p-forms, Hodge dual & Stokes' theorem in SR

1 Intro

Completely antisymmetric $\binom{0}{p}$ tensors, called p-forms, have a special geometric properties and many applications in physics. For instance, they allow for a very elegant expression of Maxwell's equations (and its generalisations) as well as nice generalisation of Stokes' Theorem to arbitrary dimensions.[1]

2 Notation

To make these notes more self-contained some notation is introduced in this section.

The components of a $\binom{0}{p}$ tensor with respect to a set of basis vectors $\{\vec{e}_i\}$ is given in term its action on the basis vectors:

Definition 2.1:

$$T_{ij\ldots k} = \tilde{T}(\vec{e}_i, \vec{e}_j, \ldots, \vec{e}_k)$$

We write the anti-symmetric part of a $\binom{0}{2}$ tensor as:

Definition 2.2:

$$T_{[ij]} = \frac{1}{2!}\left(T_{ij} - T_{ji}\right) \ .$$

More generally for a $\binom{0}{p}$ tensor we write

Definition 2.3:

$$T_{[i_1 \ldots i_p]} = \frac{1}{p!}\left(\text{alternating sum over permutations of the indices } i_1 \text{ to } i_p\right) \ .$$

So, for example

$$T_{[ijk]} = \tfrac{1}{6}\left(T_{ijk} - T_{jik} + T_{jki} - T_{kji} + T_{kij} - T_{ikj}\right) \qquad (1)$$

The tensor product of a $\binom{0}{p}$ tensor, $\tilde{S}$, and a $\binom{0}{q}$ tensor, $\tilde{T}$, is a $\binom{0}{p+q}$ tensor, $\tilde{S} \otimes \tilde{T}$, whose action on $(p+q)$ vector arguments is defined by

Definition 2.4:

$$\tilde{S} \otimes \tilde{T}(\vec{A}_1, \ldots, \vec{A}_p, \vec{B}_1, \ldots, \vec{B}_q) = \tilde{S}(\vec{A}_1, \ldots, \vec{A}_p)\tilde{T}(\vec{B}_1, \ldots, \vec{B}_q)$$

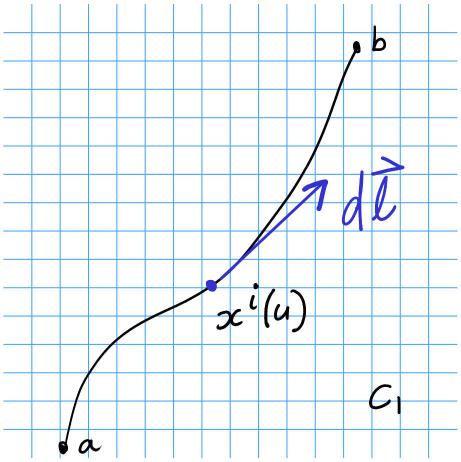
[1] After this section the enthusiastic student might like to consult Gravitation by MTW which has an extensive discussion of the geometric interpretation of forms.

3 1-Forms

Aside from mapping vectors to numbers, 1-forms also allow us to map lines to numbers using line integrals. Consider a 1-form in $\mathbb{R}^3$

$$\tilde{p} = p_i \tilde{d}x^i \tag{2}$$

and a curve, c_1, parameterised by $x^i(u)$.



- The curve has tangent $U^i = \frac{dx^i}{du}$
- If we shift $u \to u + du$, we get infinitesimal displacement along the curve in the i-th direction:

$$dx^i = \frac{dx^i}{du} du = U^i du$$

$\to$ The infinitesimal displacement vector is

$$d\vec{l} = (U^i du)\vec{e}_i$$

Divide the curve up into little line segments $\Delta\vec{l}$. It is not hard to see that we can approximate the curve using the segments

$$\Delta\vec{l} = \left(\frac{\Delta x^i}{\Delta u}\Delta u\right)\vec{e}_i. \tag{3}$$

At the start of each line segment, the form $\tilde{p}$ can map $\Delta\vec{l}$ to a number

$$\tilde{p}(\Delta\vec{l}) = p_j \frac{\Delta x^i}{\Delta u}\Delta u \underbrace{\tilde{d}x^j(\vec{e}_i)}_{\delta^j_i} = p_i \frac{\Delta x^i}{\Delta u}\Delta u \tag{4}$$

Now adding up all the numbers for all the segments and taking the limit $\Delta\vec{l} \to 0$ gives us the line integral

$$\int_{c_1} \tilde{p} = \int_{c_1} \tilde{p}(d\vec{l}) = \int_a^b p_i \frac{dx^i}{du} du \tag{5}$$

and using the chain-rule we can write

$$\int_{c_1} p_i \tilde{d}x^i = \int_a^b p_i dx^i. \tag{6}$$

Aside: Everything in this section can be easily generalised to space-time by including a time-index.

Aside: This looks like a lot of work to go from $\tilde{d} \to d$ but its useful to understand since

1. The basic steps go through for integrals over higher dimensional surfaces.
2. Having formulated things in terms of forms will mean that the generalisation to space-time will give answers independent of the frame we chose or indeed even the coordinates we choose.

4 2-forms

We see 1-forms map lines to numbers. We would like to try construct something to map areas to numbers using surface integrals — these will be called 2-forms. Just as we need to construct an infinitesimal line element to define the line-integral, we need to construct an infinitesimal area element to define the surface-integral:

$$\int_{c_2} F_{(2)} = \# \tag{7}$$

Consider the area of the a parallelogram whose sides are formed by two vectors $\vec{V}$ and $\vec{W}$. The area is given by

$$A = |\vec{V} \times \vec{W}| \tag{8}$$

and taking these vectors to be in the (x, y) plane, in terms of the components, we have

$$\pm |A| = V^x W^y - V^y W^x = \begin{vmatrix} V^x & W^x \\ V^y & W^y \end{vmatrix} = \det(\vec{V}\vec{W}). \tag{9}$$

Notice that the area, calculated this way comes with a sign. We shall interpret this sign as denoting the orientation of the area and accept negative areas.

A $\binom{0}{2}$ tensor maps two vectors to a number so we would like to construct the tensor which will map vectors to the area given by the parallelogram whose sides they form. Consider the area 2-form in the (x, y) plane which we define as

Definition 4.1: $\tilde{d}x \wedge \tilde{d}y := \tilde{d}x \otimes \tilde{d}y - \tilde{d}y \otimes \tilde{d}x$.

The area two form acting on two vectors gives

$$\tilde{d}x \wedge \tilde{d}y(\vec{V}, \vec{W}) = \tilde{d}x(\vec{V})\tilde{d}y(\vec{W}) - \tilde{d}y(\vec{V})\tilde{d}x(\vec{W}) = V^x W^y - V^y W^x = A. \tag{10}$$

Aside: We could have defined the area two form as $\tilde{d}y \wedge \tilde{d}x$ which would have given the opposite sign for the area. Choosing a particular form as the area form defines an orientation for the plane.

Aside: It is not hard to see that for vectors with components outside the (x, y) plane, $\tilde{d}x \wedge \tilde{d}y$ gives us the area defined by their projection on the plane.

For general one-forms we can define the wedge product as the anti-symmetric tensor product

Definition 4.2: $\tilde{p} \wedge \tilde{q} := \tilde{p} \otimes \tilde{q} - \tilde{q} \otimes \tilde{p}$

so that

$$\tilde{p} \wedge \tilde{q}(\vec{A}, \vec{B}) = -\tilde{p} \wedge \tilde{q}(\vec{B}, \vec{A}) \tag{11}$$

In terms of components this becomes:

$$\begin{aligned} \tilde{p} \wedge \tilde{q} &= (p_\alpha \tilde{d}x^\alpha) \otimes (q_\beta \tilde{d}x^\beta) - (q_\beta \tilde{d}x^\beta) \otimes (p_\alpha \tilde{d}x^\alpha) \\ &= p_\alpha q_\beta (\tilde{d}x^\alpha \otimes \tilde{d}x^\beta - \tilde{d}x^\beta \otimes \tilde{d}x^\alpha) \\ &= p_\alpha q_\beta (\tilde{d}x^\alpha \wedge \tilde{d}x^\beta). \end{aligned}$$

It is convenient to introduce the notation

Definition 4.3: Short-hand $\tilde{d}x^{\alpha\beta} := \tilde{d}x^\alpha \wedge \tilde{d}x^\beta$.

Now

$$\begin{aligned}(p_\alpha \tilde{d}x^\alpha) \wedge (q_\beta \tilde{d}x^\beta) &= p_\alpha q_\beta (\tilde{d}x^\alpha \wedge \tilde{d}x^\beta) \\ &= \tfrac{1}{2}(p_\alpha q_\beta \tilde{d}x^{\alpha\beta} + p_\beta q_\alpha \tilde{d}x^{\beta\alpha}) \\ &= \tfrac{1}{2}(p_\alpha q_\beta - p_\beta q_\alpha)\tilde{d}x^{\alpha\beta} \\ &= p_{[\alpha} q_{\beta]} (\tilde{d}x^\alpha \wedge \tilde{d}x^\beta)\end{aligned}$$

so that the wedge product picks out the anti-symmetric part. Notice that

$$(\tilde{p} \wedge \tilde{q})_{\alpha\beta} = (\tilde{p} \wedge \tilde{q})(\vec{e}_\alpha, \vec{e}_\beta) = p_\alpha q_\beta - q_\alpha p_\beta = 2 p_{[\alpha} q_{\beta]} \tag{12}$$

The most general 2-form in 3-D can be written

$$\tilde{A} = \tfrac{1}{2} A_{ij} \tilde{d}x^i \wedge \tilde{d}x^j = \tfrac{1}{2} A_{[ij]} \tilde{d}x^i \wedge \tilde{d}x^j \tag{13}$$

with A_{ij} antisymmetric. The factor of $\frac{1}{2}$ is chosen so that $A_{ij} = \tilde{A}(e_i, e_j)$ which we can check

$$\begin{aligned}\tilde{A}(e_k, e_l) &= \tfrac{1}{2} A_{ij} \tilde{d}x^i \wedge \tilde{d}x^j (e_k, e_l) && (14)\\ &= \tfrac{1}{2} A_{ij} (\delta^i_k \delta^j_l - \delta^j_k \delta^i_l) && (15)\\ &= \tfrac{1}{2}(A_{kl} - A_{lk}) = A_{[kl]} = A_{kl} && (16)\end{aligned}$$

It is useful to define a generalised Kronecker delta as follows:

Definition 4.4: $\delta^{ij}_{kl} := \det \begin{vmatrix} \delta^i_k & \delta^i_l \\ \delta^j_k & \delta^j_l \end{vmatrix} = \delta^i_k \delta^j_l - \delta^j_k \delta^i_l = \tilde{d}x^i \wedge \tilde{d}x^j (\vec{e}_k, \vec{e}_l)$

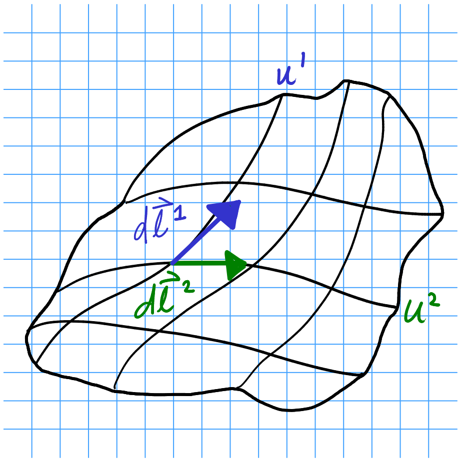
Aside: Properties of Generalised Kronecker delta

- Takes on values $-1, 0$ or 1.
- Completely antisymmetric in its upper and lower indices
  - So if any upper or lower index repeats it gives zero
- $= 0$ if the upper indices are not a permutation of the lower indices
- $= \pm 1$ if the upper indices are a permutation of the lower indices
  - Sign = Sign of permutation
  - Eg $\delta^{12}_{34} = \delta^{12}_{11} = 0$, $\delta^{12}_{12} = -\delta^{34}_{43} = 1$

We are now in a position to consider surface integrals in the language of forms:

$$\int_{c_2} F_{(2)} = \# \tag{17}$$

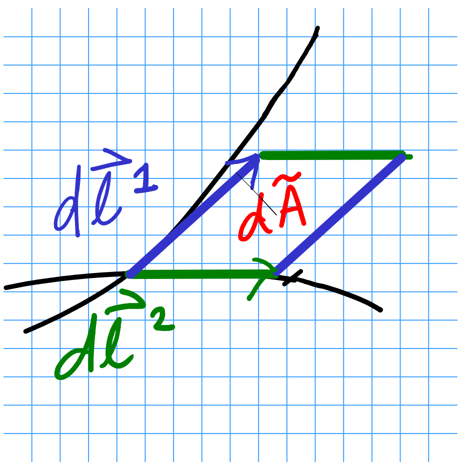
Aside: We will sometimes us a subscript (p) to denote a p-form and subscripts to denote the dimensionality of a region of integration.



- Consider a surface, c_2, parametrically defined by $x^i(u^1, u^2)$.
- $$d\vec{l^1} = du^1 \frac{\partial x^m}{\partial u^1} \vec{e}_m$$
$$d\vec{l^2} = du^2 \frac{\partial x^m}{\partial u^2} \vec{e}_m$$

Along the curves of constant u^i we have infinitesimal line elements

$$d\vec{l^j} = du^j \frac{\partial x^m}{\partial u^j} \vec{e}_m \tag{18}$$



These line elements can be used to define infinitesimal area elements

$$dA^{ij} = dx^i \wedge \tilde{d}x^j(d\vec{l^1}, d\vec{l^2}) \tag{19}$$

so that for some 2-form (in analogous way to how we defined the line-integral) we can define

$$\begin{aligned} \int_{c_2} F_{(2)} &= \int_{c_2} F_{(2)}(d\vec{l^1}, d\vec{l^2}) \\ &= \int du^1 du^2 \left(\frac{1}{2} F_{ij} \delta^{ij}_{mn} \frac{\partial x^m}{\partial u^1} \frac{\partial x^n}{\partial u^2} \right) \\ &= \int du^1 du^2 \left(F_{mn} x^m_{,1} x^n_{,2} \right) \end{aligned}$$

which adds what we get from the action of F on the dls defining the infinitesimal area elements.

Notice that the area-form just calculates the Jacobian for a change of variables

$$\begin{aligned}
\int_{c_2} F_{(2)} &= \int du^1 du^2 \left(\frac{1}{2} F_{ij} \delta^{ij}_{mn} \frac{\partial x^m}{\partial u^1} \frac{\partial x^n}{\partial u^2} \right) \\
&= \int du^1 du^2 \frac{1}{2} F_{ij} \underbrace{\det \begin{vmatrix} \frac{\partial x^i}{\partial u^1} & \frac{\partial x^i}{u^2} \\ \frac{\partial x^j}{\partial u^1} & \frac{\partial x^j}{\partial u^2} \end{vmatrix}}_{\frac{\partial(x^i, x^j)}{\partial(u^1, u^2)} = |J|} \\
&= \int dx^i dx^j \left(\frac{1}{2} F_{ij} \right)
\end{aligned}$$

and we've once again gone through a bit of work to go from $\tilde{d} \to d$

Aside: You can often just "follow your nose" when calculating these integrals by just substituting

$$\begin{aligned}
\tfrac{1}{2} F_{ij} dx^i \wedge dx^j &= \tfrac{1}{2} F_{ij} (x^i_{,1} du^1 + x^i_{,2} du^2) \wedge (x^j_{,1} du^1 + x^j_{,2} du^2) \\
&= \tfrac{1}{2} F_{ij} (x^i_{,1} x^j_{,2} - x^i_{,2} x^j_{,1}) du^1 \wedge du^2 \\
&= (F_{ij} x^i_{,1} x^j_{,2}) du^1 \wedge du^2
\end{aligned}$$

5 3-forms

Just as we defined area forms we can define a volume form

Definition 5.1: $\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 = \{\text{alternating sum } \tilde{d}x^i \otimes \tilde{d}x^j \otimes \tilde{d}x^k\}$

so that

$$\begin{aligned}
\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 (\vec{A}, \vec{B}, \vec{C}) &= A^1 B^2 C^3 - A^1 B^3 C^2 + \dots \\
&= \det \begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix} \\
&= \det |\vec{A} \vec{B} \vec{C}| \\
&= \text{Volume of parallelepiped with sides } (\vec{A}, \vec{B}, \vec{C})
\end{aligned}$$

In particular the volume form acting on the basis vector gives us another generalised Kronecker delta

$$\tilde{d}x^i \wedge \tilde{d}x^j \wedge \tilde{d}x^k (\vec{e}_l, \vec{e}_m, \vec{e}_n) = \det \begin{vmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{vmatrix} =: \delta^{ijk}_{lmn} \tag{20}$$

In a similar fashion we can define the wedge product of three 1-forms as an alternating sum of tensor products

$$\tilde{p} \wedge \tilde{q} \wedge \tilde{r} = \tilde{p} \otimes \tilde{q} \otimes \tilde{r} + \tilde{q} \otimes \tilde{r} \otimes \tilde{p} + \tilde{r} \otimes \tilde{p} \otimes \tilde{q} - \tilde{p} \otimes \tilde{r} \otimes \tilde{q} - \tilde{q} \otimes \tilde{p} \otimes \tilde{r} - \tilde{r} \otimes \tilde{q} \otimes \tilde{p} \tag{21}$$

which will be anti-symmetric in all its arguments, that is

$$\begin{aligned}
\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{A}, \vec{B}, \vec{C}) &= -\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{B}, \vec{A}, \vec{C}) \quad (\text{ odd permutation }) \\
&= -\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{A}, \vec{C}, \vec{B}) \quad (\text{ odd permutation }) \\
&= -\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{C}, \vec{B}, \vec{A}) \quad (\text{ odd permutation }) \\
&= +\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{B}, \vec{C}, \vec{A}) \quad (\text{ even permutation }) \\
&= +\tilde{p} \wedge \tilde{q} \wedge \tilde{r} (\vec{C}, \vec{A}, \vec{B}) \quad (\text{ even permutation })
\end{aligned}$$

In terms of components we have

$$(\tilde{p} \wedge \tilde{q} \wedge \tilde{r})_{ijk} = 3! p_{[i} q_j r_{k]} \tag{22}$$

The most general 3-form can be written as

$$F_{(3)} = \frac{1}{3!} F_{ijk} \tilde{d}x^i \wedge \tilde{d}x^j \wedge \tilde{d}x^k \tag{23}$$

where F_{ijk} is completely antisymmetric in all indices and we can check

$$F_{(3)}(\vec{e}_l, \vec{e}_m, \vec{e}_n) = \frac{1}{3!} F_{ijk} \delta^{ijk}_{lmn} = F_{lmn} \tag{24}$$

Defining the integral of a 3-form over a volume, c_3 parameterised as $x^i(u^1, u^2, u^3)$ gives

$$\int_{c_3} F_{(3)} = \int_{c_3} F_{(3)}(d\vec{l}_1, d\vec{l}_2, d\vec{l}_3) = \int du^1 du^2 du^3 x^i_{,1} x^j_{,2} x^k_{,3} F_{ijk} \tag{25}$$

which can also be written using the chain-rule in differential notation as

$$\int dx^i dx^j dx^k \left(\frac{1}{3!} F_{ijk} \right) = \int du^1 du^2 du^3 \underbrace{\frac{\partial(x^i, x^j, x^k)}{\partial(u^1, u^2, u^3)}}_{|J|} \left(\frac{1}{3!} F_{ijk} \right) \tag{26}$$

6 General forms

Recall that a p-form is just a completely anti-symmetric $\binom{0}{p}$ tensor.

In general, for a particular frame, we can define higher dimensional p-volume forms

$$\tilde{d}x^{i_1} \wedge \tilde{d}x^{i_2} \wedge \ldots \wedge \tilde{d}x^{i_p} = \{ \text{ alternating sum over } \tilde{d}x^{i_1} \otimes \tilde{d}x^{i_2} \otimes \ldots \otimes \tilde{d}x^{i_p} \} \tag{27}$$

and space-time volume p-forms

$$\tilde{d}x^{\mu_1} \wedge \tilde{d}x^{\mu_2} \wedge \ldots \wedge \tilde{d}x^{\mu_p} = \{ \text{ alternating sum over } \tilde{d}x^{\mu_1} \otimes \tilde{d}x^{\mu_2} \otimes \ldots \otimes \tilde{d}x^{\mu_{(p)}} \} \tag{28}$$

and write a generic p-form as

$$F_{(p)} = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} \tag{29}$$

So that

$$F_{(p)}(\vec{e}_{\nu_1}, \ldots, \vec{e}_{\nu_p}) = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} \delta^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} = F_{\nu_1 \ldots \nu_p} \tag{30}$$

The integral of a p-form over a p-volume given by

$$\int_{c_p} F_{(p)} = \int F_{(p)}(d\vec{l}_1, \ldots, d\vec{l}_p) = \int du^1 \ldots du^p x^{\mu_1}_{,1} \ldots x^{\mu_p}_{,p} F_{\mu_1 \ldots \mu_p} \tag{31}$$

which can be written in differential notation using the chain-rule as

$$\int dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} \left(\frac{1}{p!} F_{\mu_1 \mu_2 \ldots \mu_p} \right) = \int du^1 du^2 \ldots du^p \underbrace{\frac{\partial(x^{\mu_1}, x^{\mu_2}, \ldots, x^{\mu_p})}{\partial(u^1, u^2 \ldots, u^p)}}_{|J|} \left(\frac{1}{p!} F_{\mu_1 \mu_2 \ldots \mu_p} \right) \tag{32}$$

Aside: In space-time, the $\tilde{d\vec{l}}$'s are not necessarily space-like.

It is not hard to see that in p-dimensional space there are no $p+1$ forms. For instance in three dimensions the four form

$$F_{(4)} = \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 \wedge \tilde{d}x^3 = -\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 \wedge \tilde{d}x^3 \tag{33}$$

has to be zero. Similarly in four dimensional space-time there are no 5-forms.

Finally we end with general formulae for the components of the wedge product of two forms. In general

$$\begin{aligned} A_{(p)} \wedge B_{(q)} &= \left(\frac{1}{p!} A_{\mu\ldots\nu} \tilde{d}x^\mu \wedge \ldots \wedge \tilde{d}x^\nu\right) \wedge \left(\frac{1}{q!} B_{\alpha\ldots\beta} \tilde{d}x^\alpha \wedge \ldots \wedge \tilde{d}x^\beta\right) \\ &= \frac{1}{p!q!} A_{[\mu\ldots\nu} B_{\alpha\ldots\beta]} \tilde{d}x^\mu \wedge \ldots \wedge \tilde{d}x^\nu \wedge \tilde{d}x^\alpha \wedge \ldots \wedge \tilde{d}x^\beta \\ &= \frac{1}{(p+q)!} (A_{(p)} \wedge B_{(q)})_{\mu\ldots\beta} \tilde{d}x^\mu \wedge \ldots \wedge \tilde{d}x^\beta \end{aligned}$$

so that we must have

$$(A_{(p)} \wedge B_{(q)})_{\mu\ldots\beta} = \frac{(p+q)!}{p!q!} A_{[\mu\ldots} B_{\ldots\beta]} = \binom{p+q}{p} A_{[\mu\ldots} B_{\ldots\beta]}.. \tag{34}$$

☞ From the second line of the above derivation, we can also see that

$$A_{(p)} \wedge B_{(q)} = (-1)^{(pq)} B_{(q)} \wedge A_{(p)} \tag{35}$$

since we have to move the q dx's of B through p dx's of A to rearrange the formula picking up $p \times q$ sign changes. This means that **even forms commute with everything** and **odd forms anti-commute with each other**.

7 Hodge dual

We summarises some of the forms of (3+1) dimensional special relativity below

$\mathbb{R}^2$	p	p-form basis	dimension
	0	1	1
	1	$\tilde{d}x$, $\tilde{d}y$	2
	2	$\tilde{d}x \wedge \tilde{d}y$	1

$\mathbb{R}^3$	p	p-form basis	dimension
	0	1	1
	1	$\tilde{d}x$, $\tilde{d}y$, $\tilde{d}z$	3
	2	$\tilde{d}x \wedge \tilde{d}y$, $\tilde{d}y \wedge \tilde{d}z$, $\tilde{d}z \wedge \tilde{d}x$	3
	3	$\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$	1

$\mathbb{R}^{3,1}$	p	p-form basis	dimension
	0	1	1
	1	$\tilde{d}t$, $\tilde{d}x$, $\tilde{d}y$, $\tilde{d}z$	4
	2	$\tilde{d}t \wedge \tilde{d}x$, $\tilde{d}t \wedge \tilde{d}y$, $\tilde{d}t \wedge \tilde{d}z$, $\tilde{d}x \wedge \tilde{d}y$, $\tilde{d}y \wedge \tilde{d}z$, $\tilde{d}z \wedge \tilde{d}x$	6
	3	$\tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y$, $\tilde{d}t \wedge \tilde{d}y \wedge \tilde{d}z$, $\tilde{d}t \wedge \tilde{d}z \wedge \tilde{d}x$, $\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$,	4
	4	$\tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$	1

Aside: The notation $\mathbb{R}^{s,t}$ is often used to denote a flat space-time with s spacial and t time-like directions.

Looking at the tables above we note that the number of forms follows Pascal's triangle so that

$$\#p-\text{forms} \;=\#(n-p)-\text{forms} \;=\binom{n}{p}=\binom{n}{n-p}=\frac{n!}{p!(n-p)!} \tag{36}$$

which suggests that it is possible to construct a map between p-forms and $(n-p)$-forms.

Definition 7.1: The Hodge Dual, which maps p-forms to $(n-p)$-forms is given in terms of the basis forms as,

$$*\left(\tilde{d}x^{\mu_1}\wedge\ldots\wedge\tilde{d}x^{\mu_p}\right)=\frac{1}{(n-p)!}\varepsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}(\tilde{d}x^{\mu_{p+1}}\wedge\ldots\wedge\tilde{d}x^{\mu_n}) \tag{37}$$

where ε, which is called the Levi-Cevita Symbol, is completely antisymmetric in all its indices. This means that it has only one independent component, which in our conventions is:

$$\varepsilon_{012\ldots p}=1=-\varepsilon^{012\ldots p} \tag{38}$$

and in Euclidean space we take

$$\varepsilon^{12\ldots p}=1=\varepsilon_{12\ldots p} \tag{39}$$

Aside: We could have taken

$$\tilde{\varepsilon}=\tilde{d}x^0\wedge\tilde{d}x^1\wedge\tilde{d}x^2\wedge\ldots\tilde{d}x^p \tag{40}$$

which has the correct components

$$\varepsilon_{\mu\ldots\nu}=\tilde{d}x^0\wedge\tilde{d}x^1\wedge\tilde{d}x^2\ldots\wedge\tilde{d}x^p(\vec{e}_\mu,\ldots,\vec{e}_\nu) \tag{41}$$

We define the Hodge dual to act linearly on its arguments so that

Definition 7.2: $*F_{(p)}=\frac{1}{p!}F_{\mu_1\ldots\mu_p}*\left(\tilde{d}x^{\mu_1}\wedge\ldots\wedge\tilde{d}x^{\mu_p}\right)$

For $\mathbb{R}^2$ we have

$$\begin{aligned}
*1 &= \frac{\varepsilon_{ij}}{2!}\tilde{d}x^i\wedge\tilde{d}x^j=\tilde{d}x\wedge\tilde{d}y\\
*\tilde{d}x &= *\tilde{d}x^1=\varepsilon^1{}_i\tilde{d}x^i=dy\\
*\tilde{d}y &= -\tilde{d}x\\
*\tilde{d}x\wedge\tilde{d}y &= \varepsilon^{12}1=1
\end{aligned}$$

so that

$$\begin{aligned}
*^2 1 &= 1\\
*^2\tilde{d}x^i &= -\tilde{d}x^i\\
*^2\tilde{d}x\wedge\tilde{d}y &= \tilde{d}x\wedge\tilde{d}y
\end{aligned}$$

For $\mathbb{R}^3$ we have

$$\begin{aligned}
*1 &= \tilde{d}x\wedge\tilde{d}y\wedge\tilde{d}z\\
*\tilde{d}x &= \tilde{d}y\wedge\tilde{d}z \text{ and cyclic perms.}\\
*\tilde{d}x\wedge\tilde{d}y &= \tilde{d}z \text{ and cyclic perms.}\\
*\tilde{d}x\wedge\tilde{d}y\wedge\tilde{d}z &= 1
\end{aligned}$$

so that $*^2 = 1$ in $\mathbb{R}^3$.

Finally in $\mathbb{R}^{3,1}$ we have

$$\begin{aligned}
*1 &= \tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z \\
*\tilde{d}t &= -\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}x, \qquad *\tilde{d}x = -\tilde{d}t \wedge \tilde{d}y \wedge \tilde{d}z, \text{ etc.} \\
*\tilde{d}x \wedge \tilde{d}y &= \tilde{d}t \wedge \tilde{d}z, \qquad *\tilde{d}t \wedge \tilde{d}x = -\tilde{d}y \wedge \tilde{d}z, \text{ etc.} \\
*\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z &= -\tilde{d}t, \qquad *\tilde{d}t \wedge \tilde{d}y \wedge \tilde{d}z = -\tilde{d}x \text{ etc.} \\
*\tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z &= -1
\end{aligned}$$

so that

$$*^2 F_{(p)} = (-1)^{p-1} F_{(p)} \tag{42}$$

in $\mathbb{R}^{3,1}$

The general formula is

$$\mathbb{R}^n : \qquad *^2 F_{(p)} = (-1)^{p(n-p)} F_{(p)} \tag{43}$$

$$\mathbb{R}^{n-1,1} : \qquad *^2 F_{(p)} = (-1)^{p(n-p)+1} F_{(p)} \tag{44}$$

8 Generalised Stokes' theorem and the exterior derivative

The exterior derivative allows us to generalise the curl to higher dimensions and gives us an elegant expression of a generalised Stokes' theorem.

Given a p-form

$$F_{(p)} = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} \tag{45}$$

we define the exterior derivative $\tilde{d}$, which gives us a p+1 form, as follows

Definition 8.1:

$$\begin{aligned}
\tilde{d}F_{(p)} &:= \frac{1}{p!} (\tilde{d}F_{\mu_1 \ldots \mu_p}) \wedge \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} \\
&= \frac{1}{p!} F_{\mu_1 \ldots \mu_p, \nu} \tilde{d}x^{\nu} \wedge \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} \\
&= \frac{1}{p!} F_{[\mu_1 \ldots \mu_p, \nu]} \tilde{d}x^{\nu} \wedge \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} \\
&= \frac{1}{(p+1)!} (\tilde{d}F_{(p)})_{\nu_1 \ldots \nu_{p+1}} dx^{\nu_1 \ldots \nu_{p+1}}
\end{aligned}$$

and reading off components we find that for a p-form

$$(\tilde{d}F)_{\alpha \ldots \beta\gamma} = (p+1) F_{[\alpha \ldots \beta, \gamma]} \tag{46}$$

Acting twice with $\tilde{d}$ on a function gives 0 since partial derivatives commute:

$$\tilde{d}^2 f = f_{,\mu\nu} \tilde{d}x^{\mu} \wedge \tilde{d}x^{\nu} = f_{,[\mu\nu]} \tilde{d}x^{\mu} \wedge \tilde{d}x^{\nu} = 0 \tag{47}$$

so that $\tilde{d}^2 = 0$ acting on any form gives zero since

$$\tilde{d}^2 F_{(p)} := \frac{1}{p!} \underbrace{(\tilde{d}^2 F_{\mu_1 \ldots \mu_p})}_{0} \wedge \tilde{d}x^{\mu_1} \wedge \ldots \wedge \tilde{d}x^{\mu_p} = 0. \tag{48}$$

8.1 Vector Calculus revisited

The fact that the exterior derivative is nilpotent (ie. squares to zero) can be used to rederive some familiar results from vector calculus

Consider a one-form

$$\tilde{p} = p_x \tilde{d}x + p_y \tilde{d}y + p_z \tilde{d}z. \tag{49}$$

By explicit calculation we find that

$$\begin{aligned}
\tilde{d}\tilde{p} &= p_{x,x} \underbrace{\tilde{d}x \wedge \tilde{d}x}_{0} + p_{x,y}\tilde{d}y \wedge \tilde{d}x + p_{x,z}\tilde{d}z \wedge \tilde{d}x \\
&\quad + p_{y,y} \underbrace{\tilde{d}y \wedge \tilde{d}y}_{0} + p_{y,x}\tilde{d}x \wedge \tilde{d}y + p_{y,z}\tilde{d}z \wedge \tilde{d}y \\
&\quad p_{z,z} \underbrace{\tilde{d}z \wedge \tilde{d}z}_{0} + p_{z,y}\tilde{d}y \wedge \tilde{d}z + p_{z,x}\tilde{d}x \wedge \tilde{d}z \\
&= (p_{y,x} - p_{x,y})\tilde{d}x \wedge \tilde{d}y + (p_{z,y} - p_{y,z})\tilde{d}y \wedge \tilde{d}z + (p_{x,z} - p_{z,x})\tilde{d}z \wedge \tilde{d}x
\end{aligned}$$

which has the components of $\nabla \times \vec{p}$.

Now let

$$F_{(2)} = b_x \tilde{d}y \wedge \tilde{d}z + b_y \tilde{d}z \wedge \tilde{d}x + b_z \tilde{d}x \wedge \tilde{d}y \tag{50}$$

then

$$\tilde{d}F_{(2)} = b_{x,x}\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z + b_{y,y}\tilde{d}y \wedge \tilde{d}z \wedge \tilde{d}x + b_{z,z}\tilde{d}z \wedge \tilde{d}x \wedge \tilde{d}y = (\nabla \cdot \vec{b})\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z \tag{51}$$

Now since $\tilde{d}^2\tilde{p} = 0$ we conclude that $\nabla \cdot \nabla \times \vec{p} = 0$

As another example recall that $\tilde{d}f$ has the components of ∇f so that $\tilde{d}^2 f = 0$ implies that $\nabla \times \nabla f = 0$.

8.2 Liebnitz for forms

Forms satisfy the generalised Liebnitz property

$$\tilde{d}(A_{(p)} \wedge B_{(q)}) = \tilde{d}A_{(p)} \wedge B_{(q)} + (-1)^p A_{(p)} \wedge \tilde{d}B_{(q)} \tag{52}$$

To derive this property consider and p-form, A, and a q-from, B,[2]

$$A = a_{\mu_1 \ldots \mu_p} \tilde{d}x^{\mu_1 \ldots \mu_p} \qquad B = b_{\nu_1 \ldots \nu_q} \tilde{d}x^{\nu_1 \ldots \mu_q} \tag{53}$$

then

$$\begin{aligned}
\tilde{d}(A \wedge B) &= \tilde{d}(a_{\mu_1 \ldots \mu_p} b_{\nu_1 \ldots \nu_q}) \wedge \tilde{d}x^{\mu_1 \ldots \mu_p \nu_1 \ldots \mu_q} \\
&= (a_{\mu_1 \ldots \mu_p,\gamma} b_{\nu_1 \ldots \nu_q} + a_{\mu_1 \ldots \mu_p} b_{\nu_1 \ldots \nu_q,\gamma})\tilde{d}x^{\gamma\mu_1 \ldots \mu_p \nu_1 \ldots \mu_q} \\
&= (a_{\mu_1 \ldots \mu_p,\gamma} b_{\nu_1 \ldots \nu_q})\tilde{d}x^{\gamma\mu_1 \ldots \mu_p \nu_1 \ldots \mu_q} + (-1)^p (a_{\mu_1 \ldots \mu_p} b_{\nu_1 \ldots \nu_q,\gamma})\tilde{d}x^{\mu_1 \ldots \mu_p \gamma \nu_1 \ldots \mu_q} \\
&= \tilde{d}A \wedge B + (-1)^p A \wedge \tilde{d}B
\end{aligned}$$

[2] We've dropped the pesky combinatorial factors for clarity here

8.3 Boundary operator: ∂

The boundary operator, ∂ , maps a p-volume to its $(p-1)$ dimensional boundary. For instance

$$\partial[a,b] = \{a\} \cup \{b\}$$

and

$$\partial \text{ Disc } = \text{Circle}$$

Just like the exterior derivative, $\tilde{d}$, the boundary operator is nilpotent: "A boundary has no boundary".

8.4 Stokes' theorem

We state with out proof a Generalised Stokes' Theorem

$$\int_{(\partial c)_p} A_{(p)} = \int_{c_{p+1}} \tilde{d} A_{(p)}. \tag{54}$$

Notice that on the left hand side we have a p-form integrated over the p-dimensional boundary, and on the right hand side we have $(p+1)$-form integrated over a (p+1)-volume.

9 Exercises

1. If

$$\tilde{q}_\mu = (0,2,1,0) \qquad T_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{55}$$

   find

   (a) $T_A(\ , \ )$ if $T_A(\vec{V},\vec{W}) = \frac{1}{2}(T(\vec{V},\vec{W}) - T(\vec{W},\vec{V}))$

   (b) $\tilde{q} \otimes T$

   (c) $\tilde{q} \wedge T_A$

   (d) $q_{[\alpha}T_{\beta\gamma]}$

2. Consider the one-form

$$\tilde{A} = -\frac{Q}{r}\tilde{d}t, \qquad r = \sqrt{x^2+y^2+z^2} \tag{56}$$

   find

$$\int_{c_1} \tilde{A} \tag{57}$$

   where

   (a) c_1 is worldline of an observer at $x = 1$ m for $t \in [0,1]$ m

   (b) c_1 is the world-line of an observer, starting at $(0,1)$ m, travelling in the x-direction at $v_x = \frac{1}{3}$ for a proper time interval of 1 m.

   (c) c_1 is the world-line of an observer, starting at $(0,1)$ m, travelling in the x-direction with a constant acceleration (in SI units) of 10 m/s^2 for a proper time interval of 1 m.

3. Let $I_{c_2} = \int_{c_2} x\tilde{d}x \wedge \tilde{d}y$.

   (a) Find I_{c_2} with $c_2 = \{(x,y) : x \in [0,1] \text{ and } y \in [0,1]\}$.

   (b) Find a one-form such that $x\tilde{d}x \wedge \tilde{d}y = \tilde{d}A$ and check your answer to the first part using Stokes' Theorem

4. (a) Find $\int_{c_2} \frac{\tilde{d}x\wedge\tilde{d}y}{r}$ where c_2 is the unit disc in the (x,y)-plane centred on the origin.

   (b) Show that $\frac{\tilde{d}x\wedge\tilde{d}y}{r} = \tilde{d}(r^{-1}(x\tilde{d}y - y\tilde{d}x))$ and check your answer to the first part using Stokes' Theorem.

5. Find $\int_{c_2} \frac{\tilde{d}y\wedge\tilde{d}z}{r}$ where c_2 is southern hemisphere of the unit sphere centred on the origin.

6. Let

$$\delta^{\alpha\beta\ldots\gamma}_{\mu\nu\ldots\lambda} = \tilde{d}x^\alpha \wedge \tilde{d}x^\beta \wedge \ldots \wedge \tilde{d}x^\gamma(\vec{e}_\mu, \vec{e}_\nu, \ldots, \vec{e}_\lambda) = \det \begin{vmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \ldots & \delta^\alpha_\lambda \\ \delta^\beta_\mu & \delta^\beta_\nu & \ldots & \delta^\beta_\lambda \\ \vdots & \vdots & \ddots & \vdots \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \ldots & \delta^\gamma_\lambda \end{vmatrix} \tag{58}$$

(a) Show that in $\mathbb{R}^3$

$$\begin{aligned}
\delta^{ijk}_{lmn} &= \varepsilon^{ijk}\varepsilon_{lmn} && (59)\\
\delta^{ij}_{lm} &= \delta^{ijk}_{lmk} = \varepsilon^{ijk}\varepsilon_{lmk} && (60)\\
\delta^{i}_{l} &= \frac{1}{2}\delta^{ij}_{lj} = \frac{1}{2}\delta^{ijk}_{ljk} = \frac{1}{2}\varepsilon^{ijk}\varepsilon_{ljk} && (61)\\
1 &= \frac{1}{3}\delta^{i}_{i} = \frac{1}{6}\delta^{ij}_{ij} = \frac{1}{6}\delta^{ijk}_{ijk} = \frac{1}{6}\varepsilon^{ijk}\varepsilon_{ijk} && (62)\\
& && (63)
\end{aligned}$$

(b) Show that in $\mathbb{R}^{3,1}$

$$\begin{aligned}
\delta^{\alpha\beta\gamma\delta}_{\mu\nu\lambda\rho} &= -\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\lambda\rho} && (64)\\
\delta^{\alpha\beta\gamma}_{\mu\nu\lambda} &= \delta^{\alpha\beta\gamma\rho}_{\mu\nu\lambda\rho} = -\varepsilon^{\alpha\beta\gamma\rho}\varepsilon_{\mu\nu\lambda\rho} && (65)\\
\delta^{\alpha\beta}_{\mu\nu} &= \frac{1}{2}\delta^{\alpha\beta\lambda}_{\mu\nu\lambda} = -\frac{1}{2}\varepsilon^{\alpha\beta\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} && (66)\\
\delta^{\alpha}_{\mu} &= \frac{1}{3}\delta^{\alpha\beta}_{\mu\beta} = \frac{1}{6}\delta^{\alpha\beta\lambda}_{\mu\beta\lambda} = -\frac{1}{6}\varepsilon^{\alpha\beta\lambda\rho}\varepsilon_{\mu\beta\lambda\rho} && (67)\\
1 &= \frac{1}{4}\delta^{\alpha}_{\alpha} = \frac{1}{12}\delta^{\alpha\beta}_{\alpha\beta} = \frac{1}{24}\delta^{\alpha\beta\lambda}_{\alpha\beta\lambda} = -\frac{1}{24}\varepsilon^{\alpha\beta\lambda\rho}\varepsilon_{\alpha\beta\lambda\rho} && (68)
\end{aligned}$$

7. Confirm the expressions for Hodge dual of basis forms for $\mathbb{R}^{3,1}$ shown in the notes

8. Consider the Maxwell 2-form and current 1-forms:

$$\begin{aligned}
F_{(2)} &= (E_x\tilde{d}x + E_y\tilde{d}y + E_z\tilde{d}z)\wedge\tilde{d}t\\
&\quad B_x\tilde{d}y\wedge\tilde{d}z + B_y\tilde{d}z\wedge\tilde{d}x + B_z\tilde{d}x\wedge\tilde{d}y\\
\tilde{J} &= -\rho\tilde{d}t + J_i\tilde{d}x^i
\end{aligned}$$

(a) What are the components $F_{\mu\nu}$ and J^μ?

(b) Find the components of $F_{\bar{\mu}\bar{\nu}}$ in a frame travelling a velocity v relative to the original frame by

i. Performing a Lorentz transformation on $F_{\mu\nu}$

ii. Performing a Lorentz transformation on the basis forms of $F_{(2)}$ in the expression above

(c) Find $*F$ and $*\tilde{J}$

(d) Determine $\tilde{d}F$ and $\tilde{d}*F$

(e) Show that Maxwell's equations are equivalent to

$$\begin{aligned}
\tilde{d}F &= 0\\
\tilde{d}*F &= 4\pi*\tilde{J}
\end{aligned}$$

(f) Show that in component form the above equations reduce can be written

$$\begin{aligned}
F_{[\alpha\beta,\gamma]} &= 0\\
F^{\mu\nu}{}_{,\nu} &= 4\pi J^\mu
\end{aligned}$$

(g) Rewrite the forms so that we obtain Maxwell's equations in SI units.

(h) Use the nilpotency of $\tilde{d}$ and the equations above to show that

$$\tilde{d}*J = 0. \qquad (69)$$

Write this equation out explicitly and show that it is the law of charge conservation. Show that in component form it can be written $J^\mu{}_{,\mu}$=0.

(i) Find the components of the 1-form $q\ F_{(2)}(\vec{U},\ )$ where $\vec{U}$ is the four-velocity of a particle with charge q. How do you interpret this form?

(j) Find the components of $F \wedge F$ and $F \wedge *F$. What do the components correspond to ?

(k) **Bonus:** Show that their is a frame in which the Maxwell tensor can be written

$$F_{(2)} = E_{\bar{z}} \tilde{d\bar{z}} \wedge \tilde{d\bar{t}} + B_{\bar{z}} \tilde{d\bar{x}} \wedge \tilde{d\bar{y}} \tag{70}$$

9. Given a p-form $A_{(p)}$ and a vector $\vec{X}$ we can define an interior product i_X that maps p-forms to p-1 forms by

$$i_X A(\ ) = A(\vec{X},\ ) \tag{71}$$

(a) Find the components of $i_X A$

(b) Show that

$$i_X(A_{(p)} \wedge B_{(q)}) = (i_X A_{(p)}) \wedge B_{(q)} + (-1)^p A_{(p)} \wedge (i_X B_{(q)}) \tag{72}$$

10. Show that the Generalised Stokes' theorem implies

(a) Stokes' theorem in 3d:

$$\int_{\partial c} \vec{A} \cdot d\vec{l} = \int_c (\nabla \times \vec{A}) \cdot \vec{n}\ dA \tag{73}$$

(b) Gauss' theorem in 3d:

$$\int_{\partial V} (\vec{B} \cdot \vec{n}) dA = \int_V (\nabla \cdot \vec{B})\ dV \tag{74}$$

(c) The 4d Gauss theorem:

$$\int_{\partial V_4} (V^\alpha n_\alpha)\ d^3S = \int_{V_4} (V^\alpha{}_{,\alpha}) d^4x \tag{75}$$

10 Bibliography

These notes are partially based on Ch3 of Ryder with additional inspiration from MTW