

Operator Growth and Complexity in Krylov space

(What We Talk About When We Talk About “Complexity”)

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Outline

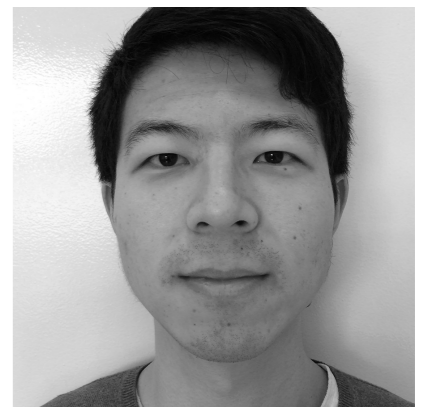
- Introduction and Motivation
- Krylov basis and “complexity” of operators and states
- Examples/Applications
- Conclusions/Questions

Based on:

“Quantum chaos and the complexity of spread of states” with V. Balasubramanian, J.M. Magan, Q. Wu, Phys. Rev. D. 106 (2022) 4, 046007

“Geometry of Krylov Complexity” with J.M. Magan, D. Patramanis Phys. Rev. Res. **4**, 013041

works with D. Patramanis (UW) and Sinong Liu (UW) and Dongsheng Ge (UW -> Osaka)



Also work with Shajid Haque, Jeff Murugan, Nitin Gupta and Jaco Van Zyl (local experts) !

General Problem

Unitary evolution of operators and states (QM or QFT):

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$$

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle$$

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}(0) e^{-iHt}$$

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

Generically, a “simple” operator $\mathcal{O}(0)$ “grows” and becomes “complex” (in operator space)

Similarly, a “simple” reference quantum state $|\Psi(0)\rangle$ “spreads” and becomes “complex” (in Hilbert space)

How to quantify this “Complexity”?

Motivation/Intuition:

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}(0) e^{-iHt} = \mathcal{O}(0) + it[H, \mathcal{O}(0)] + \frac{(it)^2}{2} [H, [H, \mathcal{O}(0)]] + \dots$$

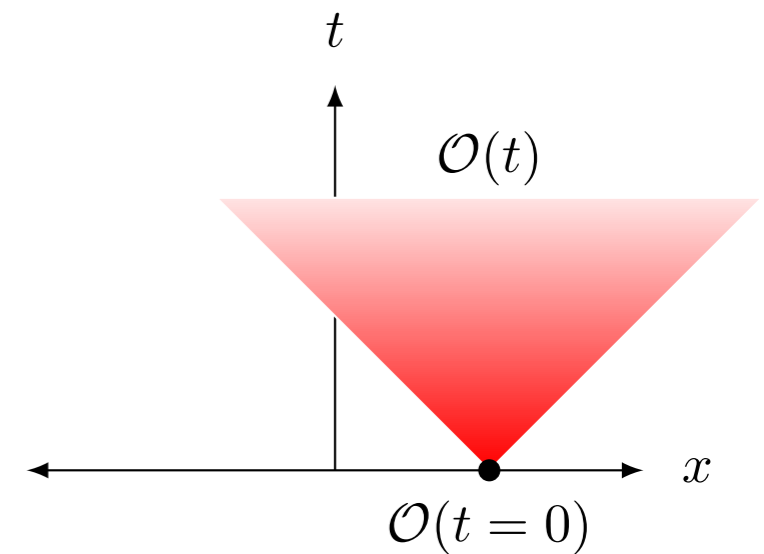
E.g.

$$H = \sum_i (Z_i \cdot Z_{i+1} + B_x X_i + B_z Z_i) \quad \mathcal{O}(0) = X_1$$

$$\mathcal{O}(t) = X_1 - 2t(Y_1 \cdot Z_2 + B_z Y_1)$$

$$-2t^2(B_x Y_1 \cdot Y_2 - B_x B_z Z_1 - B_x Z_1 \cdot Z_2 + 2B_z X_1 \cdot Z_2 + B_z^2 X_1 + X_1 \cdot Z_2^2)$$

$$+t^3(\dots\dots\dots)$$



Common lore: the more “chaotic” H, the faster the operator grows.

How to quantify this: A universal definition of the operator size/complexity?

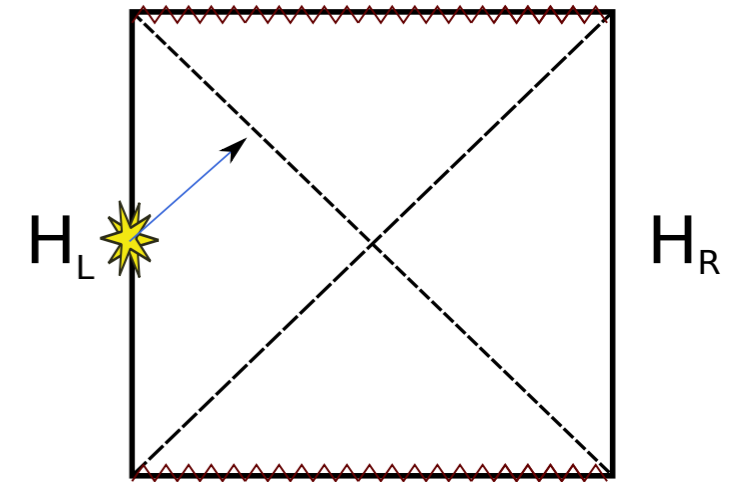
Motivation: Operator Growth and Holography (BH & QChaos)

1. Butterfly effect

$$|\psi'\rangle = e^{-iH_L t_w} O_L(x) e^{iH_L t_w} |\psi\rangle$$

2. Growth of “Precursors”: $W(t) = U^\dagger(t) W U(t)$

3. Goals: Universal/working definition of the “Operator Size”? Operator Complexity? Quantum Chaos?



[Shenker, Stanford'13]

[Roberts, Stanford'14]

[+ Susskind]

Partial answers” from **O**ut-of-**T**ime **O**rdered **C**orrelators:

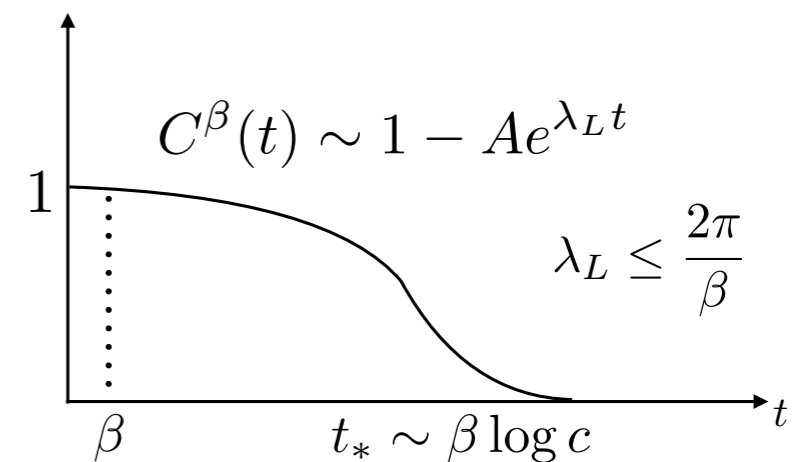
$$C^\beta(t) = \frac{\langle W(t) V W(t) V \rangle_\beta}{\langle W(t) W(t) \rangle_\beta \langle V V \rangle_\beta}$$

“Maximal chaos” (OTOC) for Einstein BH dual in the bulk.

Related:

Many-body QChaos?, Thermalisation (or lack thereof)

“Central Dogma”: Black Hole = Strongly Interacting Qubits



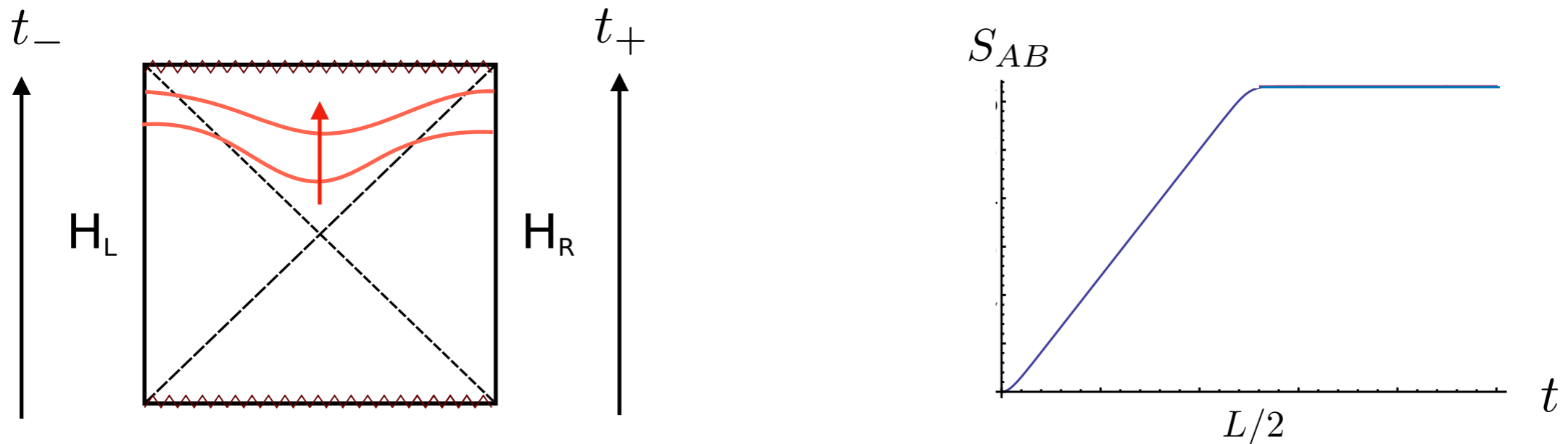
[Maldacena, Shenker, Stanford'15]

Motivation: Complexity in Holography?

[Hartman&Maldacena '13] (2d CFT)

Time-evolved Thermofield-double state

$$|\Psi_\beta(t)\rangle = e^{-i(H_L+H_R)t} \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n, n\rangle$$



BH (ERB) continues to grow with t but entanglement entropy saturates (“not enough”)

What is the “CFT dual” of this (ERB) growth? “Complexity” of the TFD state? [Suskind, '14]

Is there some universal (useful) notion of complexity (number)? Unexplored in QFT...

Attempts and Hopes for “Complexity”

States (Formation, Evolution):

Geometric Approaches (“Nielsen”)

AdS/TN (Path Integral Complexity)

“Distance measures” (Inf. metric)

....

Operators (Growth, Chaos)

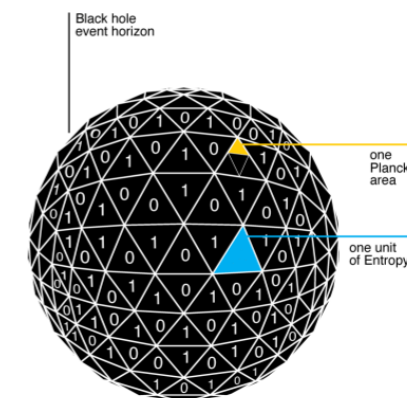
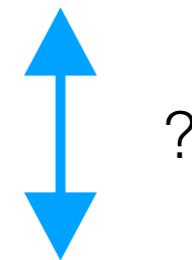
Operator Size/Complexity?

“Operator Size” in SYK

OTOC

....

Growth of the ERB,
late time physics of BH,
singularity?

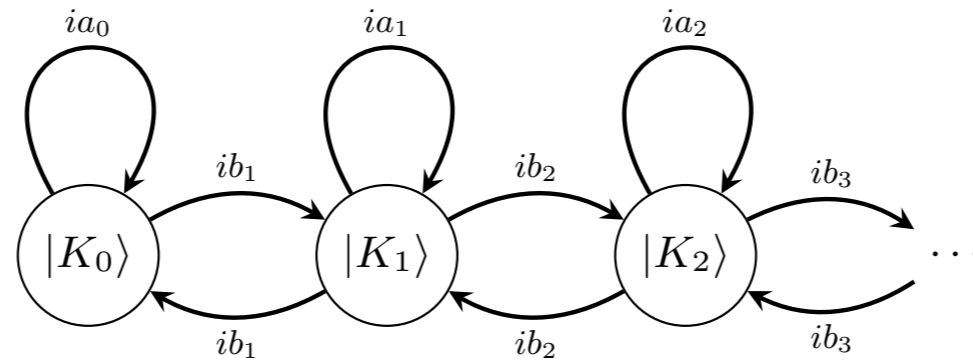


Near (behind?)
horizon of BH....

Goal: Better understanding of $\mathcal{H}_{bulk} \simeq \mathcal{H}_{bdr}$

Universal framework for “Complexity”?

[Balasubramanian, PC, Magan, Wu '22]



Today: discuss a notion(s) of “complexity” based on Krylov basis

that can be universally defined (and computed) in systems from QM to QFT

and show some recent results for both, operators and states.

References (Operator Growth):

[Qi, Streicher '18] [Roberts, Stanford, Streicher '18]

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19] [Barbon, Rabinovici, Shir, Sinha '19] [Dymarsky, Gorsky '19]

[Rabinovici, Sanchez-Garrido, Shir, Sonner '20] [Magan, Simon '20] [Jian, Swingle, Xian '21]

[Kar, Lamprou, Rozali, Sully '21] [Dymarsky, Smolkin '21] [Rabinovici, Sanchez-Garrido, Shir, Sonner '21'22]...

Basic Idea

Given

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}(0) e^{-iHt} \equiv e^{i\mathcal{L}t} \mathcal{O}(0)$$

$\mathcal{L} = [H, \cdot]$

More generally we can think about quantum circuits (circuit H and circuit t)

We can expand them in a certain basis (Krylov basis):

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_n \phi_n(t) |K_n\rangle$$

$$|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t} |\mathcal{O}_0\rangle = \sum_n \phi_n(t) |\mathcal{O}_n\rangle$$

Unitarity: Probability distribution

$$p_n(t) = |\phi_n(t)|^2$$

$$\sum_n |\phi_n(t)|^2 = 1$$

Use this probability to characterise the evolution/growth and “complexity”

Aleksey Nikolaevich Krylov (1863-1945)

Russian naval engineer and applied mathematician.

His mother Sofya Lyapunova came from the famous “Lyapunov” family and Alekandr Lyapunov was his cousin.

He became famous for pioneering “Theory of oscillating motions of the ship”.

In 1904 he built the first machine in Russia for integrating ODEs.

In 1931 he wrote a paper on Krylov subspace: A $n \times n$ matrix and b n -vec.

$$\mathcal{K}_r(A, b) = \text{span} \{b, Ab, A^2b, \dots, A^{r-1}b\}$$

He was interested in efficient diagonalization of matrices and computation of characteristic polynomial coefficients.

“... he was concerned with efficient computations and counted computational work as the number of separate numerical multiplications ”

Here with his daughter Anna (later married Pyotr Kapitsa)



Krylov Basis

[Recursion Method: Viswanath, Muller '63]

Unitary evolution/Q-circuit

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\Psi_n\rangle$$

Goal: Given states

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, \dots, H^n|\Psi_0\rangle, \dots\}$$

construct an orthonormal basis $|K_n\rangle$ recursively (Lanczos algorithm, G-S):

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \quad |K_n\rangle = b_n^{-1}|A_n\rangle$$

with “Lanczos coefficients”:

$$a_n = \langle K_n | H | K_n \rangle, \quad b_n = \langle A_n | A_n \rangle^{1/2}$$

Such that $b_0 = 0$ and $|K_0\rangle = |\Psi_0\rangle$

Krylov Basis details

[Recursion Method: Viswanath, Muller '63]

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \quad |K_n\rangle = b_n^{-1}|A_n\rangle$$

Start from normalized $|K_0\rangle = |\psi_0\rangle$

n=0

$$|A_1\rangle = (H - a_0)|K_0\rangle \quad |K_1\rangle = b_1^{-1}(H - a_0)|K_0\rangle$$

$$0 = \langle K_0 | K_1 \rangle = b_1^{-1} (\langle K_0 | H | K_0 \rangle - a_0)$$

$$a_0 = \langle K_0 | H | K_0 \rangle$$

$$1 = \langle K_1 | K_1 \rangle = b_1^{-2} \langle A_1 | A_1 \rangle$$

$$b_1 = \langle A_1 | A_1 \rangle^{1/2}$$

n=1

$$|A_2\rangle = (H - a_1)|K_1\rangle - b_1|K_0\rangle \quad |K_2\rangle = b_2^{-1}|A_2\rangle$$

$$0 = \langle K_1 | K_2 \rangle = b_2^{-1} [(\langle K_1 | H | K_1 \rangle - a_1) - b_1 \langle K_1 | K_0 \rangle]$$

$$a_1 = \langle K_1 | H | K_1 \rangle$$

$$\begin{aligned} 0 &= \langle K_0 | K_2 \rangle = \langle K_0 | H | K_1 \rangle - b_1 = \langle K_0 | (H - a_0) | K_1 \rangle - b_1 \\ &= b_1 \langle K_1 | K_1 \rangle - b_1 = 0 \end{aligned}$$

$$1 = \langle K_2 | K_2 \rangle = b_2^{-2} \langle A_2 | A_2 \rangle$$

$$b_2 = \langle A_2 | A_2 \rangle^{1/2}$$

Krylov Basis

[Recursion Method: Viswanath, Muller '63]

In the Krylov basis, the Hamiltonian becomes tri-diagonal

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle \quad \langle K_m | H | K_n \rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Expanding our state in the Krylov basis

“Hessenberg form”

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_n \phi_n(t) |K_n\rangle \quad \sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

By construction, we have a Schrödinger equation for the coefficients (amplitudes)

$$i\partial_t |\Psi(t)\rangle = \sum_n i\partial_t \phi_n(t) |K_n\rangle$$

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle = \sum_n \phi_n(t) H |K_n\rangle = \sum_n [a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1} \phi_{n+1}(t)] |K_n\rangle$$

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1} \phi_{n+1}(t) \quad \phi_n(0) = \delta_{n,0}$$

Lanczos coef. from return amplitude

[Recursion Method: Viswanath, Muller '63]

[Balasubramanian, PC, Magan, Wu '22]

Lanczos coeff. are encoded in the "return amplitude" (auto-correlator, Loschmidt amp.)

$$S(t) \equiv \langle \Psi(t) | \Psi(0) \rangle = \langle \Psi_0 | e^{iHt} | \Psi_0 \rangle = \phi_0^*(t)$$

Moments

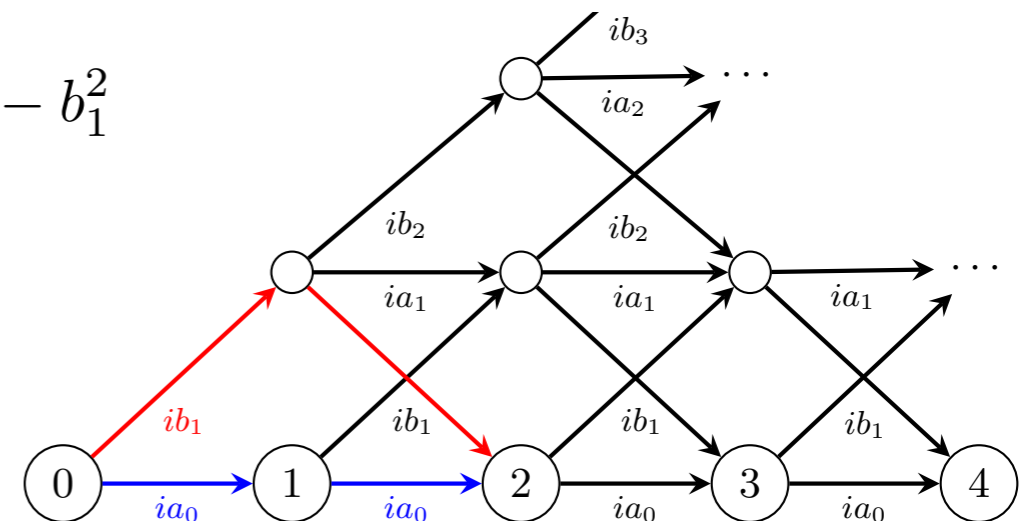
$$\mu_n = \left. \frac{d^n}{dt^n} S(t) \right|_{t=0} = \langle \psi(0) | \left. \frac{d^n}{dt^n} e^{iHt} | \psi(0) \right|_{t=0} = \langle K_0 | (iH)^n | K_0 \rangle$$

Knowing moments allows to find Lanczos coefficients (algorithm)

e.g. $\langle K_0 | (iH) | K_0 \rangle = ia_0$ $\langle K_0 | (iH)^2 | K_0 \rangle = -a_0^2 - b_1^2$

(Comment) Inverse relations:

$$a_0 = -i\mu_1, \quad b_1^2 = \mu_1^2 - \mu_2$$



Interesting for modular H: entanglement, capacity of ent.,...

Physics of Lanczos coeff?
[Balasubramanian, Magan, Wu '22]

Operator Growth in the Krylov Basis

[Recursion Method: Viswanath, Muller '63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]

Heisenberg evolution

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}(0) e^{-iHt}$$

Formally, we can write the operator as

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n \quad \tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [H, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [H, [H, \mathcal{O}]], \dots$$

Liouvillian (super)operator

$$\mathcal{L} = [H, \cdot], \quad \mathcal{O}(t) \equiv e^{i\mathcal{L}t} \mathcal{O}, \quad \tilde{\mathcal{O}}_n \equiv \mathcal{L}^n \mathcal{O}.$$

Given $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$ we need a basis (GNS) $|\mathcal{O}\rangle$ $\mathcal{L}|\mathcal{O}\rangle = |[H, \mathcal{O}]$

We must pick an inner product (freedom):

$$(A|B)_\beta^g = \int_0^\beta g(\lambda) \langle e^{\lambda H} A^\dagger e^{-\lambda H} B \rangle_\beta d\lambda, \quad \langle A \rangle_\beta = \frac{1}{Z} \text{Tr} (e^{-\beta H} A), \quad Z = \text{Tr} (e^{-\beta H})$$

$$g(\lambda) \geq 0, \quad g(\beta - \lambda) = g(\lambda), \quad \frac{1}{\beta} \int_0^\beta d\lambda g(\lambda) = 1.$$

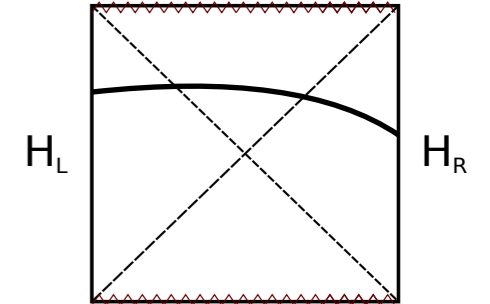
Operator Growth in the Krylov Basis

[Recursion Method: Viswanath, Muller '63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]

The most common: Wightman

$$(A|B) = \langle e^{H\beta/2} A^\dagger e^{-H\beta/2} B \rangle_\beta \quad g(\lambda) = \delta(\lambda - \beta/2)$$



Then we follow the Lanczos algorithm

$$|A_{n+1}\rangle = \mathcal{L} |O_n\rangle - b_n |O_{n-1}\rangle \quad |O_n\rangle = b_n^{-1} |A_n\rangle \quad b_n = (A_n|A_n)^{1/2}$$

$$|O_0\rangle = |O(0)\rangle = |O\rangle \quad b_0 = 0$$

Most of the inner products will involve Tr() so we don't need $a_n = 0$

$$|O(t)\rangle = e^{i\mathcal{L}t} |O\rangle \equiv \sum_n i^n \varphi_n(t) |O_n\rangle$$

Schrödinger equation:

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t) \quad \varphi_n(0) = \delta_{n,0}$$

Lanczos coefficients are encoded in the return amplitude

$$S(t) = (O_0|O(t)) = (O_0|e^{i\mathcal{L}t}|O_0) = \varphi_0(t) = \frac{1}{Z} \sum_{n,m} |\langle n|O|m\rangle|^2 e^{-\left(\frac{\beta}{2}-it\right)E_n} e^{-\left(\frac{\beta}{2}+it\right)E_m}$$

Krylov Basis Summary

States

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_n \phi_n(t) |K_n\rangle$$

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1} \phi_{n+1}(t)$$

$$S(t) \equiv \langle \Psi(t) | \Psi(0) \rangle = \langle \Psi_0 | e^{iHt} | \Psi_0 \rangle = \phi_0^*(t)$$

Operators

$$|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t} |\mathcal{O}\rangle \equiv \sum_n i^n \varphi_n(t) |\mathcal{O}_n\rangle$$

$$\sum_n |\varphi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t)$$

$$S(t) = (\mathcal{O}(0) | \mathcal{O}(t)) = (\mathcal{O}_0 | e^{i\mathcal{L}t} | \mathcal{O}_0) = \varphi_0(t)$$

Connections:

$$|\Psi(t)\rangle = \mathcal{O}(-t) |\Psi(0)\rangle$$

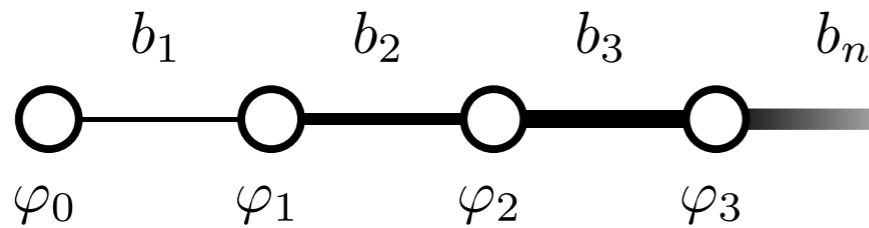
E.g. Wightman $|\psi(t)\rangle = \rho_\beta^{1/4} \mathcal{O}_L(t) \rho_\beta^{-1/4} |\psi_\beta\rangle$

Krylov/Spread Complexity

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]

[Balasubramanian, PC, Magan, Wu '22]

The physics of the growth/evolution \Leftrightarrow motion of a particle on a chain



$$\sum_n |\varphi_n(t)|^2 = 1$$

The further in the chain the particle is, the more “complex” state in the Krylov basis needs to be employed (to represent the state or the operator)

A natural (working) definition of “complexity” as an average position on the chain:

$$C_\Psi(t) = \sum_n n |\phi_n(t)|^2$$

$$K_O = \sum_n n |\varphi_n(t)|^2$$

Evolution can be also characterised with other QI/Probability tools:

K-entropy $S_K = - \sum_n p_n \log p_n$ K-variance, K-capacity, $C_K = e^{S_K} \dots$

[Barbon, Rabinovici, Shir, Sinha '19] [PC, Datta '21] [Patramanis '21].

Comment

[Balasubramanian, PC, Magan, Wu '22]

Starting from the state: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

Complexity = “Spread in Hilbert space”

Take a basis: $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, \dots\}$ and a “cost function” (a family, $c_n = n$)

$$C_{\mathcal{B}}(t) = \sum_n c_n |\langle \psi(t) | B_n \rangle|^2 \equiv \sum_n c_n p_{\mathcal{B}}(n, t)$$

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t)$$

minimum (finite t) for the
Krylov basis!

Intuition: For discrete time evolution, assume $N-1$ vectors equal to the Krylov basis.
Then in the next step:

$$|\psi_N\rangle = p_{\perp} |K_N\rangle + p_{\parallel} |\chi_{\parallel}\rangle$$

Ex.1 q-SYK and “Universal Operator Growth Hypothesis”

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]

[Roberts, Stanford, Streicher '18]

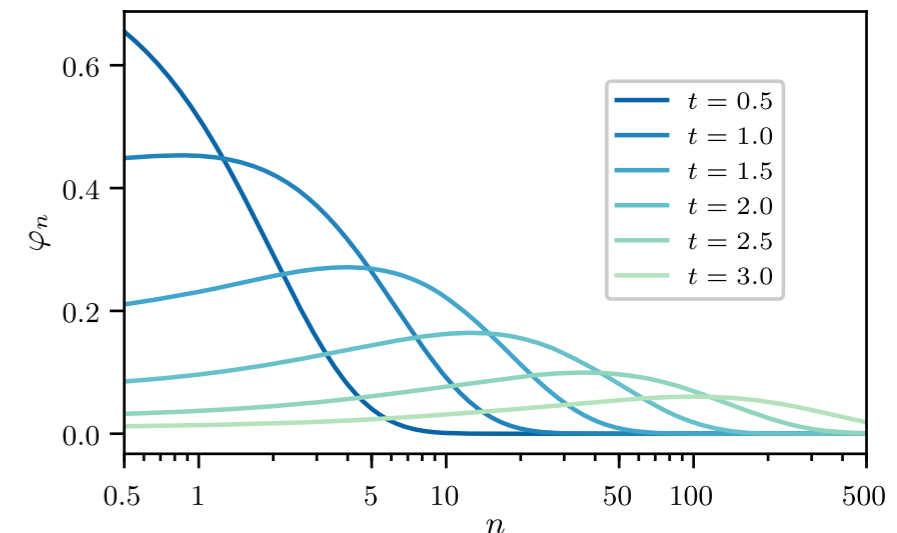
$$H_{\text{SYK}}^{(q)} = i^{q/2} \sum_{1 \leq i_1 < i_2 < \dots < i_q \leq N} J_{i_1 \dots i_q} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_q} \quad \mathcal{O} = \sqrt{2} \gamma_1$$

At large q this growth is represented by return amplitude (low T, $\eta \sim 2/q$)

$$S(t) = (\mathcal{O}(t) | \mathcal{O}(0)) \simeq \frac{1}{\cosh^\eta \left(\frac{\pi t}{\beta} \right)} \sim e^{-\frac{\eta \pi}{\beta} t}$$

Lanczos coefficients and amplitudes ($a_n=0$)

$$b_n = \frac{\pi}{\beta} \sqrt{n(\eta + n - 1)} \quad \varphi_n(t) = \sqrt{\frac{\Gamma(\eta + n)}{n! \Gamma(\eta)} \frac{\tanh^n \left(\frac{\pi t}{\beta} \right)}{\cosh^\eta \left(\frac{\pi t}{\beta} \right)}}$$



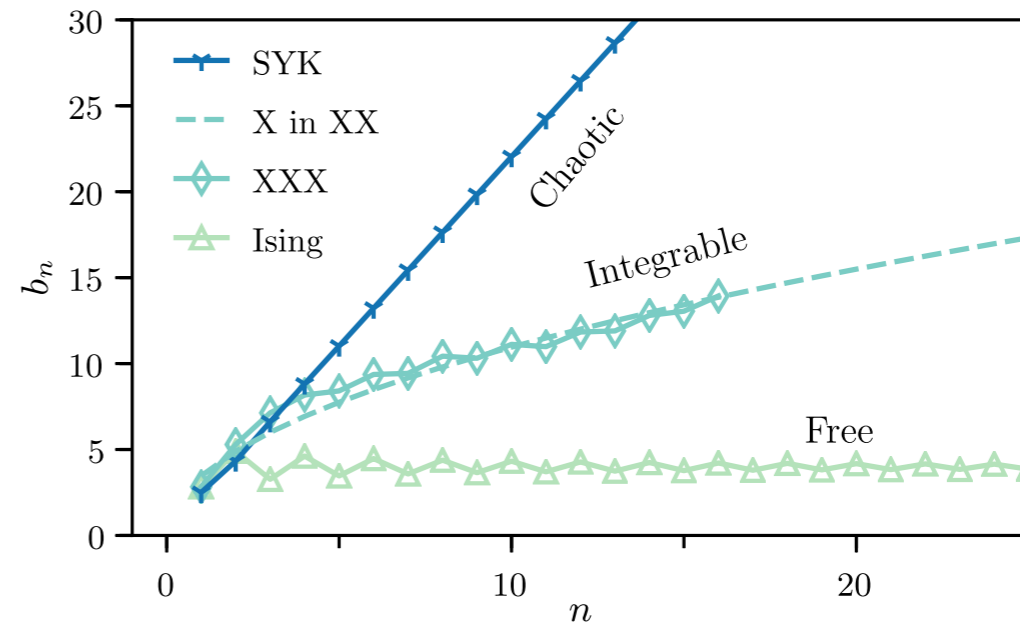
$$K_{\mathcal{O}} = \eta \sinh^2 \left(\frac{\pi t}{\beta} \right) \sim \frac{\eta}{4} e^{\frac{2\pi}{\beta} t}$$

Ex.1 q-SYK and “Universal Operator Growth Hypothesis”

[Parker et al. '19]

Hypothesis: “Maximal growth of Lanczos coefficients”

$$b_n \leq \alpha n + O(1)$$



Claim: Saturated for “maximally chaotic” systems (OTOC)

Saturation \Leftrightarrow exponential growth of Krylov Complexity

$$b_n = \alpha n + O(1) \quad \Rightarrow \quad K_O \sim e^{\lambda t} \quad \lambda = 2\alpha$$

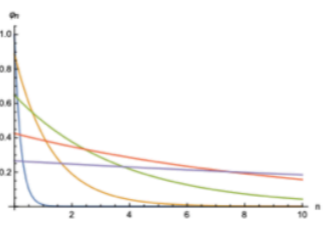
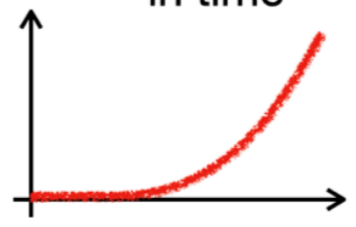
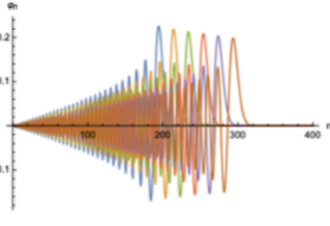
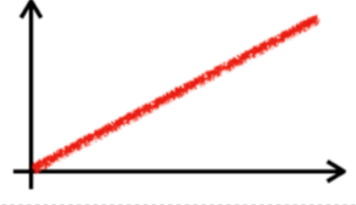
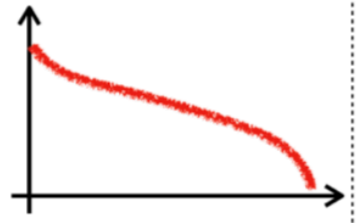
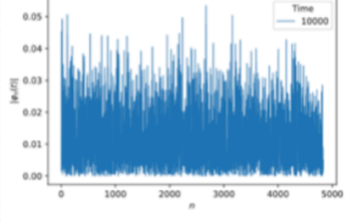
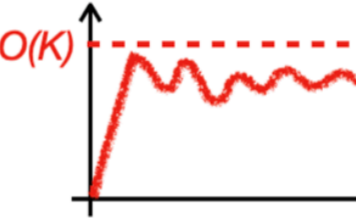
Ex.2 Extensive studies of operator growth

[Barbon, Rabinovici, Shir, Sinha '19]
 [Rabinovici, Sanchez-Garrido, Shir, Sonner '21'22]

Continuum limit: $x = \epsilon n$, $\varphi(x, t) = \varphi_n(t)$, $v(x) = 2\epsilon b_n = 2\epsilon b(\epsilon n)$

$$\partial_t \varphi(x, t) + v(x) \partial_x \varphi(x, t) + \frac{1}{2} v'(x) \varphi(x, t) = 0 \quad (\text{cont. eq for } p = |\varphi|^2)$$

Numerics (Operator growth in XXZ chain + Integrability breaking terms, RMT)

n	Lanczos coefficients	wavefunction	K-complexity	time scales
$1 \ll n < S$	Linear growth in n $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in n $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

$S = \# \text{dof}$

Fig from: [Rabinovici, Sanchez-Garrido, Shir, Sonner '22]

Ex.3 Symmetry approach: e.g. SL(2,R)

Take the SL(2,R) algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0$$

And the algebra basis

$$|h, n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^n |h\rangle$$

$$L_0 |h, n\rangle = (h+n) |h, n\rangle,$$

$$L_{-1} |h, n\rangle = \sqrt{(n+1)(2h+n)} |h, n+1\rangle,$$

$$L_1 |h, n\rangle = \sqrt{n(2h+n-1)} |h, n-1\rangle,$$

Consider a class of models/states where the (state/operator) evolution in the Krylov space can be represented by

$$H = \gamma L_0 + \alpha(L_{-1} + L_1) \quad |K_n\rangle = |h, n\rangle$$

Then we can immediately read-off the Lanczos coefficients

$$a_n = \gamma(n+h) \quad b_n = \alpha\sqrt{n(2h+n-1)}$$

Related: Toda system, Orthogonal polynomials

Ex.3 Symmetry approach: SL(2,R)

[PC, J.M.Magan, D.Patramanis '21]
[Balasubramanian, PC, Magan, Wu '22]

The state evolution can be represented as a generalised coherent state (“driven CFT”)

$$|\Psi(t)\rangle = e^{-iHt} |h\rangle = e^{AL_{-1}} e^{BL_0} e^{CL_1} |h\rangle = \sum_n \phi_n(t) |K_n\rangle$$

These “amplitudes” solve the Schrödinger equation with SL(2,R) Lanczos coeff.

We can also derive a general result for complexity for this symmetry setup

$$C_\Psi(t) = \sum_{n=0}^{\infty} np_n = \frac{2h}{1 - \frac{\gamma^2}{4\alpha^2}} \sinh^2 \left(\alpha t \sqrt{1 - \frac{\gamma^2}{4\alpha^2}} \right)$$

Exponential growth for: $\gamma < 2\alpha$

Oscillating: $\gamma > 2\alpha$

Quadratic for: $\gamma = 2\alpha$ ($a_n \sim 2\alpha n$, $b_n \sim \alpha n$)

Complexity and Variation of L_0 :

$$C_\Psi(t) = \langle \Psi(t) | L_0 | \Psi(t) \rangle - \langle \Psi(0) | L_0 | \Psi(0) \rangle$$

Application to SYK (~2d CFT)

[PC, J.M.Magan, D.Patramanis '21]

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta + n)}{n!\Gamma(\eta)}} \frac{\tanh^n\left(\frac{\pi t}{\beta}\right)}{\cosh^\eta\left(\frac{\pi t}{\beta}\right)} \quad b_n = \frac{\pi}{\beta} \sqrt{n(\eta + n - 1)}$$

Displacement operator and generalised coherent states

[Perelomov'72]

$$D(\xi) = e^{\xi L_{-1} - \bar{\xi} L_1} \quad |z, h\rangle \equiv D(\xi) |h\rangle \quad \xi = \frac{1}{2} \rho e^{i\phi} \quad z = \tanh\left(\frac{\rho}{2}\right) e^{i\phi}$$

$$|z, h\rangle = \sum_{n=0}^{\infty} e^{in\phi} \sqrt{\frac{\Gamma(2h + n)}{n!\Gamma(2h)}} \frac{\tanh^n(\rho/2)}{\cosh^{2h}(\rho/2)} |h, n\rangle$$

“Trajectory in Phase Space”: $\rho = 2\alpha t, \quad \phi = \pi/2 \quad \alpha = \pi/\beta$

$$|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t} |\mathcal{O}\rangle \quad \mathcal{L} = \alpha (L_{-1} + L_1) \quad |K_n\rangle = |h = \eta/2, n\rangle$$

Relation to geometric complexity?

[PC, J.M.Magan, D.Patramanis '21] [PC, Datta '21]
[Miyaji et al. '15]

With coherent states we can associate a natural “information metric”

$$ds_{FS}^2 = \langle dz|dz \rangle - \langle dz|z \rangle \langle z|dz \rangle \quad |z, h\rangle = \sum_{n=0}^{\infty} e^{in\phi} \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}} \frac{\tanh^n(\rho/2)}{\cosh^{2h}(\rho/2)} |h, n\rangle$$

E.g. for $SL(2, R)$ this becomes a hyperbolic disc metric

$$ds_{FS}^2 = \frac{2hdzd\bar{z}}{(1-z\bar{z})^2} = \frac{h}{2} (d\rho^2 + \sinh^2(\rho)d\phi^2) \quad R = -\frac{4}{h}$$

Operator growth is a geodesic in this manifold (phase space): $\rho = 2\alpha t, \quad \phi = \pi/2$

Observe a universal relation between the Volume and Krylov complexity

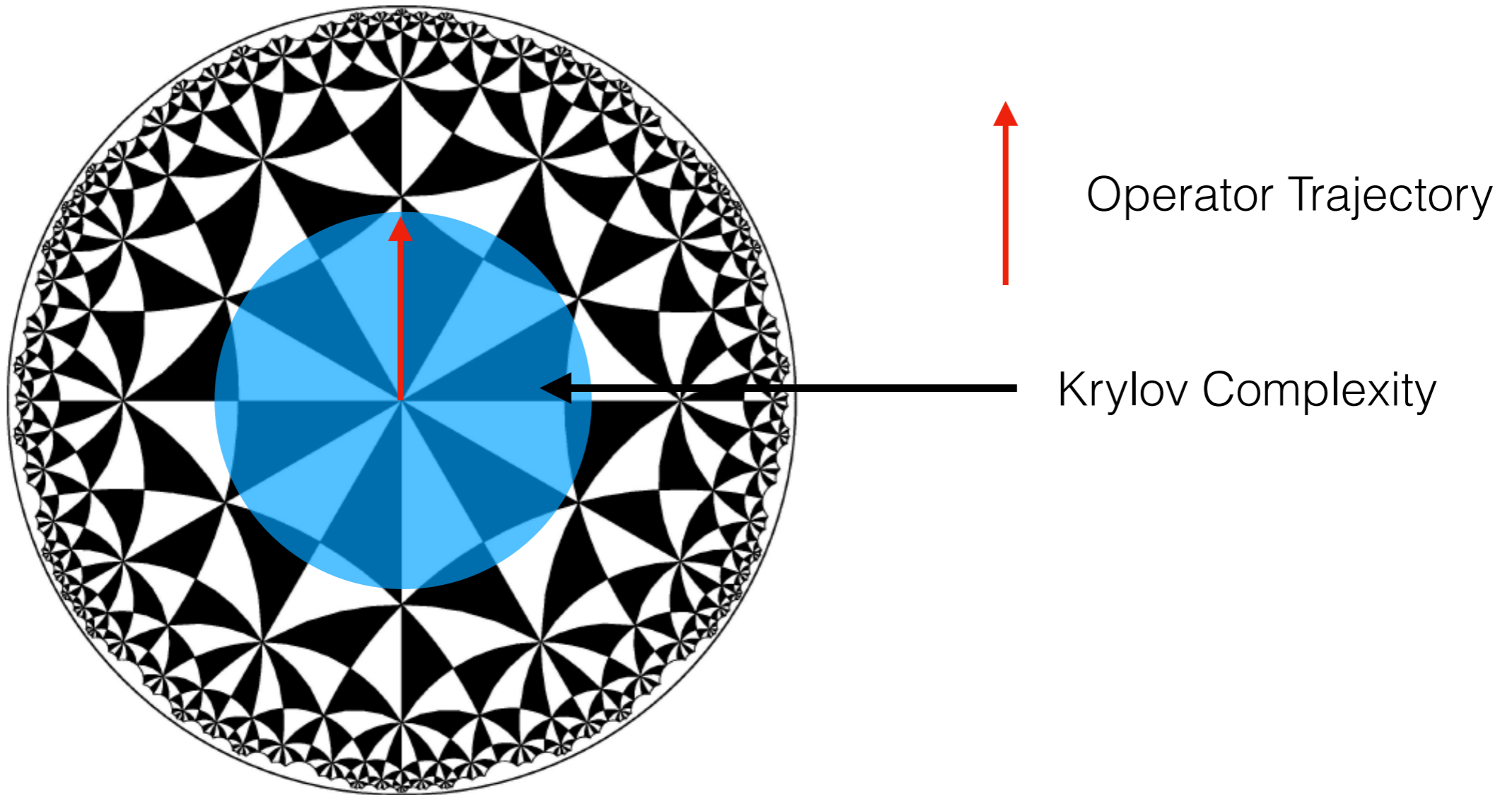
$$V_t = \int_0^{2\alpha t} d\rho \int_0^{2\pi} d\phi \sqrt{g} = 2\pi h \sinh^2(\alpha t) = \pi K_{\mathcal{O}}$$

In all the symmetry examples that we studied ($SU(2)$, HW)

Krylov complexity operator and L_0 (Symmetry generator)

$$L_0 = i\partial_\phi \quad L_{-1} = -ie^{-i\phi} [\coth(\rho)\partial_\phi + i\partial_\rho]$$
$$L_1 = -ie^{i\phi} [\coth(\rho)\partial_\phi - i\partial_\rho].$$

Cartoon:



Phase Space “Information Geometry”

“Complexity Algebra”

[PC, J.M.Magan, D.Patramanis '21]

More generally lessons from the symmetry approach

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\mathcal{L} = \tilde{L}_+ + \tilde{L}_-$$

$$\mathcal{B}|\mathcal{O}_n) = -b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\mathcal{B} = \tilde{L}_+ - \tilde{L}_-$$

Lets commute: From these definitions

$$\tilde{K} \equiv [\mathcal{L}, \mathcal{B}]|\mathcal{O}_n) = 2(b_{n+1}^2 - b_n^2)|\mathcal{O}_n)$$

We can demand that the algebra closes at this first step. This gives

$$2(b_{n+1}^2 - b_n^2) = An + B \qquad b_n = \sqrt{\frac{1}{4}An(n-1) + \frac{1}{2}Bn + C}$$

What if it doesn't? Number of steps to the closure? Classification?

“Complexity Algebra”

[PC, J.M.Magan, D.Patramanis '21]

For $SL(2, \mathbb{R})$

$$\mathcal{L} = \alpha(L_{-1} + L_1), \quad \mathcal{B} = \alpha(L_{-1} - L_1), \quad \tilde{K} = 4\alpha^2 L_0,$$

Geometrically, these are simply combinations of the isometry generators

$$ds^2 = \frac{h}{2} (d\rho^2 + \sinh^2(\rho) d\phi^2)$$

$$\begin{aligned} L_0 &= i\partial_\phi, \\ L_{-1} &= -ie^{-i\phi} [\coth(\rho)\partial_\phi + i\partial_\rho], \\ L_1 &= -ie^{i\phi} [\coth(\rho)\partial_\phi - i\partial_\rho]. \end{aligned}$$

In particular

$$\tilde{K} = 4\alpha^2 (\hat{K}_\mathcal{O} + h) \sim \partial_\phi$$

Relation between complexity and Isometries (Momentum/Boost)

[Lin, Maldacena, Zhao '19]

Ex.4 Complexity of the TFD evolution

[Balasubramanian, PC, Magan, Wu '22]

Consider the TFD state

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |n, n\rangle$$

$$Z(\beta) = \sum_n e^{-\beta E_n}$$

and its time evolution

[Hartman, Maldacena '13]

$$|\psi_\beta(t)\rangle = e^{-iHt} |\psi_\beta\rangle$$

$$H = H_L + H_R \quad H = H_{L/R}$$

Goal: expand this state in the Krylov basis and compute complexity.

Lanczos coefficients from the moments of

$$S(t) = \langle \Psi_\beta(t) | \Psi_\beta \rangle = \frac{Z(\beta - it)}{Z(\beta)} \quad (\sim \text{SFF})! \quad [\text{Polchinski et al. '16}]$$

Non-universal, can be extracted once we know Z (also in some limits).

See [Balasubramanian, PC, Magan, Wu '22]

Ex.5 Evolution of TFD for 2 HO

$$H_L = H_R = \omega(\hat{n} + \frac{1}{2}), \quad E_n = \omega(n + \frac{1}{2}) \quad Z(\beta) = \frac{1}{2 \sinh(\frac{\beta\omega}{2})}$$

return amplitude

$$\mathcal{S}(t) = \frac{\sinh(\frac{\beta\omega}{2})}{\sinh(\frac{(\beta-it)\omega}{2})}$$

Lanczos coefficients

$$a_n = \gamma(n + \frac{1}{2}), \quad b_n = \alpha n, \quad \gamma = \frac{\omega}{\tanh(\frac{\omega\beta}{2})}, \quad \alpha = \frac{\omega}{2 \sinh(\frac{\omega\beta}{2})}$$

Amplitudes (solutions of Schrodinger eq.)

$$\phi_n(t) = \frac{\sinh(\frac{\beta\omega}{2})}{\sinh(\frac{(\beta+it)\omega}{2})} \left(\frac{-i \sin(\frac{\omega t}{2})}{\sinh(\frac{(\beta+it)\omega}{2})} \right)^n$$

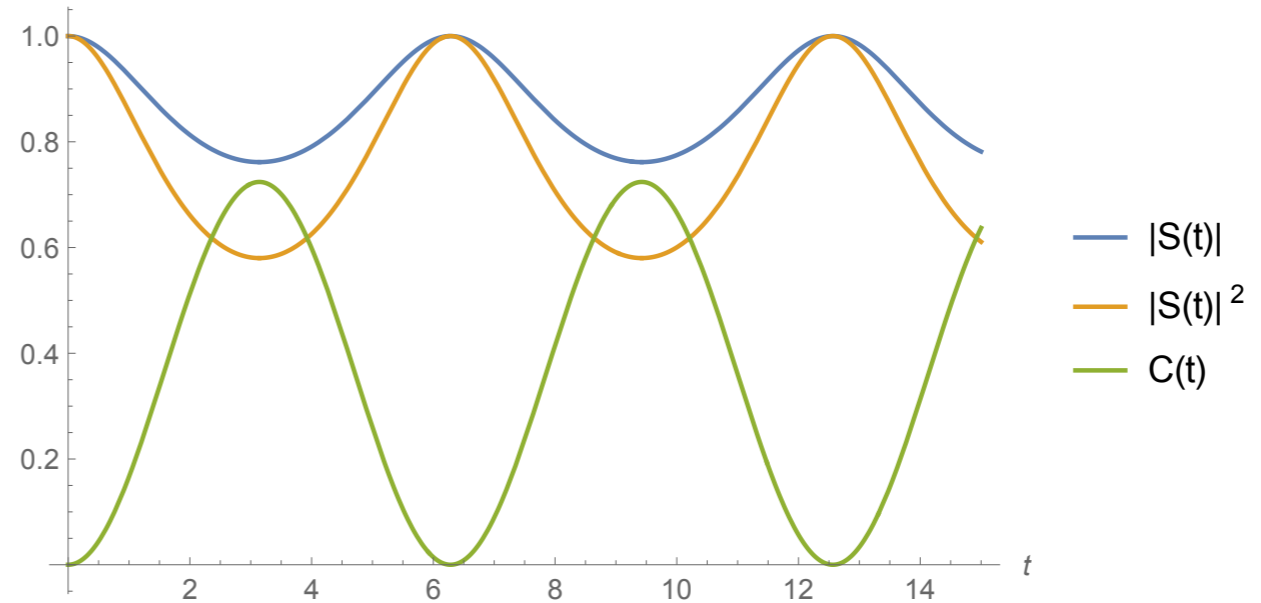
Spread/Krylov complexity

$$\mathcal{C}_\Psi(t) = \frac{\sin^2(\omega t/2)}{\sinh^2(\beta\omega/2)}$$

Ex.6 Evolution of TFD for 2 HO

$$\mathcal{S}(t) = \frac{\sinh\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{(\beta-it)\omega}{2}\right)}$$

$$\mathcal{C}_\Psi(t) = \frac{\sin^2(\omega t/2)}{\sinh^2(\beta\omega/2)}$$



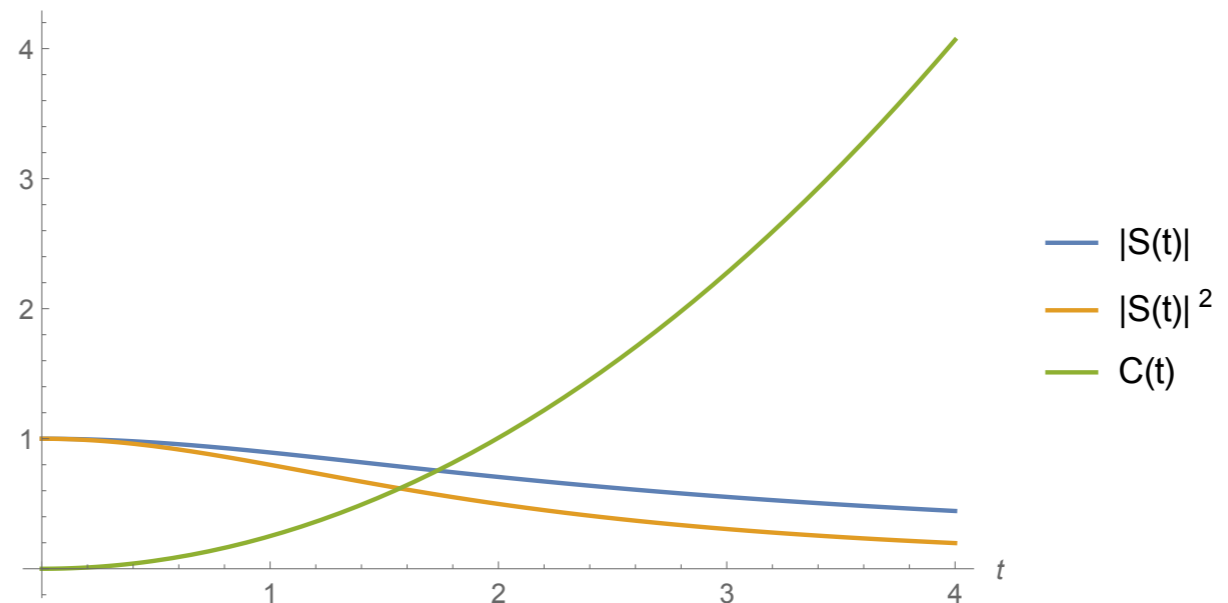
“Inverted” h.o.

$$\omega \rightarrow i\omega$$

$$\mathcal{S}(t) = \frac{-i \sin\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{\omega(t-i\beta)}{2}\right)}$$

$$\mathcal{C}_\Psi(t) = \frac{\sinh^2(\omega t/2)}{\sin^2(\beta\omega/2)} \sim e^{\omega(t-t_s)}$$

$$t_s = \frac{2}{\omega} \log\left(2 \sin \frac{\beta\omega}{2}\right)$$



Ex.7 Evolution of the TFD for RMT

[Balasubramanian, PC, Magan, Wu '22]

Late Times: “Black Holes and RM”

[Polchinski et al. '16]

Consider a random Hamiltonian (NxN, Hermitian matrix, GUE,...)

$$H = \begin{pmatrix} -0.625778 + 0.i & 0.0534572 - 0.238692i & -0.106837 + 0.170713i \\ 0.0534572 + 0.238692i & 0.518485 + 0.i & 0.995288 - 0.0813202i \\ -0.106837 - 0.170713i & 0.995288 + 0.0813202i & -0.589891 + 0.i \end{pmatrix}$$

We can easily diagonalise it, compute SFF, moments, Lanczos, etc.

We want to put it into the tri-diagonal form

and exponentiate

$$\begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

There exist very efficient algorithms/libraries (Python or Mathematica) to put a matrix into this form (Hessenberg). So we can also read off Lanczos coeff. this way.

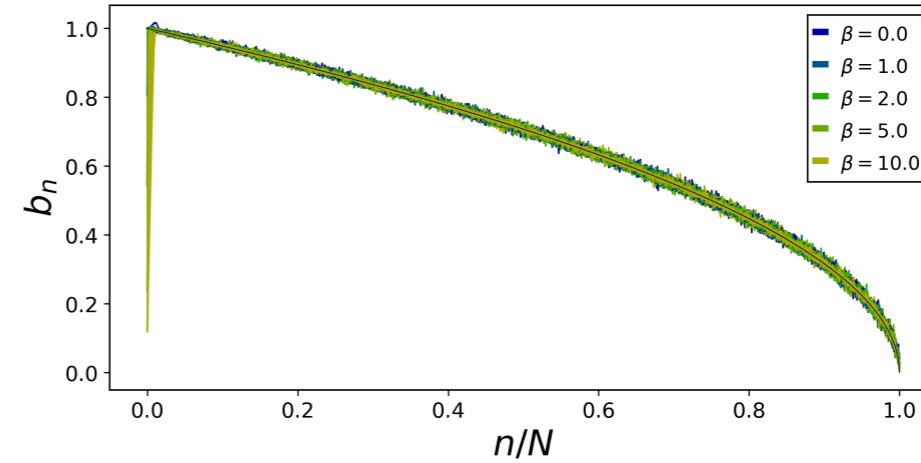
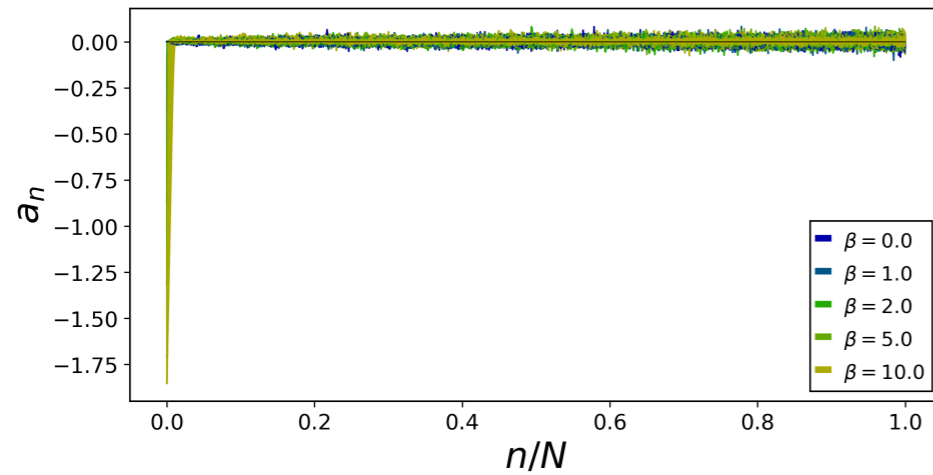
We also need to “rotate” a TFD into vec: $\{1,0,0,\dots\}$

Then applying $\exp(-iHt)$ to the initial state gives all the $\phi_n(t)$

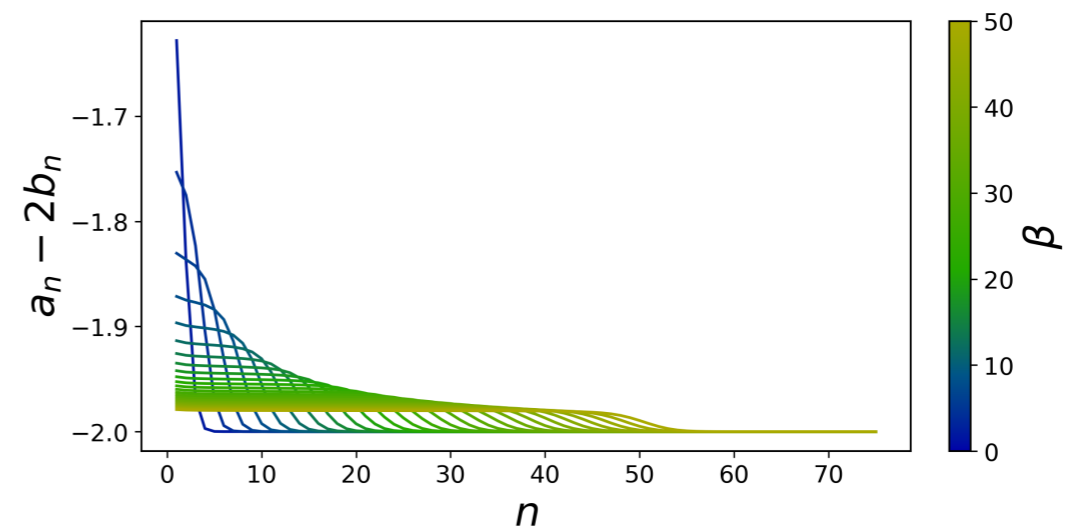
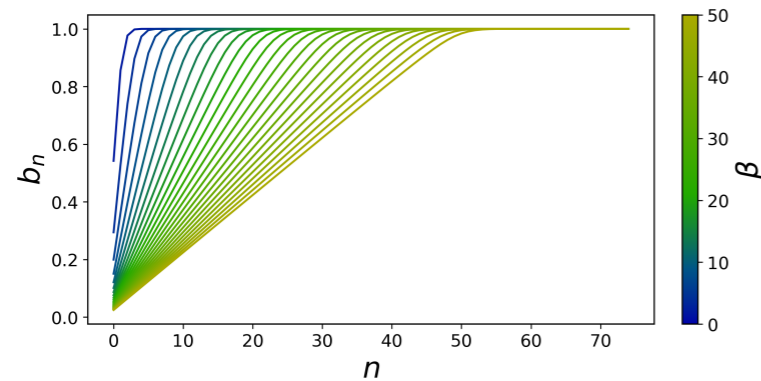
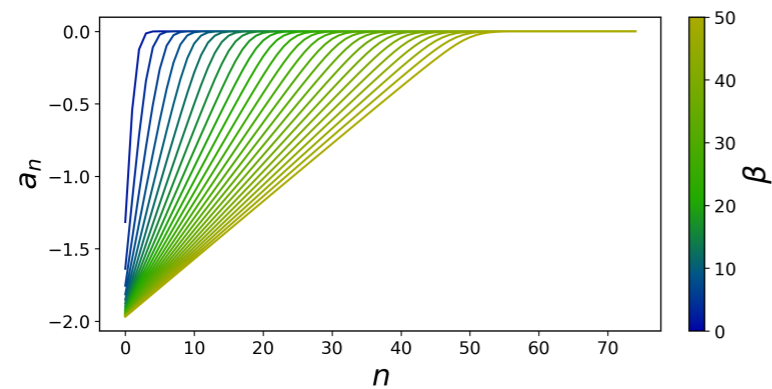
Ex.7 Evolution of the TFD for RMT

[Balasubramanian, PC, Magan, Wu '22]

Examples Lanczos: GUE, N (up to 4096)



More detailed (early time)

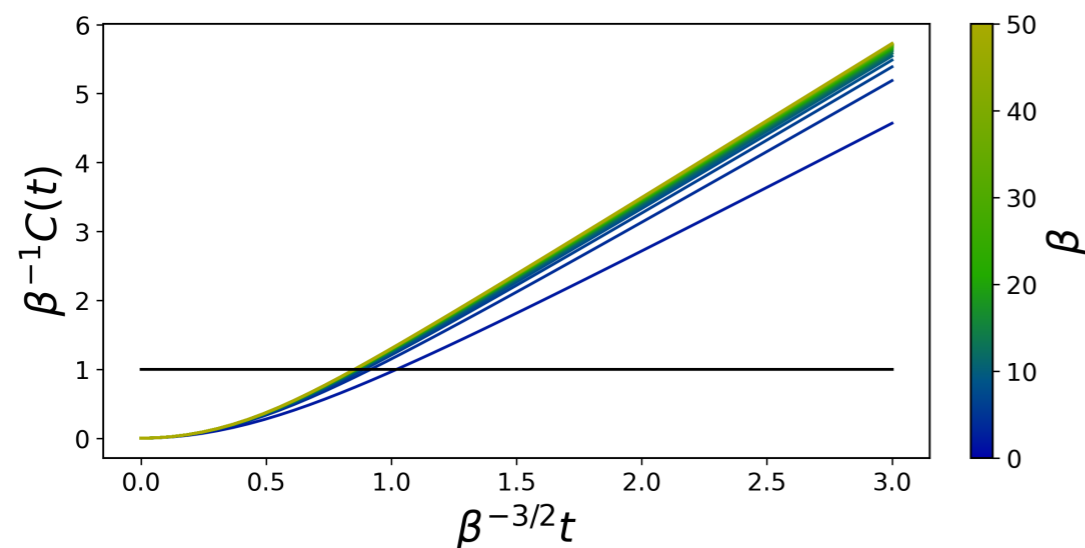


Ex.7 Evolution of the TFD for RMT

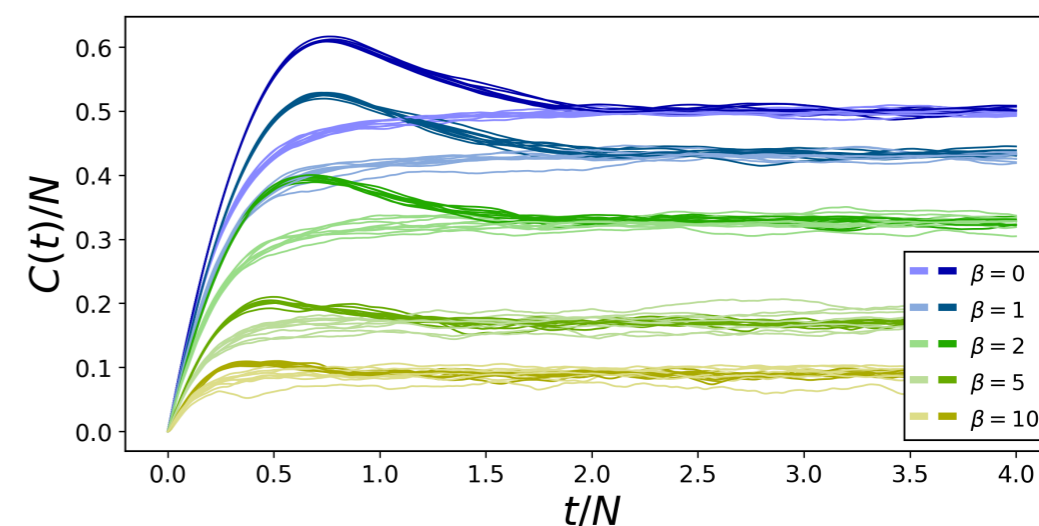
[Balasubramanian, PC, Magan, Wu '22]

Complexity for TFD evolved with GUE Hamiltonian (Similar for GOE, GSE, SYK)

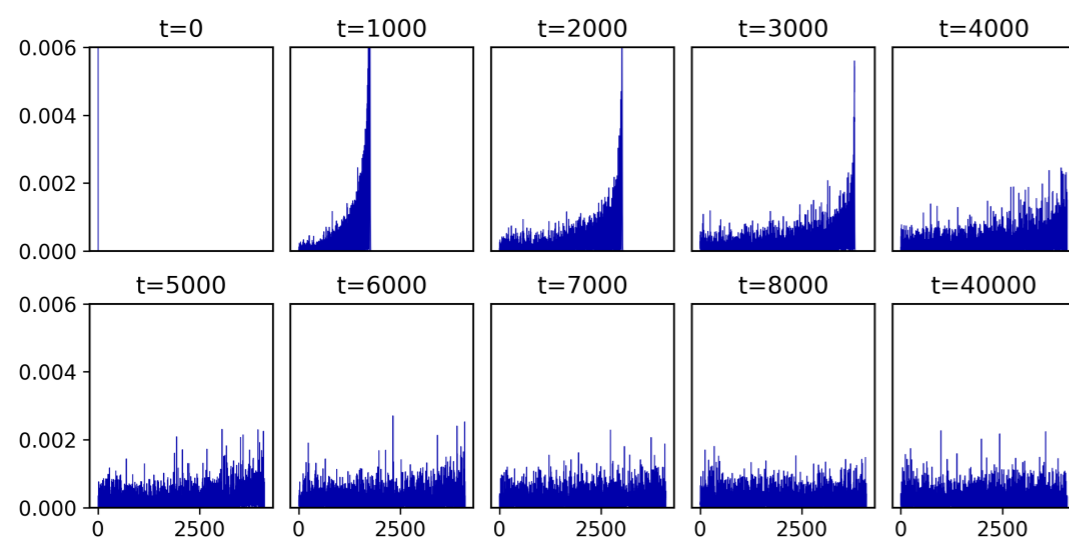
Early time



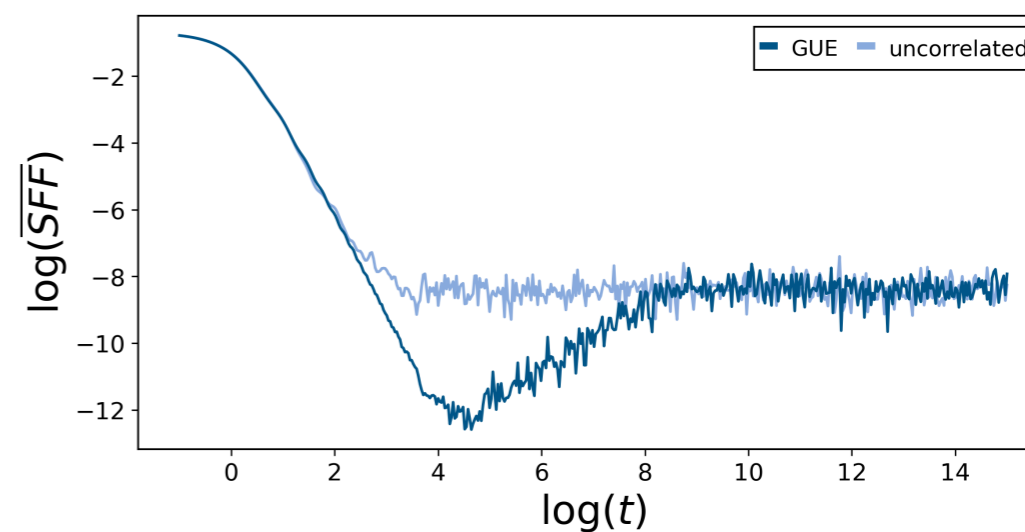
Ramp, Peak, Slope, Plateau



$N = \{1024, 1280, 1536, 1792, 2048, 2560, 3072, 3584, 4096\}$



Slope, Dip, Ramp, Plateau



$N = 4096$ and $\beta = 1$, averaged over 10 samples of the GUE


Ex. 8 Krylov complexity of formation?

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |n, n\rangle \quad H_L = H_R = \omega(\hat{n} + \frac{1}{2}), \quad E_n = \omega(n + \frac{1}{2})$$

We can write this state as

$$|\Psi_\beta\rangle = e^{ir\tilde{H}} |0, 0\rangle \quad \tilde{H} = \alpha(a_1^\dagger a_1^\dagger + a_1 a_2) \quad e^{-\beta\omega} = \tanh^2(\alpha r)$$

Action in the eigenstates $|K_n\rangle \equiv |n, n\rangle$

$$\tilde{H} |K_n\rangle = \alpha(n+1) |K_{n+1}\rangle + \alpha n |K_{n-1}\rangle$$


Expansion

$$|\Psi_\beta\rangle = \sum_n i^n \varphi_n(r) |K_n\rangle \quad \partial_r \varphi_n(r) = b_n \varphi_{n-1}(r) - b_{n+1} \varphi_{n+1}(r)$$

$$\varphi_n(r) = \frac{\tanh^n(\alpha r)}{\cosh(\alpha r)} = \frac{1}{\sqrt{Z(\beta)}} e^{-\frac{\beta}{2} E_n} \quad \sum_n |\varphi_n(r)|^2 = 1$$

Krylov complexity (of formation)

$$\mathcal{C} = \sum_n n |\varphi_n(r)|^2 = \sinh^2(\alpha r) = \frac{1}{Z} \sum_n n e^{-\beta E_n} = \frac{1}{e^{\beta\omega} - 1} \sim \Delta E$$

Similar steps for (K-)entropy

Operator Growth in 2d CFTs

[PC, Datta '21]

Recall the main observation for $SL(2,R)$ (chiral):

$$|\mathcal{O}(t)\rangle = e^{i(L_{-1}+L_1)t}|h\rangle$$

In CFT we can equivalently write this as

$$|\mathcal{O}(t)\rangle = e^{i(L_{-1}+L_1)t}|h\rangle = \mathcal{N} e^{z(t)L_{-1}} \mathcal{O}_{-h}|0\rangle = \frac{\mathcal{O}(z)}{\sqrt{\langle \mathcal{O}^\dagger(z)\mathcal{O}(z)\rangle}}|0\rangle \quad z = i \tanh(\alpha t)$$

where

$$\mathcal{O}(z)|0\rangle = \sum_{n=0}^{\infty} z^n \mathcal{O}_{-(h+n)}|0\rangle$$

Operator growth: Highest weight state \rightarrow Descendants

Can we study this evolution in the full “Virasoro basis” of 2d CFTs?

Growth of the stress tensor, KdV charges etc.?

“Oscillator” basis for Virasoro

[Polyakov;Zamolodchikov'85]

[Besken,Datta,Kraus'19]

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)$$

We can represent them as differential operators (Fock-Bargmann) $l_n f(u) \equiv \langle u | L_n | f \rangle$

Where $\{u_1, u_2, \dots\}$

$$l_0 = h + \sum_{n=1}^{\infty} n u_n \frac{\partial}{\partial u_n}, \quad c = 1 + 24\mu^2, \quad h = \mu^2 + \lambda^2$$
$$l_k = \sum_{n=1}^{\infty} n u_n \frac{\partial}{\partial u_{n+k}} - \frac{1}{4} \sum_{n=1}^{k-1} \frac{\partial^2}{\partial u_n \partial u_{k-n}} + (\mu k + i\lambda) \frac{\partial}{\partial u_k}, \quad k > 0$$
$$l_{-k} = \sum_{n=1}^{\infty} (n + k) u_{n+k} \frac{\partial}{\partial u_n} - \sum_{n=1}^{k-1} n(k - n) u_n u_{k-n} + 2k(\mu k - i\lambda) u_k, \quad k > 0$$

Inner product for the wave functions

$$(f(u), g(u)) = \int [du] \overline{f(u)} g(u), \quad [du] = \prod_{n=1}^{\infty} d^2 u_n \frac{2n}{\pi} e^{-2n u_n \bar{u}_n}$$

“Oscillator” basis for Virasoro

[Polyakov; Zamolodchikov'85]

[Besken, Datta, Kraus'19]

Orthonormal basis in the space of descendants is spanned by monomials

$$\Phi_{\{m_i\}}(u) \equiv \frac{u_1^{m_1} u_2^{m_2} \dots}{\mathcal{N}_{\{m_i\}}} \quad \sum_j j m_j = N \quad \mathcal{N}_{\{m_j\}} = \sqrt{S_{1,m_1} S_{2,m_2} \dots} = \left[\prod_{j=1}^{\infty} \frac{m_j!}{(2j)^{m_j}} \right]^{1/2},$$

Satisfying

$$(\Phi_{\{m_i\}}(u), l_0 \Phi_{\{m_i\}}(u)) = h + \sum_j j m_j = h + N$$

They are labeled by integer partitions: Young diagrams

$$|1^{m_1} 2^{m_2} \dots\rangle \mapsto \Phi_{\{m_i\}}(u), \quad \sum_{j=1}^N j m_j = N$$

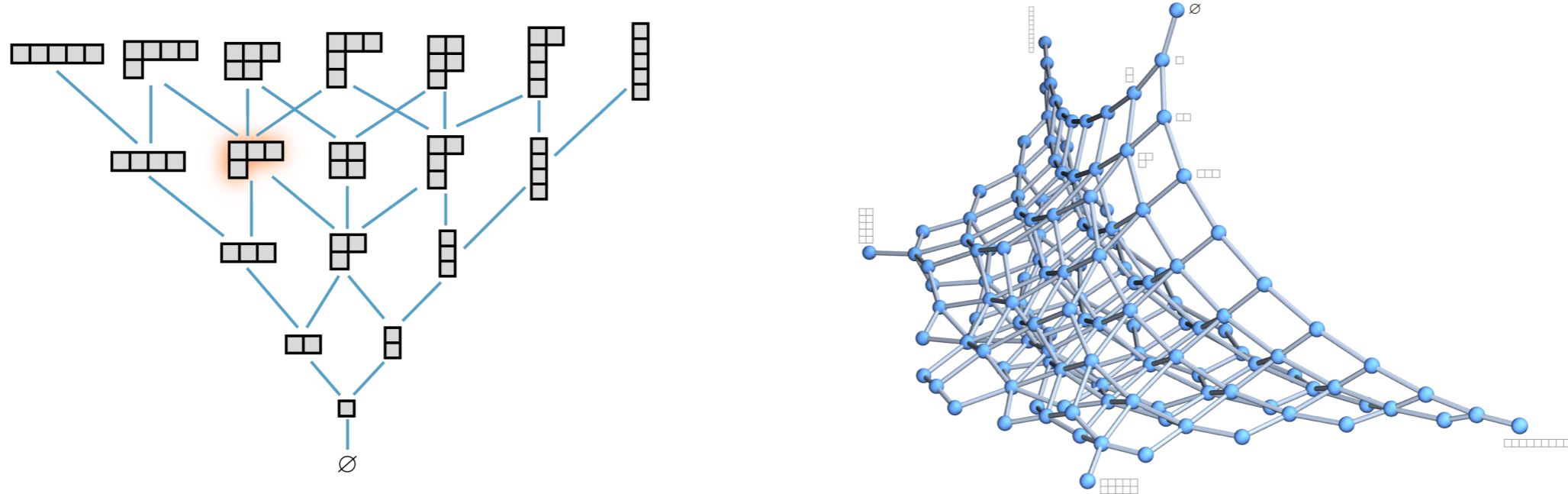
e.g. level 3

$$|1^3\rangle \mapsto \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \quad |1^1 2^1\rangle \mapsto \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \quad |3^1\rangle \mapsto \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}$$

“Oscillator” basis for Virasoro

[PC, Datta '21]

This way we can picture the descendants as a Young lattice



Moreover oscillator representation of a primary is also known

[Besken, Datta, Kraus '19]

$$\langle u | \mathcal{O}(z) | 0 \rangle = \exp \left[2(\mu - i\lambda) \sum_{n=1}^{\infty} z^n u_n \right] = 1 + \sum_{N=1}^{\infty} z^N \sum_{\sum j m_j = N} [2(\mu - i\lambda)]^{\sum m_j} \frac{u_1^{m_1} u_2^{m_2} \dots}{m_1! m_2! \dots}$$

Important: Irrational 2d CFTs with $c > 1$ (Virasoro, no null states!)

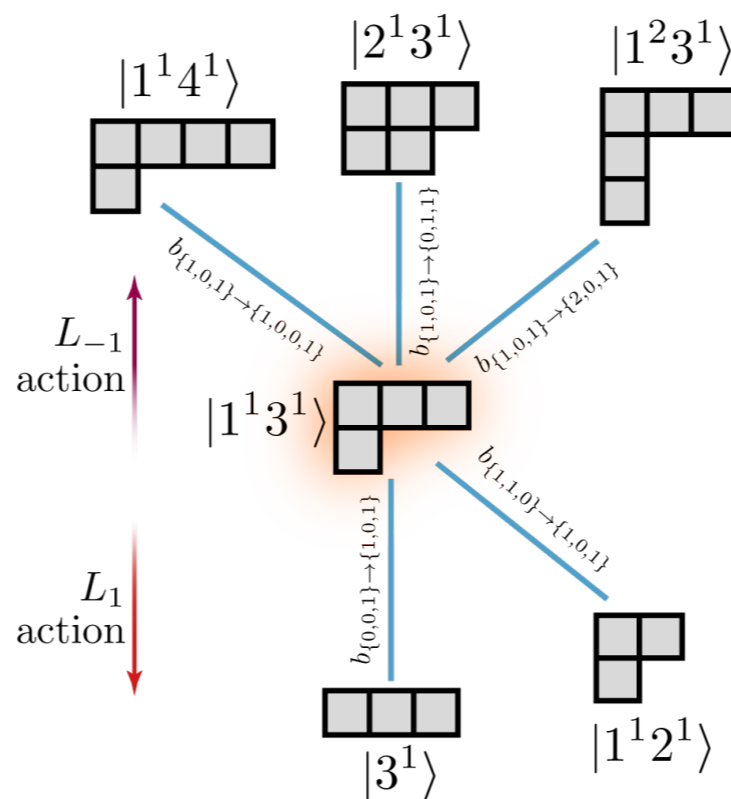
SL(2,R) Liouvillian in the full Virasoro “Oscillator” basis

[PC, Datta '21]

Our Liouvillian is built from $\mathcal{L} \equiv \alpha(L_{-1} + L_1)$

$$l_1 = \sum_{n=1}^{\infty} n u_n \frac{\partial}{\partial u_{n+1}} + (\mu + i\lambda) \frac{\partial}{\partial u_1}, \quad l_{-1} = \sum_{n=1}^{\infty} (n+1) u_{n+1} \frac{\partial}{\partial u_n} + 2(\mu - i\lambda) u_1$$

Defines a local Hamiltonian on the Young Lattice that adds and subtracts only a single box from YT.



SL(2,R) Liouvillian in the full Virasoro “Oscillator” basis

[PC, Datta '21]

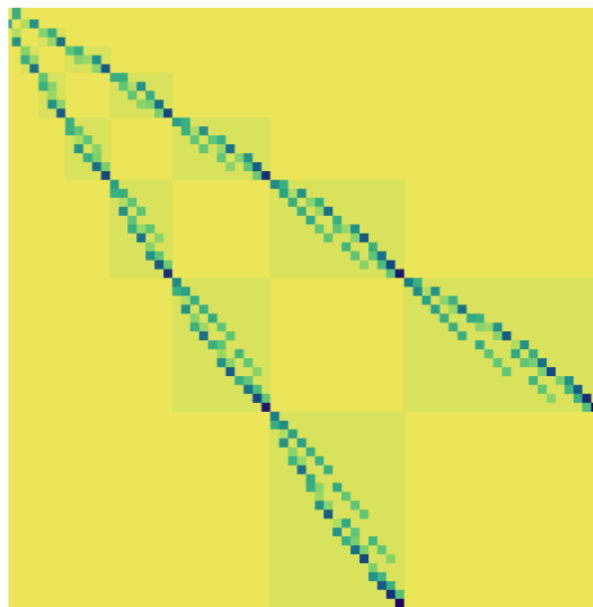
More precisely we have in the Bargmann space $\mathcal{L} \equiv \alpha(L_{-1} + L_1)$

$$\langle u | \mathcal{L} \Phi_{\{m_k\}} \rangle = \alpha(l_{-1} + l_1) \Phi_{\{m_k\}}(u) = \sum_{\sum j r_j = N+1} b_{\{m_k\} \rightarrow \{r_j\}} \Phi_{\{r_j\}}(u) + \sum_{\sum j s_j = N-1} b_{\{s_j\} \rightarrow \{m_k\}} \Phi_{\{s_j\}}(u) ,$$

Lanczos coefficients

$$b_{\{m_k\} \rightarrow \{r_j\}} = \left(\Phi_{\{m_k\}}(u), \alpha l_{-1} \Phi_{\{r_j\}}(u) \right)$$

Liouvillian is block tri-diagonal $p(N) \times p(N \pm 1)$



We can explicitly evaluate various classes of Lanczos coefficients and their scaling

Operator Growth in 2 CFTs

[PC, Datta '21]

The operator growth with the $SL(2,R)$ Liouvillian can be interpreted in this basis:

$$\Psi_{\mathcal{O}}(t) \equiv \langle u | e^{i\alpha t(l_1+l_{-1})} \mathcal{O}(0) | 0 \rangle = e^{\alpha_0 h} \left[1 + \sum_{N=1}^{\infty} (\alpha_-)^N \sum_{\sum j m_j = N} \frac{[2(\mu - i\lambda)]^{\sum m_j}}{\sqrt{T_{1,m_1} T_{2,m_2} \cdots}} \Phi_{\{m_i\}}(u) \right]$$

$$\alpha_- = i \tanh(\alpha t)$$

$$\alpha_0 = -2 \log \cosh(\alpha t)$$

$$T_{j,m} = (2j)^m m!$$

This defines a probability distribution (t: Primary->Higher Desc.)

$$p_{\{m_j\}}(t) = |\varphi_{\{m_j\}}(t)|^2 = \frac{\tanh^{2N}(\alpha t)}{\cosh^{4h}(\alpha t)} \frac{[4h]^{\sum m_j}}{2^{m_1} m_1! 4^{m_2} m_2! 6^{m_3} m_3! \cdots}$$

Krylov complexity

$$K_{\mathcal{O}}(t) = \sum_{N=0}^{\infty} N \sum_{\sum j m_j = N} |\varphi_{\{m_j\}}(t)|^2$$

We can write also as

$$K_{\mathcal{O}}(t) = \langle \mathcal{O}(t) | L_0 | \mathcal{O}(t) \rangle - \langle \mathcal{O}(0) | L_0 | \mathcal{O}(0) \rangle$$

We recover the universal $SL(2,R)$ result and relation with Volume!

Lanczos Coefficients

[PC, Datta '21]

Linear Growth of Lanczos coefficients corresponds to

$$\emptyset \rightarrow \square \rightarrow \square\square \rightarrow \square\square\square \rightarrow \square\square\square\square \rightarrow \square\square\square\square\square \rightarrow \dots$$

Growth for “typical” states (\sim triangle shape)

$$b_{\text{typ}}^{(1)} \approx \frac{\sqrt{6N}}{\pi}$$

Slower than the initial linear growth

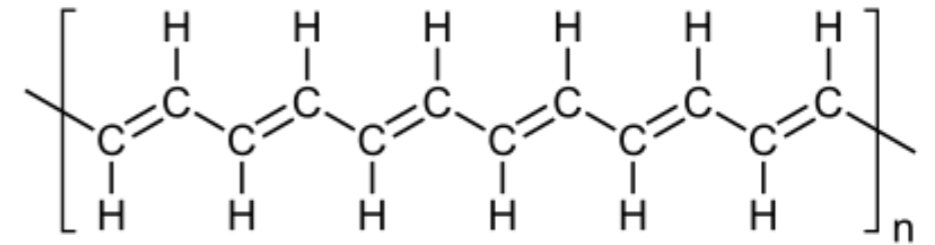
Krylov complexity is the same as for the global (SL(2,R)) case (exponential growth) slowed down to polynomial for typical states and then is expected to saturate (for constant b , regime beyond CFT(?))

This analysis can be now done for stress tensor (see paper) or more general states.

Probe of topological phases?

SSH model (polyacetylene)

$$H = t_1 \sum_i \left(c_{Ai}^\dagger c_{Bi} + \text{h.c.} \right) - t_2 \sum_i \left(c_{Bi}^\dagger c_{A,i+1} + \text{h.c.} \right)$$



Depending on t 's the ground state of the model SU(2) CS:

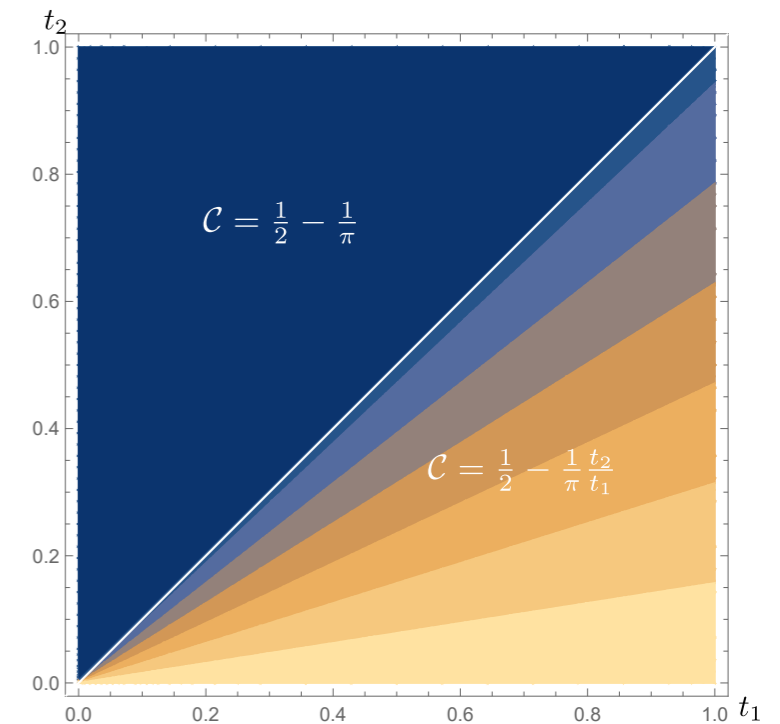
$$|\Omega\rangle = \prod_{k>0} \mathcal{N}_k e^{-i \tan\left(\frac{\phi_k}{2}\right) (J_+^{(k)} + J_+^{(-k)})} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_k$$

$$\sin \phi_k = \frac{|R_1|}{R}, \quad \cos \phi_k = \frac{R_3}{R},$$

$$R_1 = t_1 - t_2 \cos(k) \quad R_3 = t_2 \sin(k) \quad R = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos(k)}.$$

Represents non-topological phase ($t_1 > t_2$) or topological insulator ($t_1 < t_2$).

We can use Krylov methods to compute spread complexity of formation for a single momentum and then sum over.



$$\mathcal{C}(t_1, t_2) = 2 \int_0^\pi \frac{dk}{2\pi} C_k = \frac{1}{2} - \frac{t_1 + t_2 - |t_1 - t_2|}{2\pi t_1}.$$

Comments on Holography

[PC, Ge '22]

$$|\mathcal{O}(t)\rangle = e^{i(L_{-1}+L_1)t}|h\rangle = \mathcal{N}e^{z(t)L_{-1}}\mathcal{O}_{-h}|0\rangle = \frac{\mathcal{O}(z)}{\sqrt{\langle\mathcal{O}^\dagger(z)\mathcal{O}(z)\rangle}}|0\rangle$$

$$z = i \tanh(\alpha t)$$

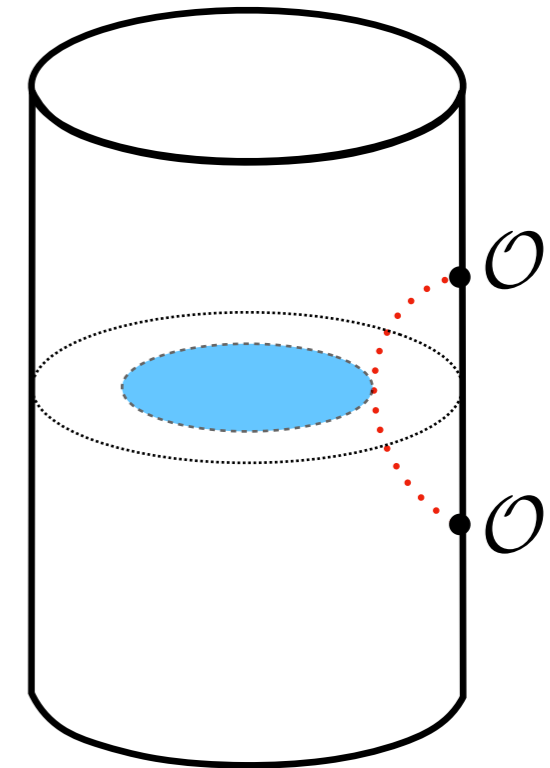
We can consider Banados geometries with stress tensor in this states

$$ds^2 = \frac{d\eta^2}{\eta^2} + \frac{(dz + \eta^2 \bar{\mathcal{L}}(\bar{z})d\bar{z})(d\bar{z} + \eta^2 \mathcal{L}(z)dz)}{\eta^2}$$

Where (generally) $|\Phi\rangle = |\mathcal{O}(t)\rangle$

$$\langle\Phi|T(z)|\Phi\rangle = \frac{c}{12}\{f(z), z\} \quad \mathcal{L}(z) = -\frac{6}{c}\langle\Phi|T(z)|\Phi\rangle$$

K-complexity= variation of $L_0 \sim$ area of the blue disc.



Euclidean geometries = back-reacted along the point particle trajectory.

Lorentzian picture?

Conclusions

- New definition of Krylov/Spread Complexity for operators/states !
- Progress on a useful notion of “Complexity” in many-body systems
- Computable for operators and states; numerically for discrete models and QFTs
- Crucial ingredient: Return amplitude (2- and higher-point function, SFF etc.)
- Evolution of TFD in RM: Ramp, Peak, Slope, Plateau
- Symmetry: new angle on Lanczos coefficients and growth in SYK, 2d CFT
- For $SL(2,R)$ (semi-simple Lie alg.) we can “geometrize” it (coherent states) and interpret as phase space volume
- Straightforward to generalise to more interesting many-body setups (topological phases)

Many Open Problems

- Universal laws for Spread/Krylov complexity? Is it useful for QI or QC?
- Integrable vs Chaotic growth? Is it sensitive? At which time regime?
- Purely Integrable models? Can we study it using integrability (not just numerics)?
- Interesting states? More complicated objects (defects, boundaries)?
- Precise connection with Holography?
- Complexity and near-horizon geometry (AdS₂)?
- More from Symmetry/Algebras? BMS, flat space, non-relativistic....?
- Late-time physics of AdS/CFT and Black-Holes?
- Why Krylov basis? Bulk understanding? (Chords in SYK? [\[Lin'22\]](#))

Thank You! Stay Tuned! Join the fun ;)

SU(2)

[PC, J.M.Magan, D.Patramanis '21]

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$J_{\pm} = J_1 \pm iJ_2$$

$$[J_0, J_{\pm}] = \pm J_{\pm},$$

$$[J_+, J_-] = 2J_0$$

Liouvillian:

$$\mathcal{L} = \alpha(J_+ + J_-)$$

Representation:

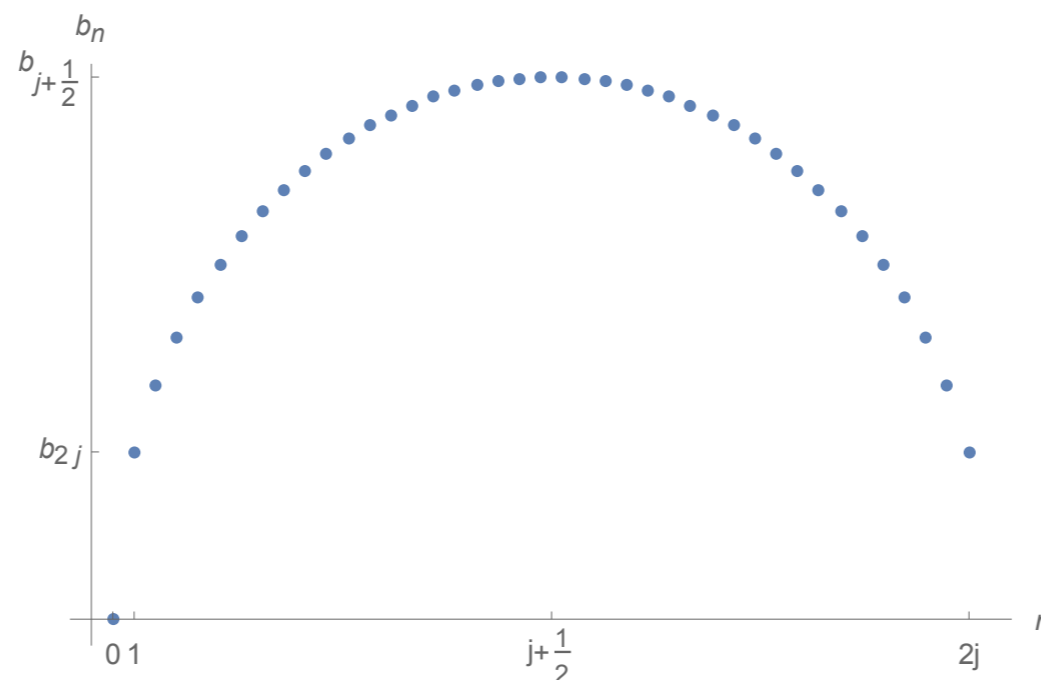
$$J_0 |j, -j + n\rangle = (-j + n) |j, -j + n\rangle,$$

$$J_+ |j, -j + n\rangle = \sqrt{(n+1)(2j-n)} |j, -j + n + 1\rangle$$

$$J_- |j, -j + n\rangle = \sqrt{n(2j-n+1)} |j, -j + n - 1\rangle.$$

$$b_n = \alpha \sqrt{n(2j-n+1)}.$$

$$n = 0, \dots, 2j$$



SU(2)

[PC, J.M.Magan, D.Patramanis '21]

Spin coherent states:

$$|z, j\rangle = (1 + z\bar{z})^{-j} \sum_{n=0}^{2j} z^n \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}} |j, -j+n\rangle$$

$$z = \tan\left(\frac{\theta}{2}\right) e^{i\phi}$$

Trajectory: $\theta = 2\alpha t$ and $\phi = \pi/2$

$$\varphi_n(t) = \frac{\tan^n(\alpha t)}{\cos^{-2j}(\alpha t)} \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}$$

Krylov complexity:

$$K_{\mathcal{O}} = \sum_{n=0}^{2j} n |\varphi_n(t)|^2 = 2j \sin^2(\alpha t)$$

Information Geometry

$$ds^2 = \frac{2j dz d\bar{z}}{(1 + |z|^2)^2} = \frac{j}{2} (d\theta^2 + \sin^2 \theta d\phi^2) \quad V_t = \int_0^{2\alpha t} d\theta \int_0^{2\pi} d\phi \sqrt{g} = 2\pi j \sin^2(\alpha t) = \pi K_{\mathcal{O}}$$