Tutorial

In this tutorial we will explore some of the representation theory of the conformal group.

1. Write down transformations corresponding to Lorentz transformations, translations, scalings and special conformal transformations. Consider an infinitesimal transformation and hence obtain the following generators

$$\begin{aligned} M_{\mu\nu} &= (-i)(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ P_\mu &= -i\partial_\mu \\ D &= ix\cdot\partial \\ K_\mu &= 2ix_\mu(x\cdot\partial) - ix^2\partial_\mu \end{aligned} \tag{1}$$

   Work in d dimensions.

2. Use the above generators to compute the Lie algebra of the conformal group.

3. Prove that the free scalar field evaluated at the origin $\phi(0)$ is a primary field. What is $\big[D,\phi(0)\big]$? What is $\big[D,\big[P_\mu,\phi(0)\big]\big]$? Work in d dimensions.

4. For this question, work in 4 dimensions (i.e. 3+1 in Minkowski signature for example). Given that $P_\mu^\dagger = K_\mu$ prove that $|\psi\rangle = P^\mu P_\mu|\phi\rangle$ (with $|\phi\rangle$ the state corresponding to the primary operator $\phi(0)$) is a zero norm state

$$\langle\psi|\psi\rangle = 0 \tag{2}$$

   Hint: $D|\phi\rangle = -i|\phi\rangle$.

5. The multiplet built on the the primary operator $\phi(0)$ is labeled by $\Delta = 1$ and $(s_L, s_R) = (0,0)$. We computed the corresponding character $\chi_{\big[1,0,0\big]}$ in class. Compute $\chi_{[2,1,1]}$. You should be careful since this module has null states in it.

6. Using $D|\gamma\rangle = -i\Delta_\gamma|\gamma\rangle$ argue that D is antihermittian. Check that with the i on the right hand side of this equation, P_μ really is a raising operator for D and that the i is needed for this to work. In QFT P_μ is observable, so it must be hermittian to ensure it has real eigenvalues. Looking at the commutator

$$\big[D,P_\mu\big] = -iP_\mu \tag{3}$$

   do you think that D is hermittian or anti-hermittian? If you find a puzzle resolve it.

7. Resolve $[\Delta,0,0]\otimes[\Delta,1,1]$ into irreducible representations.