Information measures in QFT

Marina Huerta

Fifth Mandelstam Theoretical Physics School and Workshop

Recent developments in Entanglement, Large N in QFT and String theory

Johannesburg, January 2023

H. Casini and MH: “Lectures on entanglement in quantum field theory, proceedings TASI 2021, e-Print: 2201.13310 [hep-th]

Entropy

Measure of
- disorder, randomness
- uncertainty
- information
- surprise
- entanglement

Thermodynamical Entropy: "a thermodynamic quantity representing the unavailability of a system's thermal energy for conversion into mechanical work, often interpreted as the degree of disorder or randomness in the system."

Clausius wrote that he "intentionally formed the word Entropy as similar as possible to the word Energy", basing the term on the Greek: en- 'inside' + Greek τροπή *tropē* 'transformation'. 1850

Clausius, Rudolf (1865). *Ueber verschiedene für die Anwendung bequeme Formen der Hauptgleichungen der mechanischen Wärmetheorie: vorgetragen in der naturforsch. Gesellschaft den 24. April 1865*. p. 46.

I thought of calling it "information", but the word was overly used, so I decided to call it "uncertainty". [...] Von Neumann told me, "You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage."

Conversation between Claude Shannon and John von Neumann regarding what name to give to the attenuation in phone-line signals

M. Tribus, E.C. McIrvine, Energy and information, Scientific American, 224 (September 1971), pp. 178–184

Density matrix :

What is a state ?

Something that takes an operator and gives a number!

belongs to $\mathcal{A}$ ← $\rho : \mathcal{A} \to \mathbb{C}$ ↳ algebra of op.

Definition and some properties

QUANTUM state can be characterized by a density matrix. this is a more general description, usefull principally in the case of mixed states which canno be written as vectors

$$\rho = \sum_\lambda P_\lambda |\Psi_\lambda\rangle\langle\Psi_\lambda|$$

↓ describes a statistical mixing of vector $|\Psi_\lambda\rangle$ with probability P_λ and $\sum P_\lambda = 1$

- $\rho = \rho^+$ hermitian
- spectral representation $\rho = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$ where $|\varphi_i\rangle$ are ON vectors, $\lambda_i > 0$, $\sum \lambda_i = 1$
- positive definite $\langle\Psi|\rho|\Psi\rangle \geq 0$
- $tr(\rho) = 1$
- $\langle\mathcal{O}\rangle = tr(\rho\,\mathcal{O})$ expectation values

  if $\mathcal{O} = \mathbb{I} \Rightarrow tr(\rho) = 1$
- if the P_λ are all 0, except one $\Rightarrow$

  ρ represents a pure state.

$\rho = |\alpha\rangle\langle\alpha|$ which is a projector over $|\alpha\rangle$

• In general for any hermitian, positive definite and tr 1 operator

$$\mathrm{tr}\, \rho^2 \leq 1$$

Obviously for pure states $\mathrm{tr}\, \rho^2 = 1$ ✓

REDUCED DENSITY MATRIX

(Introduced by Dirac in 1930)

two syst $A, B \to \mathcal{H}_A, \mathcal{H}_B$

operators living in A and B commute to each other

"no causal connection between laboratories"

$\to$ the global state $|\psi\rangle \in \underbrace{\mathcal{H}_A \otimes \mathcal{H}_B}$

$\mathcal{H}$ is a bipartite Tensor product

We introduce a base $|i, j\rangle = |i\rangle \otimes |j\rangle$ of the complete space $\mathcal{H}$

$$\Rightarrow \quad |\psi\rangle = \sum \lambda_{k\ell} |k, \ell\rangle$$

with $\sum |\lambda_{k\ell}|^2 = 1$

The density matrix is $\rho = |\psi\rangle\langle\psi|$

$$= \lambda_{ij} \lambda^*_{k\ell} |ij\rangle\langle k\ell|$$

We define $\boxed{\rho_A = \mathrm{tr}_B\, \rho} \Rightarrow \rho_A = \sum_\ell \lambda_{k\ell} \lambda^*_{i\ell} |k\rangle\langle i|$

Any operator $\mathcal{O} : H_A \to H_A$

$$\langle \psi \rangle = \mathrm{tr}\,(\rho_A \mathcal{O})$$

The VON NEUMANN ENTROPY (1927)

$$S = -\mathrm{tr}\rho \log \rho$$

$\longrightarrow$ QUANTUM generalization OF the statistical ENTROPY $S = k_B \ln(\Omega)$

NUMBER OF microstates FOR A GIVEN MACROSTATE

Once we have a state in an algebra represented by an operator ρ we can compute any functional $\to$ vN entropy

$S(\rho) = 0 \Rightarrow \rho$ is pure

$S(\rho)$ is maximal $S(\rho) = \ln N$ for a maximally mixed state, N being the dim. of the Hilbert space

$S(\rho)$ is concave

$S(\rho)$ is additive for independent systems

$$S(\rho_A \otimes \rho_B) = S(\rho_A \otimes \rho_B)$$

but in general $S(\rho)$ is strongley subadditive for any three systems

$$S(\rho_{ABC}) + S(\rho_B \leq S(\rho_{AB}) + S(\rho_{BC})$$

Lieb - Ruskai 1973.

ENTANGLEMENT ENTROPY: $S_A = -\mathrm{tr}\rho_A \log \rho_A$

- Any ρ can be thought or expressed as a reduced state of a pure density operator. "purification" in a bigger space

- two syst A, B and ρ_{AB} pure $\Rightarrow$ ρ_A and ρ_B have equal eigenvalues. $\Rightarrow$ $S_A = S_B$

- Simple example

system: two particles with spin 1/2 (EPR pairs)

state with spin 0 $\rightarrow$ $|\chi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$

$|\uparrow\downarrow\rangle \rightarrow |\uparrow\rangle \otimes |\downarrow\rangle$

- ENTANGLEMENT

A state in $\mathcal{H}_{AB}$ bipartite system is <u>NOT</u> entangled if

$$\rho_{AB} = \sum p_i \, \rho_A^i \otimes \rho_B^i$$

- if the global state is not entangled and pure $\Rightarrow$ $\rho_{AB} = \rho_A \otimes \rho_B$.

- if the global state is pure but entangled $\Rightarrow$ $S_A = S_B$ is A GOOD MEASURE OF ENTANGLEMENT

• ¿ connected to thermodynac entropy?

canonical ensemble $\quad P_{E_i} = \dfrac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = \dfrac{e^{-\beta E_i}}{Z}$

$\underbrace{S = k_B \ln \Omega}_{\text{Boltzmann formula}}$ or more generally $\underbrace{S = -k_B \sum p_i \ln p_i}_{\text{Gibbs formula}}$

Boltzmann formula (only when microstates are equally probable)

Gibbs formula or general Boltzmann formula.

$$\Rightarrow \quad \rho = \frac{e^{-\beta H}}{Z} \quad \rightarrow \quad S = -\mathrm{tr}\, \rho \log \rho = \beta \langle H \rangle + \log Z$$

From thermodynamics $\quad F = E - TS$

$$-T \log Z = E - TS \quad \checkmark$$

* In general $\rho = \mathbb{1} e^{-K}$; K is called modular Hamiltonian

$\downarrow$

we can think $S(\rho)$ as the canonical entropy of an equilibrium state at temperature $T = 1$ respect to a Hamiltonian $K \neq H$ (the physical Hamiltonian)

¿ Can be measured?

Measuring S implies to measure ρ: this means the complete knoledge of the state and the space of states $\mathcal{H}$ where it acts.

For example: $|\psi_{AB}\rangle$ pure

A	B
↑	↑

ρ_A gives the same expectation values as ρ

$$\mathrm{tr}\, \rho_A \mathcal{O}_A = \langle \psi_{AB} | \mathcal{O}_A | \psi_{AB} \rangle \Rightarrow$$

it cannot be distinguish from exp. values!

Neverthelen there are less operators in $\mathcal{H}_A$ than in $\mathcal{H}_{AB}$ and $S_A \geq 0$ while $S_{AB} = 0$!!

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) \rightarrow \rho_A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Now suppose a system of N spins $\Rightarrow \dim \mathcal{H} = 2^N$

$\Rightarrow \rho$ is $2^N \times 2^N$ matrix !

as N grows, ρ has an "ugly" growing!!

$\Rightarrow$ ¿how to calculate?

Gaussian systems.

for example the vacuum state of a free theory (scalars or fermions)

$$[\phi_i, \pi_j] = i\delta_{ij} \qquad [\phi_i \, \phi_j] = [\pi_i, \pi_j] = 0$$

$$\langle \phi_i \phi_j \rangle = X_{ij} \qquad \langle \pi_i \pi_j \rangle = P_{ij}$$

$$\langle \phi_i \pi_j \rangle = \langle \pi_j \phi_i \rangle^* = \frac{i}{2} \delta_{ij}$$

* For a quadratic Hamiltonian, the fundamental state is gaussian, this means

Gaussian state $\rightarrow$ that the two point functions contains all the information.

$$\langle \phi_1, \cdots, \phi_4 \rangle = \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle + \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \cdots$$

$\Rightarrow$ Introducing the anzats

$$\rho = \underbrace{e^{-[\phi_i N_{ij} \phi_j + \pi_i M_{ij} \pi_j]}}_{\text{quadratic exponential.}}$$

(which it reproduces Wick)

New variables

$$\phi_i = \alpha^*_{ij} a^\dagger_j + \alpha_{ij} a_j \quad , \quad \pi_i = -i\beta^*_{ij} a^\dagger_j + i\beta_{ij} a_j$$

with $[a_i \, a^\dagger_j] = \delta_{ij} \Rightarrow \alpha^* \beta^T + \alpha \beta^\dagger = -1$

$$\rho = \prod_{e} e^{-\varepsilon_e a^\dagger_e a_e} \underbrace{(1 - e^{-\varepsilon_e})}_{\text{normalization.}}$$

From the two point functions we obtain $\alpha, \beta, \varepsilon$

$$i/2 \, \delta_{ij} = \mathrm{tr}(\rho \phi_i \pi_j) \Rightarrow 1/2 = \alpha^* n \beta^T - \alpha (n+1) \beta^T$$

$$X_{ij} = \mathrm{tr}(\rho \phi_i \phi_j) = \alpha^* n \alpha^T + \alpha (n+1) \alpha^\dagger$$

$$P_{ij} = \mathrm{tr}(\rho \pi_i \pi_j) = \beta^* n \beta^T + \beta (n+1) \beta^T$$

In particular

$$\tfrac{1}{2}\coth(\varepsilon_k/2) = \nu_k = \text{eigenvalue of } \underset{\substack{| \\ \text{region } V}}{C_V} = \sqrt{X_V P_V}$$

$\Rightarrow$ the entropy is given by the entropy contributions of the decoupled armonic oscilators with $\beta\omega \equiv \varepsilon_k$

$$S = \sum_l \left(-\log\left(1-e^{-\varepsilon_l}\right) + \frac{\varepsilon_l\, e^{-\varepsilon_l}}{1-e^{-\varepsilon_l}}\right)$$

$$\rightarrow \quad \boxed{= \operatorname{tr}\left[(C+1/2)\log(C+1/2) - (C-1/2)\log(C-1/2)\right]}$$

- $C \geq 1/2$ always
- $C = 1/2$ when ρ_V is pure
- C is an $N\times N$ matrix!

$\Downarrow$

for a gaussian state

two point correlators
$\Updownarrow$
Entropy

this can be extended to different theories.

For fermions: $\langle \psi_i \psi_j^+ \rangle = C_{ij}$ $\quad \langle \psi_i^+ \psi_j \rangle = \delta_{ij} - C_{ji}$

$$H = -\log\left(C^{-1}-1\right)$$

$$\rightarrow \quad S(V) = \sum_l \cdots\cdots =$$

$$-\operatorname{tr}\left((1-C)\log(1-C) + C\log C\right)$$

Examples in QFT:

Free boson

- $H = \frac{1}{2} \sum \pi_i^2 + \sum_{ij} \phi_i K_{ij} \phi_j$

discrete version
lattice version

- fundamental state.

- $X_{ij} = \langle \phi_i \phi_j \rangle = \frac{1}{2} (K^{-1/2})_{ij}$
- $P_{ij} = \langle \pi_i \pi_j \rangle = \frac{1}{2} (K^{1/2})_{ij}$

In ~~two~~ spatial dimensions

$$H = \frac{1}{2} \sum \epsilon^2 \left(\pi_{m,n}^2 + \frac{(\phi_{n+1,m} - \phi_{n,m})^2}{\epsilon^2} + \frac{(\phi_{n,m+1} - \phi_{n,m})^2}{\epsilon^2} + M^2 \phi_{nm}^2 \right)$$

$$\langle \phi_{00}, \phi_{ij} \rangle = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} dp_x \int_{\pi}^{\pi} dp_y \frac{\cos(i p_x) \cos(j p_y)}{\sqrt{2(1-\cos p_x) + 2(1-\cos p_y) + M^2}}$$

Details of the integrals.

10-15 digits. (300 × 300)

References to our work

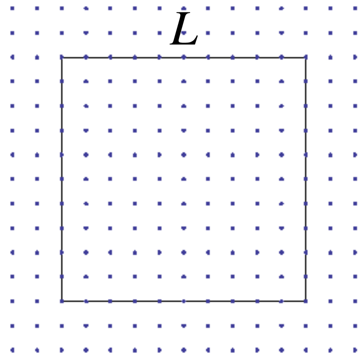
For circles → radial discretization
Srednicki 1992

Some results:

Some results:

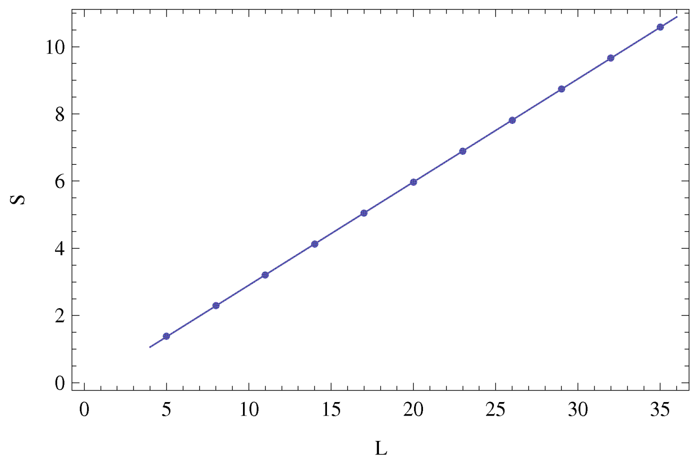
Massless scalar field model. Vacuum (fundamental) state in a square lattice

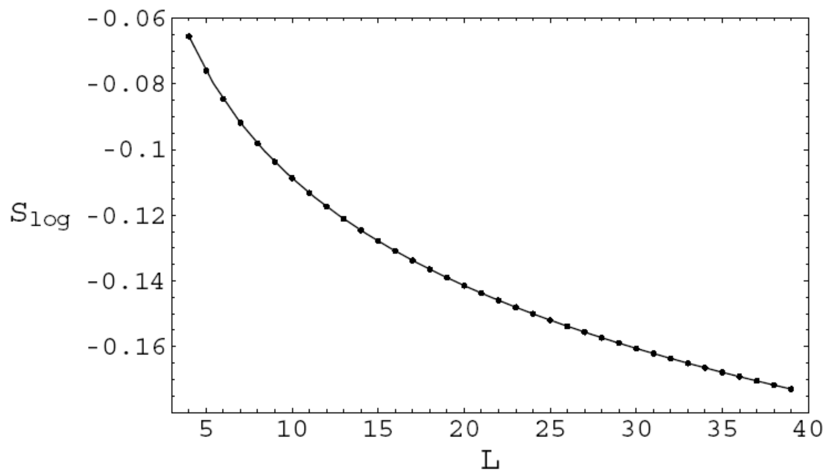
Square:



$$H = \frac{1}{2}\int d^2x \left(\dot{\phi}(x)^2 + (\nabla\phi(x))^2\right)$$

$$\rightarrow H = \frac{1}{2}\sum_i \epsilon^2 \left(\dot{\phi_i}^2 + \sum_{j\sim i} \frac{(\phi_i - \phi_j)^2}{\epsilon^2}\right)$$



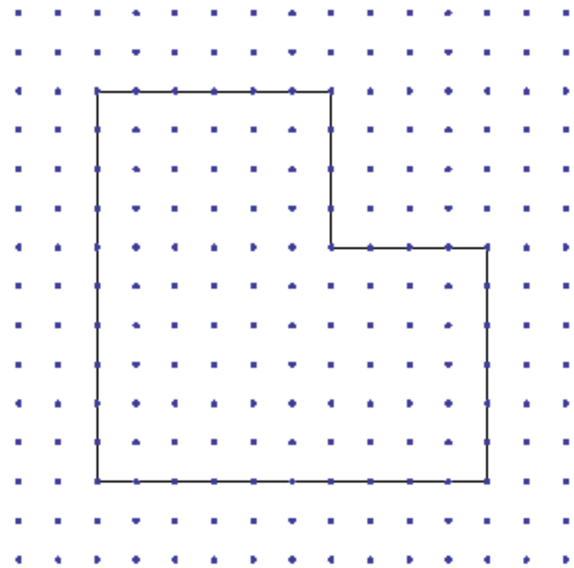


S = .075 (4 L/ε) - 0.047 Log[L/ε] + const=.075 (perimeter/ε) - 0.047 Log[L/ ε] + const

We have an «area» term and a logarithmic correction. These are divergent as ε ⟶ 0

The same perimeter but more vertices:

S = .075 (perimeter/ε) - (6/4) 0.047 Log[L/ε] + const →



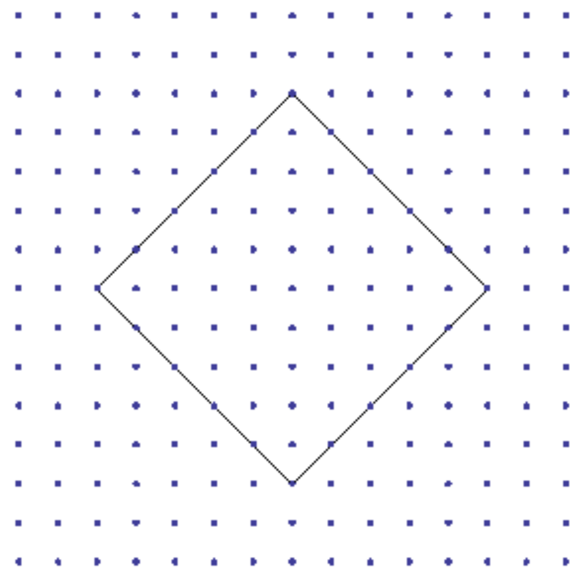
The same «area» term. A logarithmic coefficient growing with the number of vertices.
(All vertices have the same angle S(A)=S(-A) for a global pure state)

In general:

$$S(A) = c_1\,(\text{perimeter}/\epsilon) - \sum_{\text{vertices}} c_{\log}(\theta)\log(R/\epsilon) + \text{const}$$

Again the square:

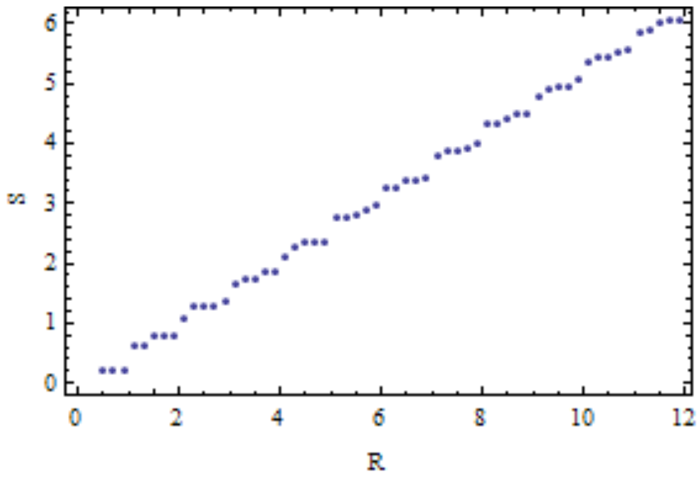
S = .085 (perimeter/ε) - 0.047 Log[L/ε] + const →



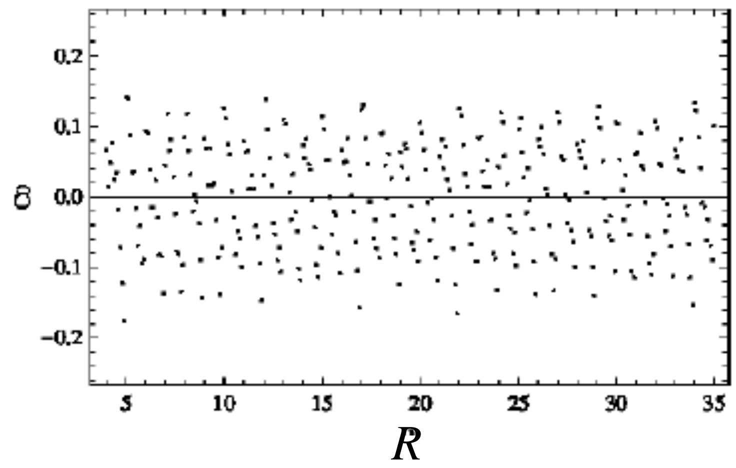
Bad: area term does not have the rotational symmetry of the theory in the continuum limit

Good: the logarithmic term does not notice the lattice

And circles?



$\longrightarrow$



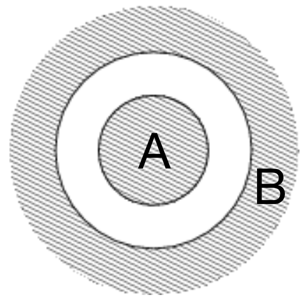
Leading area term and no log, as expected $\boxed{S(R) = c_1\, R + c_0}$

Mutual information

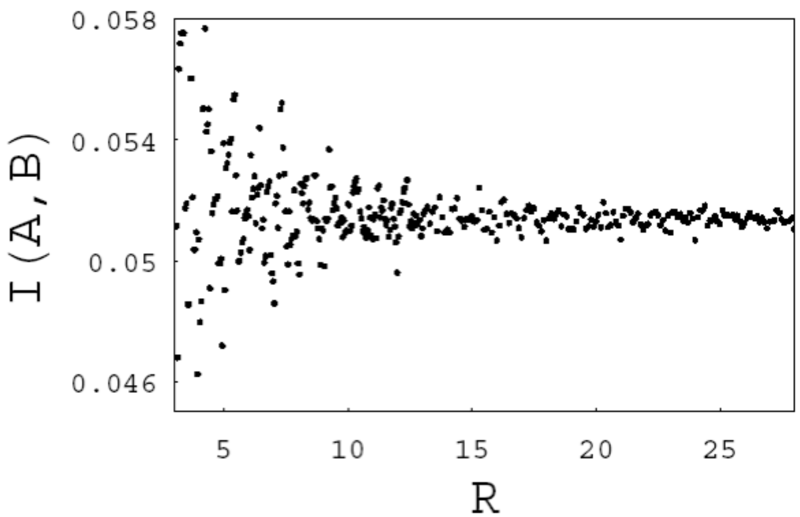
$$\boxed{I(A,B) = S(A) + S(B) - S(A \cup B)}$$

The boundary divergences cancel out in the combination.

$$S(A) = \frac{1}{2}\, I(A, -A_\epsilon)$$



$R_A + \epsilon = R_B$



$\boxed{I(A,B)\ \neq\ 0} \longrightarrow$ S cannot be finite

I(A,B) diverges when A and B touch each other