi(r)}{r^2}\right] \\ &+ \frac{\omega^2}{2}\int_0^\infty dr\, r^2\Phi(r) - \mu\left(N - \int_0^\infty dr\Phi(r)\right) \end{aligned}$$

Rescale to make powers of N explicit

$$r \to \sqrt{N}r \quad \Phi(r) \to \sqrt{N}\Phi(r) \quad \mu \to N\mu; \quad \Pi(r) \to \Pi(r)/N$$

$$V_{eff} \to N^2 V_{eff} \quad H_{kin} \to H_{kin}/N^2$$

Large N background:

$$\Phi_0(r) = \frac{1}{\pi}\left(\frac{\omega}{2}(d-1) - \omega^2 r^2 - \frac{(d-2)^2}{4}\frac{1}{r^2}\right)^{1/2} \qquad r_- \leq r \leq r_+$$

$$r_\pm^2 = \frac{(d-1)}{4\omega} \pm \sqrt{\frac{(d-1)^2}{16\omega^2} - \frac{(d-2)^2}{4\omega^2}}$$

References

- "Large N Matrix Hyperspheres and the Gauge-Gravity Correspondence", Mthokozisi Masuku, Mbavhalelo Mulokwe and João P. Rodrigues, arXiv:1411.5786 [hep-th], under review
- M. Masuku and J. P. Rodrigues, J. Phys. A **45**, 085201 (2012) [arXiv:1107.3681 [hep-th]].
- M. Masuku and J. P. Rodrigues, J. Math. Phys. **52**, 032302 (2011) [arXiv:0911.2846 [hep-th]].
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$m = 1$ Laplacian

- $$X_1 + iX_2 = Z = RU \quad , \quad Z^\dagger = U^\dagger R$$
$$Q \equiv VUV^\dagger$$
- $E^{(L)}_{ji} = Q_{jb}\frac{\partial}{\partial Q_{ib}} \quad E^{(R)}_{ji} = Q_{ai}\frac{\partial}{\partial Q_{aj}}$
- $$\begin{aligned}
\nabla^2 &= \frac{1}{\Delta^2_{MR}}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i}\left(r_i\Delta^2_{MR}\right)\frac{\partial}{\partial r_i} - \sum_i \frac{1}{r_i^2}E^{(L)}_{ii}E^{(L)}_{ii} \\
&- \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}(E^{(L)}_{ij}E^{(L)}_{ji} + E^{(R)}_{ij}E^{(R)}_{ji}) \\
&- \frac{4r_ir_j}{(r_i^2-r_j^2)^2}(E^{(L)}_{ij}E^{(R)}_{ji} + E^{(R)}_{ij}E^{(L)}_{ji})
\end{aligned}$$

"Natural metric"

Quadratic fluctuations

$$\phi(r) = \phi_0(r) + \frac{1}{N}\partial_r\psi;\ \partial_r\Pi(r) = -NP(r);$$

Then

$$H_2 = \frac{1}{2}\int_0^\infty dr\phi_0(r)P^2(r) + \frac{\pi^2}{2}\int_0^\infty dr\phi_0(r)(\partial_r\phi)^2 \qquad (3)$$

Metric

$$ds^2 = \phi_0(r)dt^2 - \frac{1}{\pi^2\phi_0(r)}dr^2 \qquad (4)$$