Generalized thermalization in integrable lattice systems

Marcos Rigol

Department of Physics
The Pennsylvania State University

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L. Vidmar and MR, *Generalized Gibbs ensemble in integrable lattice models*, J. Stat. Mech. 064007 (2016).

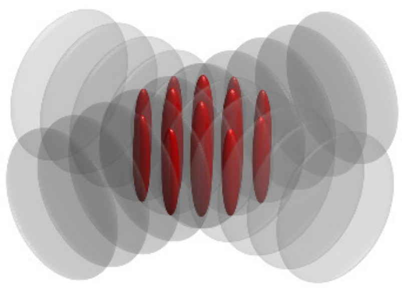
Video to be posted in the conference website!

Outline

1. Introduction
   - Experiments with ultracold gases in one dimension
   - Classical and quantum integrability
   - Hard-core bosons in one-dimensional lattices
2. Generalized Gibbs Ensemble (GGE)
   - Maximal entropy and the GGE
3. Generalized Thermalization
   - GGE vs quantum mechanics
   - Generalized eigenstate thermalization
4. Summary

Outline

Experiments in the 1D regime



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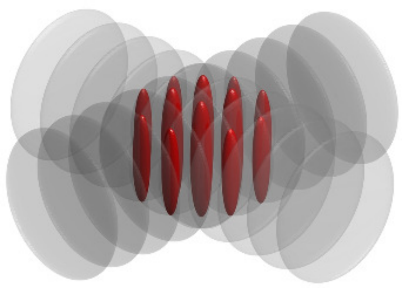

Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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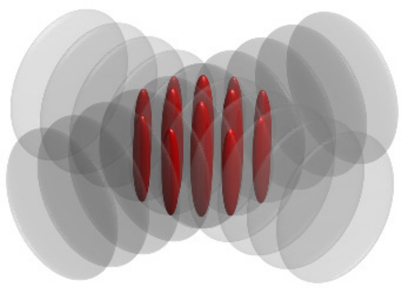

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Lieb & Liniger '63,

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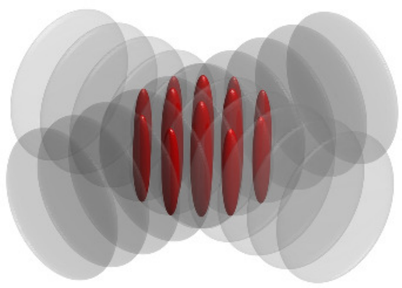

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Lieb & Liniger ’63, Girardeau ’60 ($g_{1D} = \infty$)

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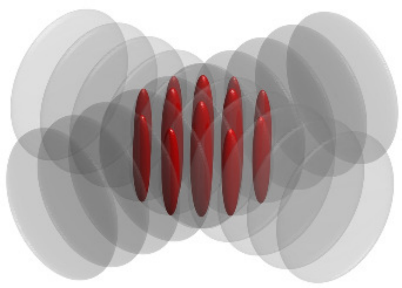

T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$g^{(2)}(x) = \frac{\langle \hat{\Psi}^{\dagger 2}(x)\Psi^2(x)\rangle}{n_{1D}^2(x)}$ and $\gamma_{\text{eff}} = \frac{m g_{1D}}{\hbar^2 n_{1D}}$ $\Leftrightarrow$

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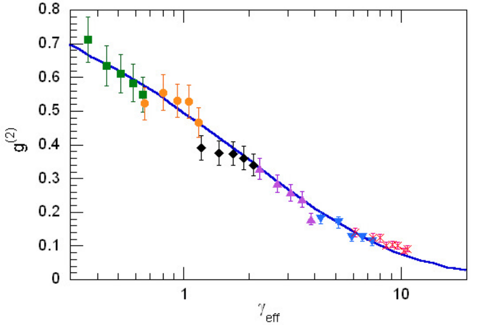

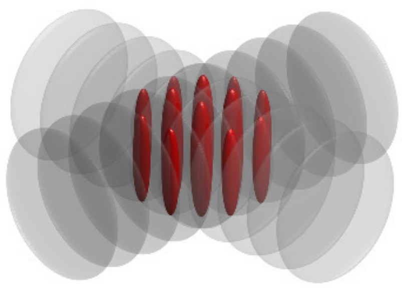

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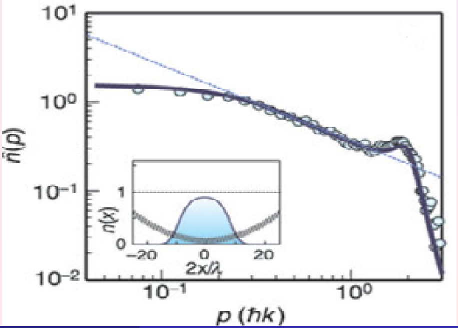
Lieb & Liniger '63, Girardeau '60 ($g_{1D} = \infty$)

Lieb, Schulz, and Mattis '61 ($U/J = \infty$)

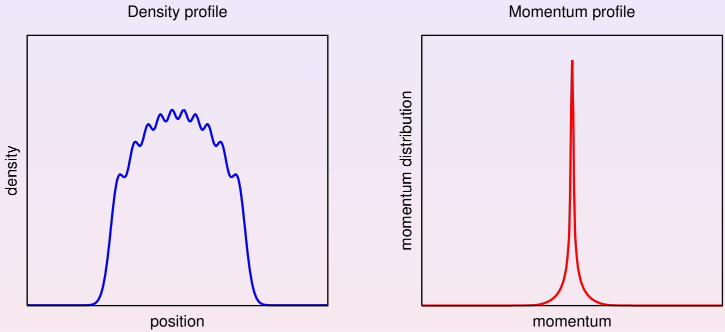
B. Paredes *et al.*,
Nature (London) **429**, 277 (2004).

$n(p)$: Momentum distribution $\Leftrightarrow$

$n(x)$: Density distribution $\Leftrightarrow$



Absence of thermalization, quantum Newton's cradle



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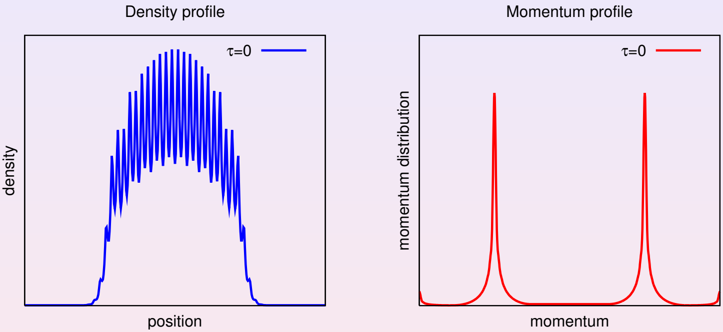

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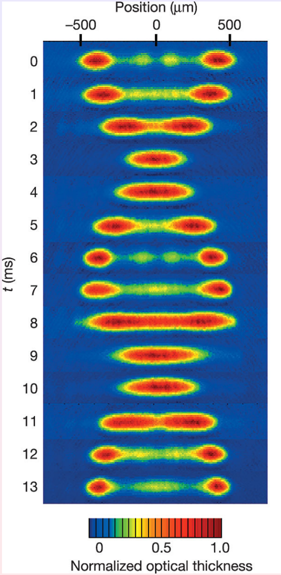
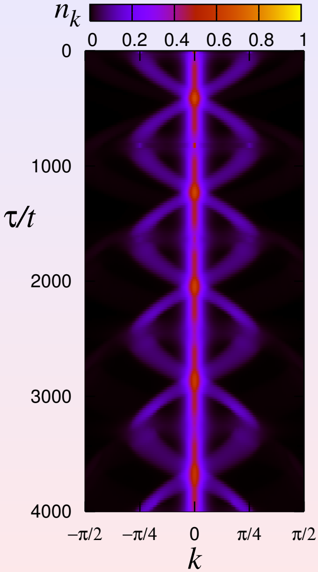
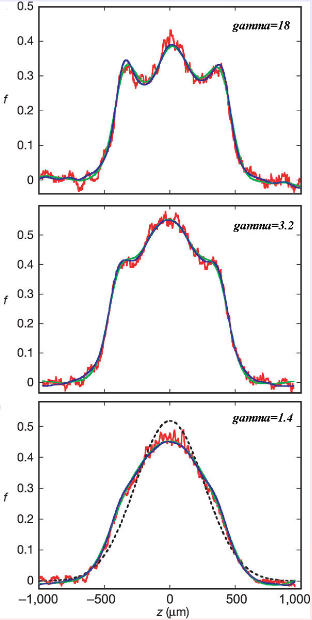
Experiment





Theory

Absence of thermalization, quantum Newton's cradle



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

$$\gamma = \frac{mg_{1D}}{\hbar^2 n_{1D}}$$

g_{1D}: Contact interaction strength
n_{1D}: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

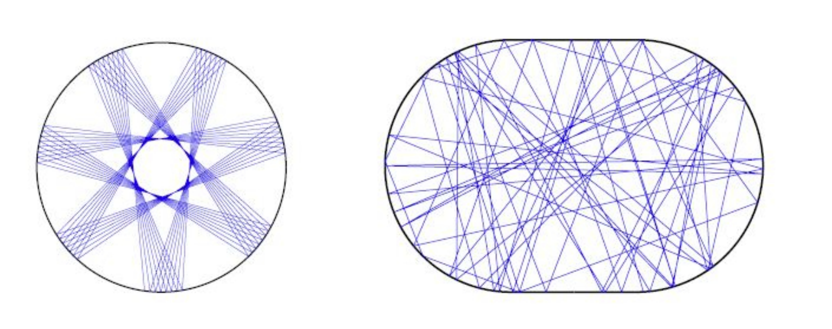
If $\gamma \ll 1$ the system is in the weakly interacting regime

Review of related work in atom chips:
T. Langen, T. Gasenzer, and J. Schmiedmayer, J. Stat. Mech. 064009 (2016).

Outline

Classical chaos and integrability

Particle trajectories in a circular cavity and a Bunimovich stadium (scholarpedia)



- Integrability: A system is said to be integrable if it has as many constants of motion as degrees of freedom
- Chaos: exponential sensitivity of the trajectories to perturbations

Liouville's integrability theorem (Classical)

Hamiltonian

$$H(p,q), \quad \text{coordinates } q=(q_1,\cdots,q_N)$$
$$\text{momenta } p=(p_1,\cdots,p_N)$$

N independent constants of the motion, $I=(I_1,\cdots,I_N)$, in involution

$$\{I_\alpha,H\}=0, \quad \{I_\alpha,I_\beta\}=0, \qquad \{f,g\}=\sum_{i=1,N}\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

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N independent constants of the motion, $I = (I_1, \cdots, I_N)$, in involution

$$\{I_\alpha, H\} = 0, \quad \{I_\alpha, I_\beta\} = 0, \qquad \{f, g\} = \sum_{i=1,N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

There is a canonical transformation $(p,q) \to (\Theta, I)$ (action-angle variables)

$$H(p,q) = H'(I)$$

Equations of motion

$$\begin{aligned} \frac{dI_\alpha}{dt} &= -\frac{\partial H'}{\partial \Theta_\alpha} = 0 \quad \Rightarrow \quad I_\alpha = \text{constant} \\ \frac{d\Theta_\alpha}{dt} &= \frac{\partial H'}{\partial I_\alpha} = \Omega_\alpha(I) \quad \Rightarrow \quad \Theta_\alpha = \Omega_\alpha(I)t + \Theta_\alpha^0 \end{aligned}$$

Scattering without diffraction (Quantum)

Impenetrable particles, interaction potential decays sufficiently fast, $x_1 \ll x_2 \ll x_3 \ll \ldots$, and $k_1 > k_2 > k_3 > \ldots$

B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).

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Two particles: $\quad K = k_1 + k_2, \quad E = \varepsilon(k_1) + \varepsilon(k_2), \quad \Psi(x_1, x_2) \rightarrow$

$$\sum_P A(P)\, e^{i(k_{P1}x_1 + k_{P2}x_2)} = A(12)\, e^{i(k_1 x_1 + k_2 x_2)} + A(21)\, e^{i(k_2 x_1 + k_1 x_2)}$$

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Three particles: $K = k_1 + k_2 + k_3, \qquad E = \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3)$
$$\Psi(x_1, x_2, x_3) \to \sum_P A(P)\, e^{i(k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3)} + \text{ diffractive scattering}$$

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Outline

Bose-Fermi mapping in a 1D lattice

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J\sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \mathrm{H.c.}\right) + \sum_i v_i \; \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

Bose-Fermi mapping in a 1D lattice

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Constraints on the bosonic operators

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$$\Downarrow$$

Map to spins and then to fermions (Jordan-Wigner transformation)

$$\hat{\sigma}_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \hat{\sigma}_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

$$\Downarrow$$

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i v_i \; \hat{n}_i^f$$

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Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J\sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}\right) + \sum_i v_i\ \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

Set of conserved quantitites

(Occupations of the single-particle energy eigenstates of the noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$

$$\left\{\hat{I}_m^f\right\} = \{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^f\}$$

One-body density matrix

One-body Green's function

$$G_{ij} = \langle \Psi_{HCB} | \hat{\sigma}_i^- \hat{\sigma}_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

Time evolution

$$|\Psi_F(t)\rangle = e^{-i\hat{H}_F t} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(t) \hat{f}_\sigma^\dagger \, |0\rangle$$

MR and A. Muramatsu, PRL **93**, 230404 (2004).

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Exact Green's function

$$G_{ij}(t) = \det \left[\left(\mathbf{P}^l(t) \right)^\dagger \mathbf{P}^r(t) \right]$$

Computation time $\propto L^2 N^3 \to$ study very large systems

$$\sim 10000 \text{ lattice sites}, \qquad \sim 1000 \text{ particles}$$

MR and A. Muramatsu, PRL **93**, 230404 (2004).

Dynamical fermionization during expansion in 1D

Dynamics after turning off the confining potential

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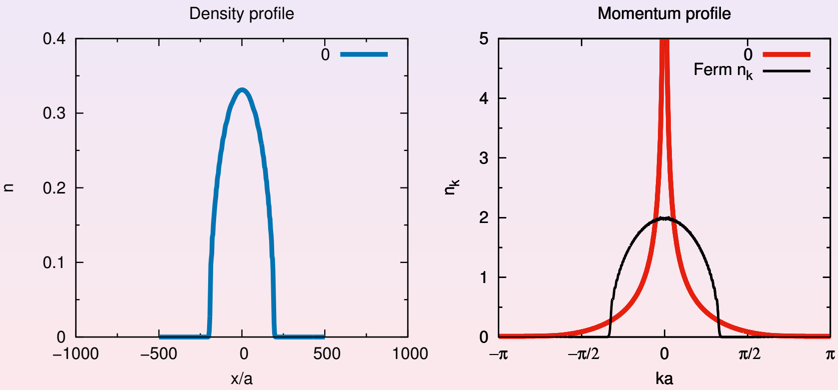

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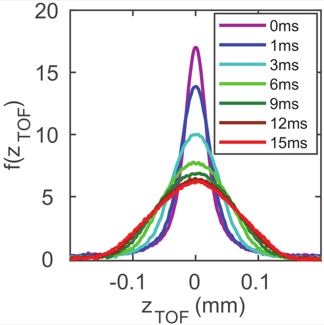
1D expansion of TG gases (fermionization)



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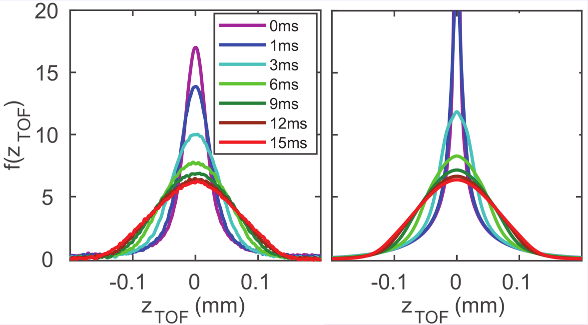

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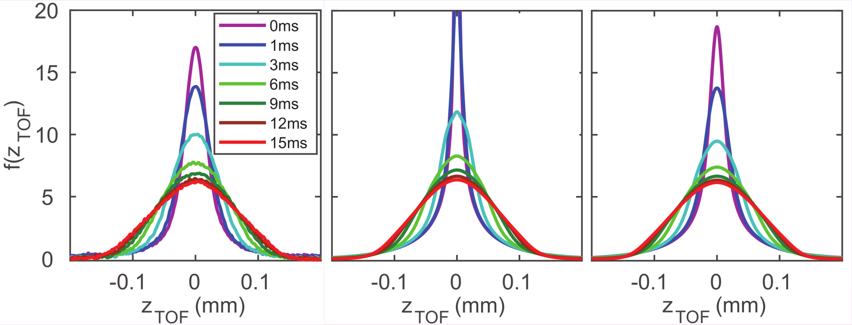

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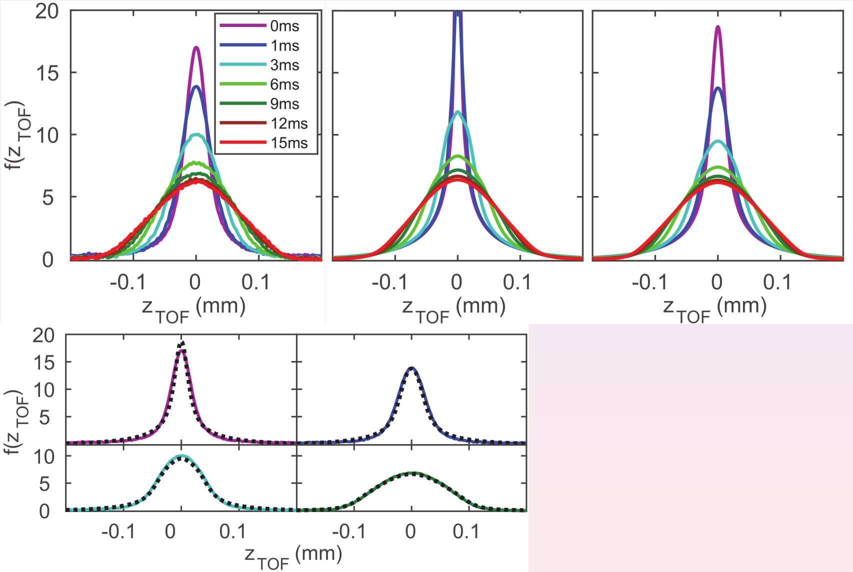

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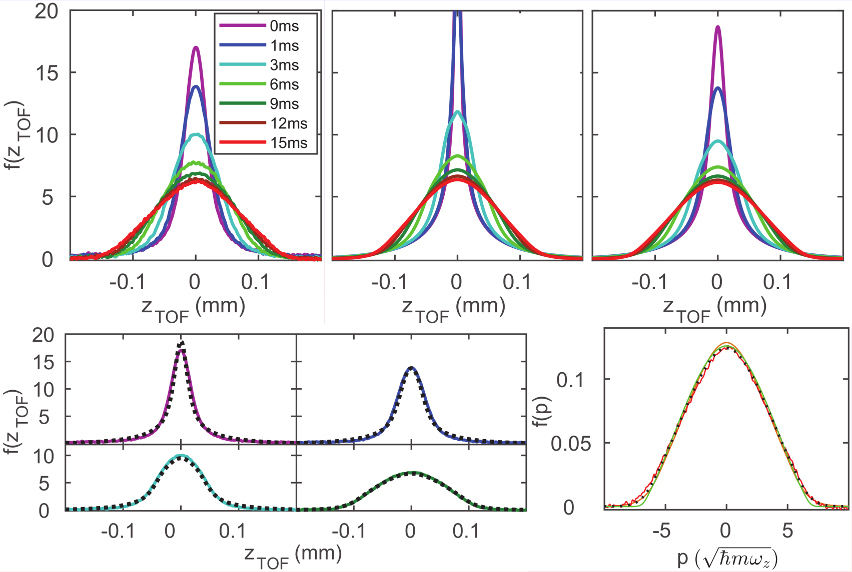

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Finite temperature

One-particle density matrix (grand-canonical ensemble)

$$\rho_{ij} \equiv \frac{1}{Z}\mathrm{Tr}\left\{\hat{b}_i^\dagger \hat{b}_j e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}, \quad Z = \mathrm{Tr}\left\{e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}$$

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$$\Downarrow$$

Mapping to noninteracting fermions

$$\rho_{ij} = \frac{1}{Z}\mathrm{Tr}\left\{\hat{f}_i^\dagger \hat{f}_j \prod_{k=1}^{j-1} e^{i\pi \hat{f}_k^\dagger \hat{f}_k} e^{-\frac{\hat{H}_F-\mu\sum_m \hat{f}_m^\dagger \hat{f}_m}{k_B T}} \prod_{l=1}^{i-1} e^{-i\pi \hat{f}_l^\dagger \hat{f}_l}\right\}$$

$$\Downarrow$$

Exact one-particle density matrix

$$\begin{aligned}\rho_{ij} \quad = \quad & \frac{1}{Z}\Big\{\det\left[\mathbf{I} + (\mathbf{I}+\mathbf{A})\mathbf{O}_1\mathbf{U}e^{-(\mathbf{E}-\mu\mathbf{I})/k_BT}\mathbf{U}^\dagger\mathbf{O}_2\right] \\ & -\det\left[\mathbf{I} + \mathbf{O}_1\mathbf{U}e^{-(\mathbf{E}-\mu\mathbf{I})/k_BT}\mathbf{U}^\dagger\mathbf{O}_2\right]\Big\}\end{aligned}$$

Computation time $\sim L^5$: 1000 sites

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Exact one-particle density matrix

$$\begin{aligned} \rho_{ij} \quad = \quad & \frac{1}{Z} \Big\{ \det \left[\mathbf{I} + (\mathbf{I} + \mathbf{A}) \mathbf{O}_1 \mathbf{U} e^{-(\mathbf{E} - \mu \mathbf{I})/k_B T} \mathbf{U}^\dagger \mathbf{O}_2 \right] \\ & - \det \left[\mathbf{I} + \mathbf{O}_1 \mathbf{U} e^{-(\mathbf{E} - \mu \mathbf{I})/k_B T} \mathbf{U}^\dagger \mathbf{O}_2 \right] \Big\} \end{aligned}$$

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MR, PRA **72**, 063607 (2005); W. Xu and MR, Phys. Rev. A **95**, 033617 (2017).

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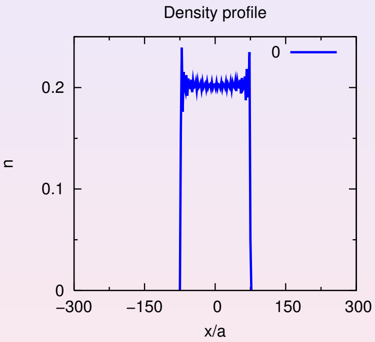
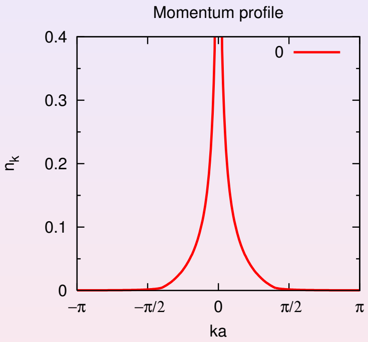
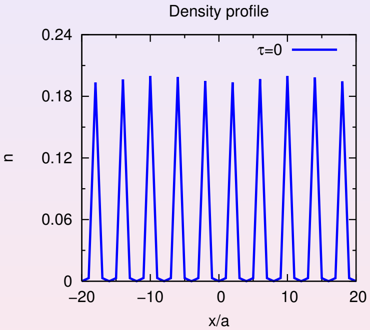
Relaxation dynamics in an integrable system

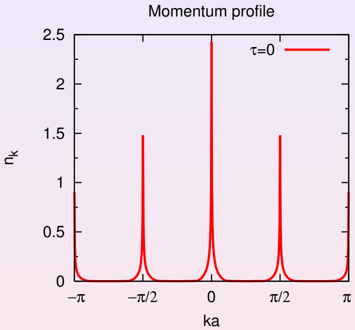




MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Relaxation dynamics in an integrable system





MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Generalized Gibbs ensemble (GGE)

Thermal equilibrium

$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu\hat{N}\right)/k_B T\right]$$

$$Z = \mathrm{Tr}\left\{\exp\left[-\left(\hat{H} - \mu\hat{N}\right)/k_B T\right]\right\}$$

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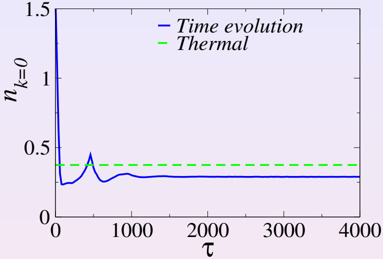
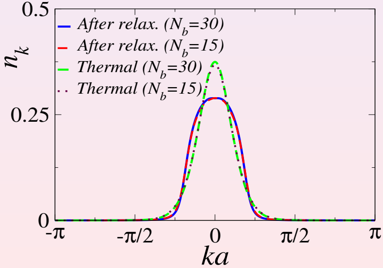
Evolution of $n_{k=0}$



n_k after relaxation



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Conserved quantities

(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$

$$\left\{\hat{I}_m\right\} = \{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^f\}$$

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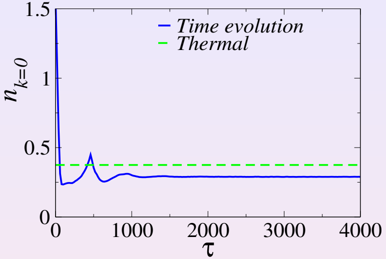

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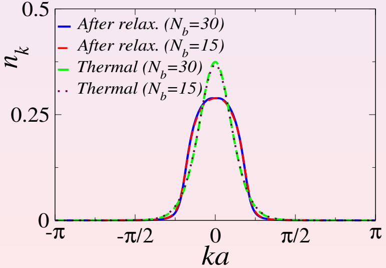

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Generalized Gibbs ensemble

$$\hat{\rho}_{\text{GGE}} = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m\right]$$

$$Z_c = \mathrm{Tr}\left\{\exp\left[-\sum_m \lambda_m \hat{I}_m\right]\right\}$$

$$\mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_{\text{GGE}}\right\} = \langle \hat{I}_m \rangle_{\tau=0}$$

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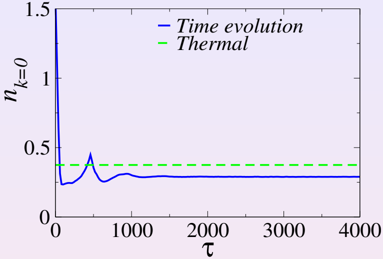

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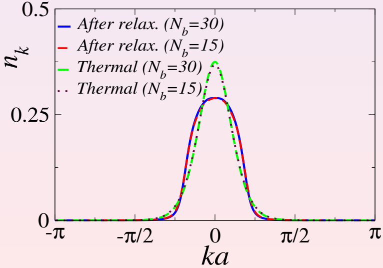

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$$\mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_{\text{GGE}}\right\} = \langle \hat{I}_m \rangle_{\tau=0}$$

result in

$$\lambda_m = \ln\left[\frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}}\right]$$

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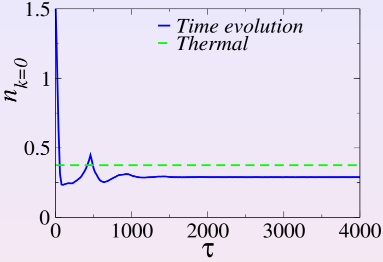

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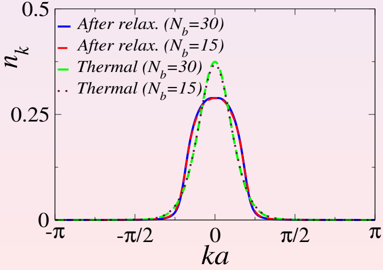

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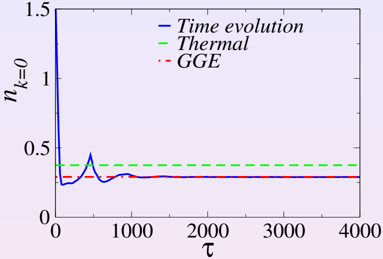

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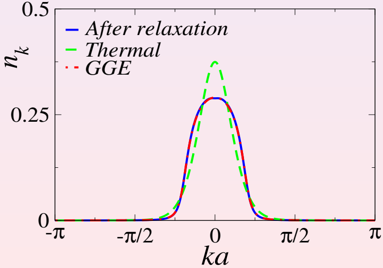

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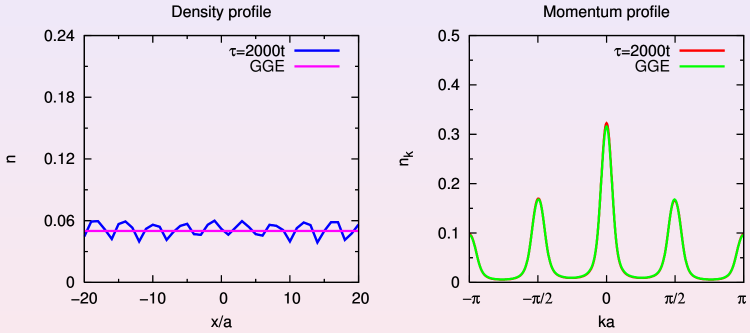

Outline

Exact results from quantum mechanics

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_{\text{ini}}\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E = \langle\psi_{\text{ini}}|\widehat{H}|\psi_{\text{ini}}\rangle,$$

then observables $\hat{O}$ evolve in time:

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_{\text{ini}}\rangle.$$

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One can rewrite

$$O(\tau) = \sum_{\alpha,\beta} C_\alpha^\star C_\beta e^{i(E_\alpha - E_\beta)\tau} O_{\alpha\beta} \quad \text{using} \quad |\psi_{\text{ini}}\rangle = \sum_\alpha C_\alpha|\alpha\rangle.$$

Taking the infinite time average (diagonal ensemble $\hat{\rho}_{\text{DE}} \equiv \sum_\alpha |C_\alpha|^2|\alpha\rangle\langle\alpha|$)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle \stackrel{?}{=} \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{DE}},$$

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_{\text{ini}}\rangle$.

Outline

Entropy of the GGE vs the diagonal entropy

$$S_{\mathsf{DE}} = -\mathsf{Tr}[\hat{\rho}_{\mathsf{DE}} \ln \hat{\rho}_{\mathsf{DE}}], \qquad S_{\mathsf{GGE}} = -\mathsf{Tr}[\hat{\rho}_{\mathsf{GGE}} \ln \hat{\rho}_{\mathsf{GGE}}]$$

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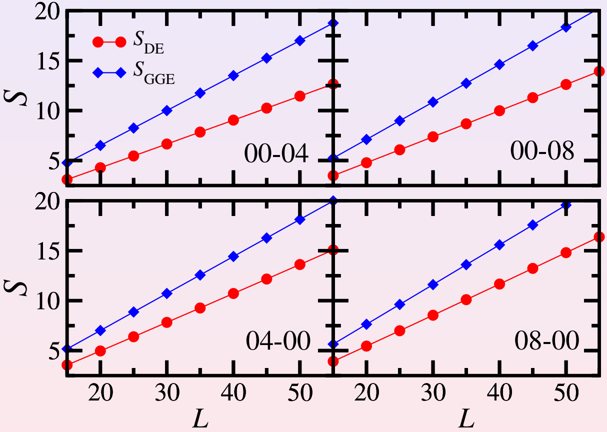
Entropies after quenches in a superlattice potential



S_{DE} & S_{GGE} are extensive but different!

Santos, Polkovnikov, and MR, Phys. Rev. Lett. **107**, 040601 (2011).

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The transverse-field Ising model

$$\hat{H}_{\mathrm{TFIM}} = -\sum_j \hat{S}^x_j \hat{S}^x_{j+1} - h \sum_j \hat{S}^z_j$$

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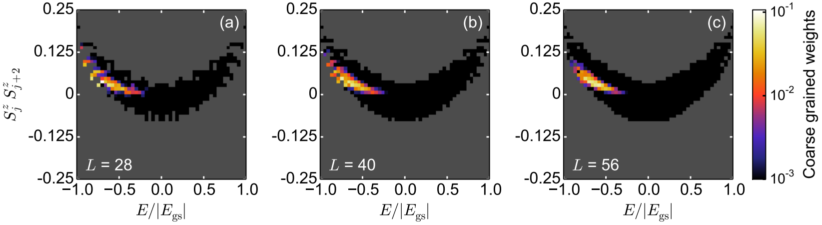
Generalized eigenstate thermalization:

A. C. Cassidy, C. W. Clark, and MR, PRL **106**, 140405 (2011).
L. Vidmar and MR, J. Stat. Mech. 064007 (2016).

Behind generalized thermodynamic Bethe ansatz approaches:
J.-S. Caux and F. H. L. Essler, PRL **110**, 257203 (2013).
B Pozsgay, J. Stat. Mech. P09026 (2014).

Generalized eigenstate thermalization (1D-TFIM)

Weight of eigenstate expectation values after equilibration



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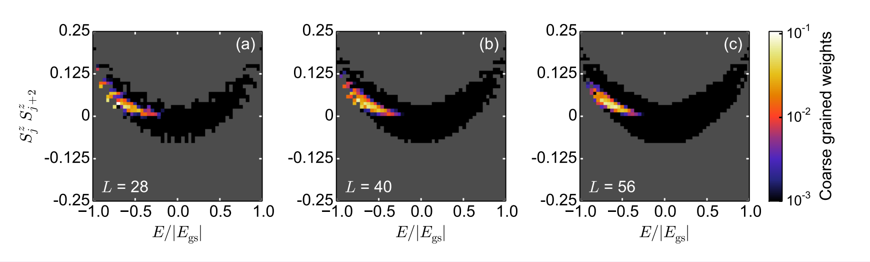
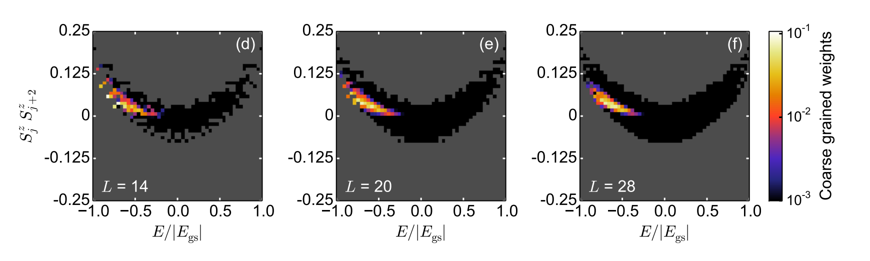


Weight of eigenstate expectation values in the GGE



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Variance of observable $\hat{\mathcal{O}}$ after quenches $h_0 \to h$ (ENS $= \mathrm{DE}, \mathrm{GGE}$)

$$\Sigma^2_{\hat{\mathcal{O}},\text{ENS}} = \sum_n \rho_n^{\text{ENS}} \langle n|\hat{\mathcal{O}}|n\rangle^2 - \left(\sum_n \rho_n^{\text{ENS}} \langle n|\hat{\mathcal{O}}|n\rangle\right)^2$$

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For all local observables for which $\Sigma^2_{\hat{\mathcal{O}},\text{ENS}}$ was calculated analytically

$$\frac{\Sigma^2_{\hat{\mathcal{O}},\mathrm{DE}}}{\Sigma^2_{\hat{\mathcal{O}},\mathrm{GGE}}} = 2, \qquad \text{and} \qquad \Sigma_{\hat{\mathcal{O}},\text{ENS}} \sim \frac{1}{\sqrt{L}}$$

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For example, for quenches across the critical field:

$$\Sigma^2_{\hat{S}^x_j \hat{S}^x_{j+1},\mathrm{DE}} = \begin{cases} \frac{1}{64L}\left[1 + h_0^2 - \frac{2h_0}{h}\right] & \text{if} \quad h > 1,\ h_0 < 1 \\ \frac{1}{64L}\left[4 - 3h^2 - \left(\frac{h}{h_0}\right)(4 - 2h^2) + \left(\frac{h}{h_0}\right)^2\right] & \text{if} \quad h < 1,\ h_0 > 1 \end{cases}$$

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- The assumption of local equilibration to the GGE is a starting point for generalized hydrodynamics theories
  ★O. A. Castro-Alvaredo, B. Doyon, T. Yoshimura, PRX **6**, 041065 (2016).
  ★B. Bertini, M. Collura, J. De Nardis, M. Fagotti, PRL **117**, 207201 (2016).

Collaborators

- Vanja Dunjko (U Mass Boston)
- Alejandro Muramatsu
  (Buenos Aires 1951- Stuttgart 2015)
- Maxim Olshanii (U Mass Boston)
- Anatoli Polkovnikov (Boston U)

- Lea F. Santos (Yeshiva U)
- Lev Vidmar (Jožef Stefan Institute)
- David Weiss & group (Penn State)
- Vladimir Yurovsky (Tel Aviv U)

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