# Multi-boundary / multi-partite Entanglement in Chern-Simons Theories


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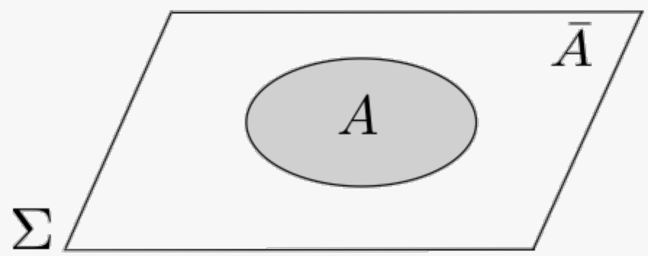
# Entanglement Structure

- **of basic interest are the patterns of entanglement in any quantum theory, especially quantum field theories**
  - in gauge/gravity duality, expected to play a central role in 'bulk emergence'
  - in condensed matter physics, a primary observable especially in topological states of matter
- **entanglement inequalities, such as the positivity and monotonicity of relative entropy, play a powerful role, constraining QFTs in interesting ways**
  - recent work on establishing ANEC is but one example **[1605.08072]**
- **reducing to the simplest terms, patterns of entanglement are understood generally only for two and three qubit systems**
  - studies of multi-partite systems are needed

# Bi-partite entanglement

- often in QFT, interested in spatial entanglement
  - standard construction presupposes $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$




- for a state $|\Psi\rangle$ on $\Sigma$, trace over degrees of freedom in $\overline{A}$

  $\longrightarrow$ reduced density matrix $\hat{\rho}_A$

$$S^{(\alpha)}(A) = \frac{1}{1-\alpha} \log tr_{\mathcal{H}_A} \hat{\rho}_A^{\alpha}$$

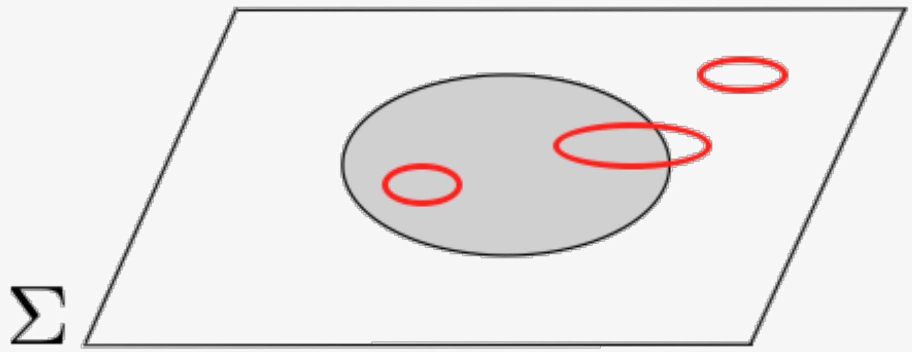
Rényi entropies

$$S_{EE}(A) = -tr_{\mathcal{H}_A} \hat{\rho}_A \log \hat{\rho}_A$$

entanglement entropy

# Bi-partite entanglement

- works well for some QFTs, such as scalars and spinor fields
- it doesn't work for gauge theories, as the Hilbert space does not factorize




- observables aren't generally local
  - cutting and gluing of regions involves degrees of freedom on cut
- in 3d CS, this is particularly familiar
  - bulk is topological, but WZW on 1+1 edges

# 3d Chern-Simons

- **the non-factorizability of the Hilbert space is strikingly evident here**
  - think of C-S theory on 3-mfld (locally) of the form $M_3 \sim \mathbb{R} \times \Sigma$
  - path integral over half-spacetime with (space-like) boundary $\Sigma$ gives a wave-functional (half-spacetime ~ solid $\Sigma$)
    - thus associate a Hilbert space $\mathcal{H}_\Sigma$ to $\Sigma$
    - the various states in $\mathcal{H}_\Sigma$ correspond to non-trivial Wilson loops

$$e.g., \qquad S^2 = D^2 \cup D^2$$

$$\dim\ \mathcal{H}_{S^2} = 1 \qquad \textbf{but} \qquad \dim\ \mathcal{H}_{D^2} > 1$$

**so**

$$\mathcal{H}_{S^2} \subset \mathcal{H}_{D^2} \otimes \mathcal{H}_{D^2}$$

# 3d C-S and Bi-partite entanglement

- nevertheless, bi-partite entanglement is well understood in 3d C-S

Kitaev & Preskill
Levin & Wen
S. Dong, E. Fradkin, S. Nowling, RGL
[0802.3231]

$$S_A \sim x\frac{L}{a} - \gamma$$

non-universal 'area law' — topological entanglement

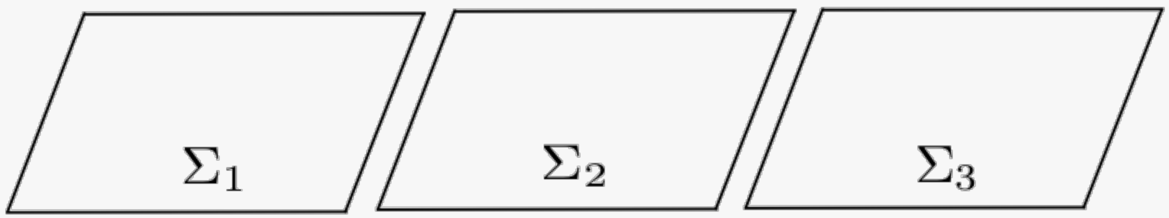
- Rényi entropies all equal

[Witten '90s]

- topological entanglement can be computed using 'surgery' methods and the replica trick, allowing for bypassing gauge issues
  - depends on data of dual CFT (modular S-matrix, etc.)
  - depends on choice of state, topological class of entanglement cut

# Multi-partite entanglement

- in that context, the spatial Riemann surface was assumed to be connected, and entanglement was associated with cutting that surface
- more generally, we can consider the spatial Riemann surface to be a disjoint union of Riemann surfaces




- gives a multi-partite system, with

$$\mathcal{H} = \otimes_j \mathcal{H}_{\Sigma_j}$$

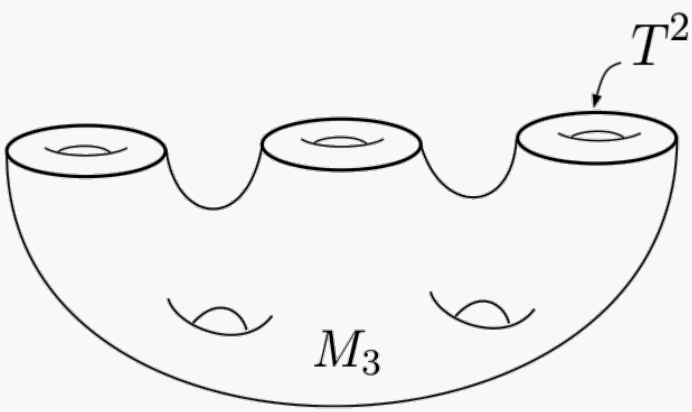
- recently, such a construction was studied in $AdS_3/CFT_2$
  - CFT on $(S^1)^n$; dual to multi-boundary wormholes
  - study entanglement via RT

[Vijay et al '14]
[Marolf et al '15]

# Multi-partite entanglement

- **here, we wish to study this directly in field theory**
  - generally very difficult, but expect significant simplification in TQFTs
  - here, we confine attention to 3d C-S theories, with gauge group $U(1)$ or $SU(2)$

$$S_{CS}[A] = \frac{k}{4\pi}\int_{M_3} Tr\left(A \wedge dA + \tfrac{2}{3}A^3\right)$$

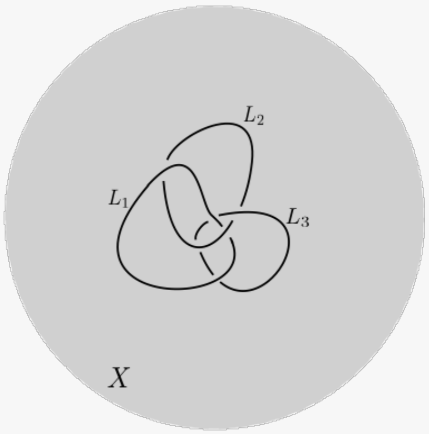



$$\Psi[A_{(0)}] = \int_{A\big|_{\Sigma}=A_{(0)}} [DA]e^{-S_{CS}[A]}$$

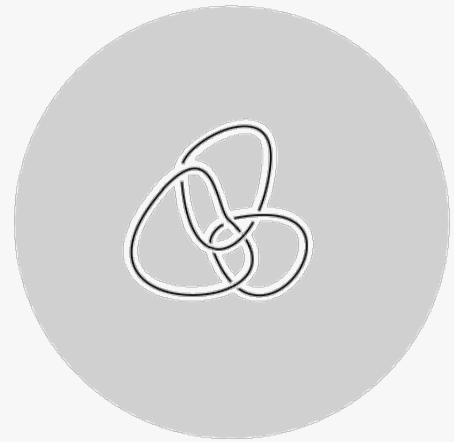
$$\partial M_3 = \Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$$

# Multi-boundary states in $CS_3$

- a simple way to generate such manifolds is to start with a closed 3-manifold X (e.g. $S^3$) and an n-component link $\mathcal{L}^n$ in X



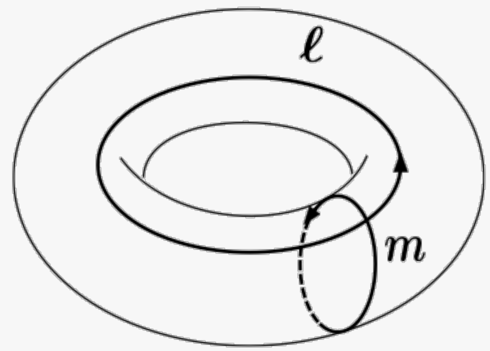

$$\mathcal{L}^n = L_1 \cup L_2 \cup \ldots \cup L_n$$



- fatten each component into a tubular neighbourhood, yielding $N(\mathcal{L}^n)$
- the *link complement* $M_{(n)} = S^3 \backslash N(\mathcal{L}^n)$ is a 3-mfld with n-component boundary
- CS path integral on $M_{(n)}$ gives a state $|\mathcal{L}^n\rangle \in \otimes_j \mathcal{H}(T^2)_j$

# Hilbert Spaces for CS

- the basic thing to understand is the Hilbert space on a torus $T^2$
  - choose a basis of cycles $m, \ell$
  - space-time is solid torus, cycle $m$ contractible
  - Wilson loops supported on $\ell$




- do CS path integral with Wilson loop in rep $R_j$ on $\ell$ $\longrightarrow$ *state* $|j\rangle$
  - $\langle j|$ associated with $R_j^*$ $\qquad \langle j|j'\rangle = \delta_{j,j'}$
- here, $R_j$ refer to the integrable representations, associated with U(1) or SU(2) characters

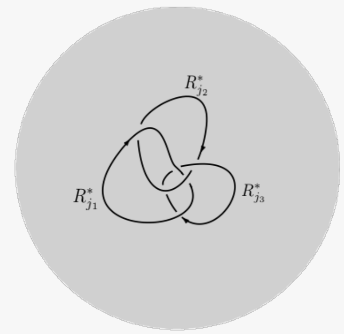
$$SU(2)_k : \quad j = 0, \tfrac{1}{2}, 1, \ldots, \tfrac{k}{2}$$
$$U(1)_k : \quad q = 0, 1, \ldots, k-1$$

- in this sense, Hilbert spaces are finite dimensional

# Link Complement States

- returning to an n-component link complement, we write

$$|\mathcal{L}^n\rangle = \sum_{j_1,\dots,j_n} C_{\mathcal{L}^n}(j_1,\dots,j_n)|j_1\rangle \otimes \dots \otimes |j_n\rangle$$



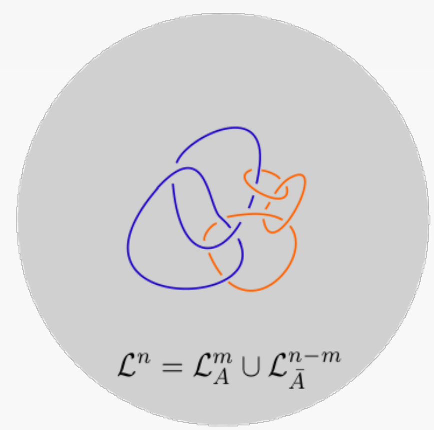

$$C_{\mathcal{L}^n}(j_1,\dots,j_n) = \langle W_{R^*_{j_1}}(L_1)\dots W_{R^*_{j_n}}(L_n)\rangle_{S^3}$$

- 'wavefunctions' are *coloured link invariants*

- we will study entanglement measures for simple partitions of the link components

$$\mathcal{L}^n = \mathcal{L}^m_A \cup \mathcal{L}^{n-m}_{\overline{A}} \qquad \mathcal{L}^m_A = L_1 \cup \dots \cup L_m$$

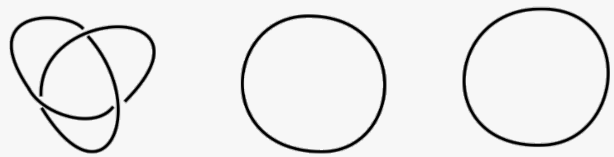
$$\rho_A = \frac{1}{\langle\mathcal{L}^n|\mathcal{L}^n\rangle} Tr_{\mathcal{L}_{\overline{A}}}|\mathcal{L}^n\rangle\langle\mathcal{L}^n|$$



# Why is this Interesting?

- in some sense (that we will explore), quantum entanglement is correlated with topological entanglement

- the first manifestation of this is seen if we take the loops to be unlinked



  - in this case, the coloured link invariant factorizes

    - so the corresponding state is a product state, all entanglement entropies vanish

- we conclude that entanglement should detect topological linking

# Why is this Interesting?

- the entanglement entropy (and other similar observables) is a topological invariant
  - in fact, it satisfies an important technical property, that is, it is *framing independent*
    - the coloured link invariants require a choice to be made for the $\ell$ cycle
    - this corresponds to a *unitary transformation* on states, and so does not affect entanglement observables

- so we will study how quantum entanglement encodes topological entanglement by studying a series of examples

# Abelian case

- for the Abelian case, it turns out that all states can be computed in closed form

[Witten '88]

  - depends only on the Gauss linking number of components

$$|\mathcal{L}^n\rangle = \sum_{q_1,\dots,q_n} exp\left(\frac{2\pi i}{k}\sum_{i<j} q_i q_j \ell_{ij}\right)|q_1\rangle \otimes \dots \otimes |q_n\rangle$$

  - here we have used the framing independence to set self-linking $\ell_{jj} \to 0$

- recall that generally we will consider partitioning

$$\mathcal{L}^n = \mathcal{L}^m_A \cup \mathcal{L}^{n-m}_{\overline{A}} \qquad \mathcal{L}^m_A = L_1 \cup \dots \cup L_m$$

$$\rho_A = \frac{1}{\langle \mathcal{L}^n | \mathcal{L}^n \rangle} Tr_{\mathcal{L}_{\overline{A}}} |\mathcal{L}^n\rangle\langle\mathcal{L}^n|$$

# U(1)$_k$: two-component links

- in this case, just one linking number

$$|\mathcal{L}^2\rangle = \frac{1}{k}\sum_{q_1,q_2=0}^{k-1} exp\Big(\frac{2\pi i}{k}q_1 q_2 \ell_{12}\Big)|q_1\rangle \otimes |q_2\rangle$$

- tracing over the $L_2$, we obtain

$$\rho_1 = Tr_{L_2}|\mathcal{L}^2\rangle\langle\mathcal{L}^2|$$

$$\langle q_1|\rho_1|q_1'\rangle = \frac{1}{k^2}\sum_{q_2=0}^{k-1} e^{2\pi i(q_1-q_1')\ell_{12}q_2/k} \equiv \frac{1}{k}\eta_{q_1,q_1'}(k,\ell_{12})$$

- this matrix element vanishes unless

$$(q_1 - q_1')\ell_{12} = 0 \ (mod\ \ k)$$

# U(1)$_k$: two-component links

- to compute entanglement entropy, we can directly compute $\rho_1$ and determine its spectrum of eigenvalues $\{p_j, \ j = 1, ..., k\}$

$$S_{EE} = -\sum_{j=1}^{k} p_j \log p_j$$

- the form of $\rho_1$ depends on $g = gcd(k, \ell_{12})$

$$\begin{pmatrix} 1 & 1 & \dots \\ \vdots & \ddots & \\ 1 & & 1 \end{pmatrix}_{g \times g} \otimes \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}_{\frac{k}{g} \times \frac{k}{g}}$$

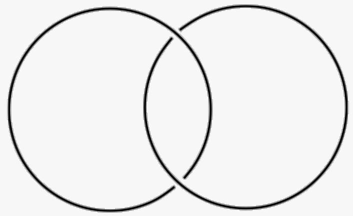
$$\boxed{S_{EE} = \log\left(\frac{k}{gcd(k, \ell_{12})}\right)}$$

# $U(1)_k$: two-component links

$$S_{EE} = \log\left(\frac{k}{gcd(k,\ell_{12})}\right)$$

- for example, the unlink gives zero entropy, while the Hopf link gives

$$S_{EE}^{Hopf} = \log k$$



- thus the Hopf link is a *maximally entangled* state
  - analogous to a Bell pair

- alternative derivation: replica trick — compute Rényi entropies

# $U(1)_k$: *n*-component links

- for *n*-component links, we partition (m|n-m)
- find entanglement entropy

$$S_{EE}^{(m|n-m)} = \log \frac{k^m}{|ker\ \ G|}$$

- where G is the linking matrix between $A$ and $\overline{A}$

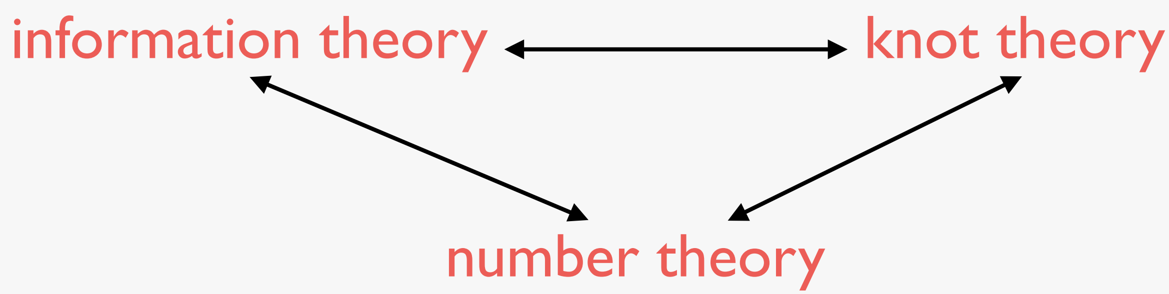
$$\mathbf{G} = \begin{pmatrix} \ell_{1,m+1} & \ell_{2,m+1} & \cdots & \ell_{m,m+1} \\ \ell_{1,m+2} & \ell_{2,m+2} & \cdots & \ell_{m,m+2} \\ \vdots & \vdots & & \vdots \\ \ell_{1,n} & \ell_{2,n} & \cdots & \ell_{m,n} \end{pmatrix}$$

"Diophantine equations"

- $|ker\ \ G|$ = # solutions to $G \cdot \vec{x} = 0 \ (mod\ \ k), \quad \vec{x} \in \mathbb{Z}_k^m$

# U(1)$_k$: *n*-component links

- in the (1|1) case, $|ker\ \ G| = gcd(k, \ell_{12})$ , but more generally a concrete formula is not available

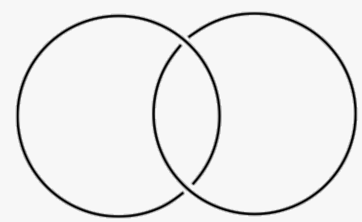



- but at least we can say:
  - entanglement entropy for (m|n) vanishes iff $G = 0\ (mod\ \ k)$
  - so Abelian quantum entanglement detects Gauss linking between sublinks

# Entanglement for SU(2)$_k$ link states

- **entanglement link invariants for non-Abelian CS theories probe more precise information about link states**
    - unfortunately, there is no known general formula for the state corresponding to a generic link
    - so we are forced to try to draw conclusions from studying example by example
- **again, the Hopf link is a maximally entangled state**

$$|Hopf\rangle = \frac{1}{\sqrt{k+1}} \sum_{j_1, j_2} S_{j_1, j_2} |j_1\rangle \otimes |j_2\rangle$$



- **where S is the (unitary) modular S-matrix (implements $\tau \to -1/\tau$)**

$$S_{j_1, j_2} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2j_1+1)(2j_2+1)\pi}{k+2}\right)$$

# SU(2)$_k$: Hopf links

$$|Hopf\rangle = \frac{1}{\sqrt{k+1}} \sum_{j_1, j_2} S_{j_1, j_2} |j_1\rangle \otimes |j_2\rangle$$

- tracing over the second loop gives a reduced density matrix

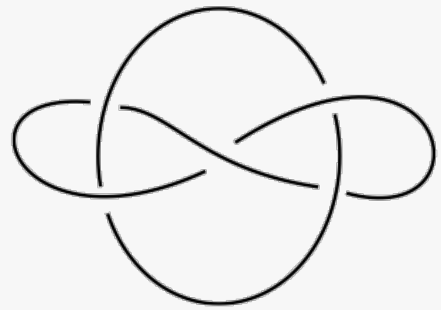
$$\langle j_1 | \hat{\rho}_1 | j_1' \rangle = \frac{1}{k+1} \sum_{j_1, j_2} S^*_{j_1, j_2} S_{j_1', j_2} = \frac{1}{k+1} \delta_{j_1, j_1'}$$

$$\longrightarrow \qquad S_{EE}^{Hopf} = \log(k+1)$$

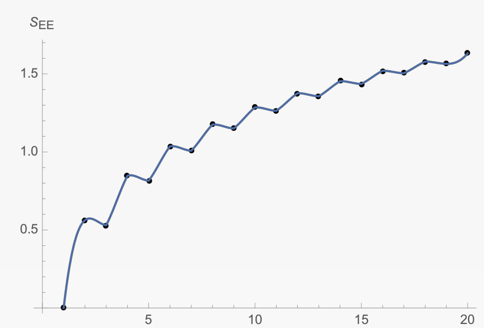
maximally entangled

# $SU(2)_k$: Whitehead ($5^2_1$) link

- this is a 2-component link, with Gauss linking number *zero*

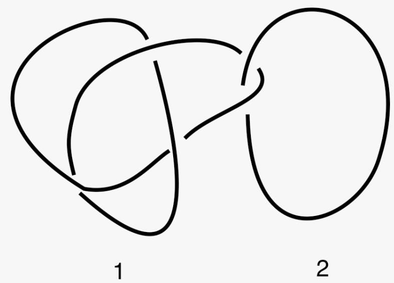


- so, in the Abelian case, we get zero entanglement entropy upon reducing one of the link components
- for $SU(2)_k$, this can be computed systematically, and the entanglement entropy does not vanish
  - computation simplified by knot theory formula due to K. Habiro
  - $SU(2)_k$ has access to more information about links



# SU(2)$_k$: "Hopf-linked knots"

- entanglement entropy depends on knotting of individual components



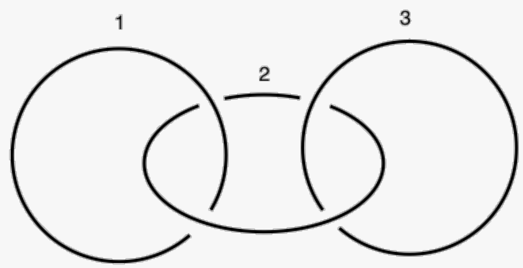

Wilson loop on $K_1$

$$S_{EE;1} = -\sum_j p_j \log p_j \qquad p_j = \frac{|C_{K_1}(j)/S_{oj}|^2}{\sum_{j'} |C_{K_1}(j')/S_{oj'}|^2}$$

- (recall Abelian version was insensitive to details of knot)

# SU(2)$_k$: 3-component links

- 3-chain: reduction on any of the three components gives the same entropy, determined by *quantum dimensions*




$$p_j = \frac{d_j^{-2}}{\sum_{j'} d_{j'}^{-2}} \qquad d_j = \frac{S_{0j}}{S_{00}} = [2j+1]$$

- *this link state is GHZ-like:*
  - trace over any link gives a separable state (unentangled mixed state)

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \qquad tr_1|GHZ\rangle\langle GHZ| = \frac{1}{2}\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

# SU(2)$_k$: 3-chain

- so entanglement entropy does not distinguish the components of the 3-chain, even though they are clearly inequivalent
- relative entropy for different traces can be employed to study this

$$S(\rho||\sigma) = tr\rho \log\rho - tr\rho\log\sigma$$

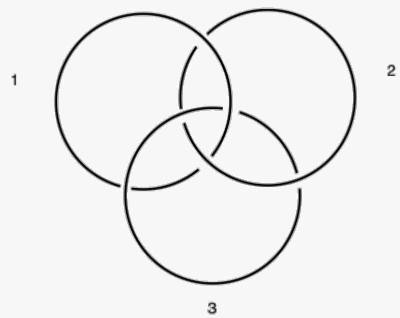
- here one finds

$$S(\rho_{L_1}||\rho_{L_2}) = \sum_i p_i \Big( \log p_i - \sum_j |S_{i,j}|^2 \log p_j \Big)$$

- this comes about because, although the diagonal forms of $\rho_{L_1}$ and $\rho_{L_2}$ are the same, they are diagonal in *different bases*
- so generally, relative entropy can be used as a basis independent measure of the distinguishability of components

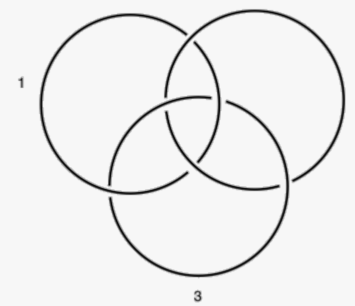
# $SU(2)_k$: 3-component links

$6^3_3$ is again GHZ-like




can be distinguished from 3-chain by looking at relative entropies

$6^3_2$ (Borromean rings) have zero Gauss linking, but are W-like




i.e., tracing over any component gives a state that is still entangled

for 3-qubit states, there are two distinct classes of states, GHZ and W, which cannot be transformed into one another by local quantum operations

$$|W\rangle = \frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle\Big)$$

$$tr_1|W\rangle\langle W| = \frac{1}{3}\Big(|00\rangle + (|01\rangle + |10\rangle)(\langle 01| + \langle 10|)\Big)$$

(still entangled)

# Summary

- **it is a compelling idea (and not original to us) that quantum entanglement should be related to topological entanglement**
  - this is realized directly in 3d CS
- **multi-boundary link states in $TQFT_3$ gives a useful multi-partite system that can be studied using entanglement notions**
  - entanglement entropy is a framing-independent link invariant
  - other entanglement measures can be used to study properties of a given state, such as the distinguishability of link components
  - in the case of 3-component links, both GHZ- and W-type are found
- **continuing to explore notions of multi-partite entanglement**

# Summary$_2$

- **perhaps ideas of quantum information theory can be used to useful effect in knot theory**
  - and vice versa
  - e.g., there are infinite classes of non-trivial links that are not distinguished by their Jones polynomial
    - perhaps entanglement invariants can be used here
  - there are various conjectures that might be informed by entanglement inequalities

# Summary3

- **can this be useful for 3d gravity?**
  - would be interesting to extend the analysis to $SL(2, \mathbb{C})$
  - one quickly gets embroiled in problems due to non-compactness
  - nevertheless, one might hope to find a geometric interpretation for multi-boundary entanglement in general
    - in this context, many links are *hyperbolic* — they admit a geodesically complete hyperbolic metric on the link complement
    - in this context, there is the *volume conjecture,* concerning the behaviour of the Jones polynomial at large k [Kashaev '97, Gukov '04]
  - it seems plausible that quantum information theory might lead to insight into such subjects, and that motivated by Bekenstein-Hawking and Ryu-Takayanagi, the entropies might be related to volumes of some minimal surface/horizon