# Holography of Information in AdS/CFT

## Fifth Mandelstam School and Workshop
## (based on arXiv:2210.11066 with Garry Kemp)

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# Outline

The principle of the holography of information.

Vector model/higher spin duality.

Bilocal holography for the vector model.

Convergence of the (Lorentzian) OPE.

Conclusions.

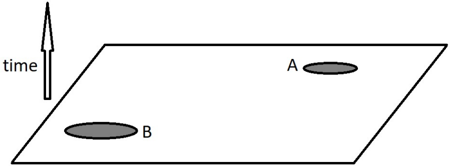
# Locality

Locality is a cherished principle in physics.

Relativistic causality says no information carrying signal propagates faster than light. Implemented by ensuring spacelike separated fields commute.

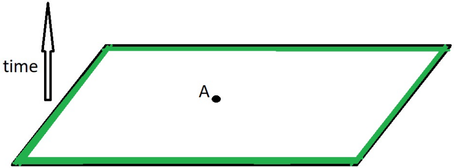
Ensures that laboratories in spacelike separated regions function independently.

Formalized in algebraic formulation of QFT as the split property: we can specify the state of quantum fields independently on different parts of a Cauchy slice.



# The Principle of the Holography of Information

*In a theory of quantum gravity, a copy of all the information available on a Cauchy slice is also available near the boundary of the Cauchy slice. This redundancy in description is already visible in the low-energy theory.*




Laddha, Prabhu, Raju, Shrivastava, arXiv:2002.02448;
Chowdhury, Papadoulaki, Raju, arXiv:2008.01740;
Raju, arXiv:2110.05470.

# The Principle of the Holography of Information

*In a theory of quantum gravity, a copy of all the information available on a Cauchy slice is also available near the boundary of the Cauchy slice. This redundancy in description is already visible in the low-energy theory.*


Changing the values of fields within $R$ changes field values in the green band. The split property fails.

Raju, arXiv:2110.05470.

# "Reeh-Schleider" for Quantum Mechanics

Consider a maximally entangled state of two qubits

$$|\psi\rangle = \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}$$

Define the operator which acts only on the first qubit

$$\vec{\sigma}_{(1)} \equiv \vec{\sigma} \otimes \mathbf{1}$$

Then

$$\sigma^{-}_{(1)}|\psi\rangle \quad \propto \quad |\downarrow\rangle\, |\downarrow\rangle \qquad \sigma^{+}_{(1)}|\psi\rangle \quad \propto \quad |\uparrow\rangle\, |\uparrow\rangle$$

$$(\sigma^{z}_{(1)} + \mathbf{1})|\psi\rangle \quad \propto \quad |\uparrow\rangle\, |\downarrow\rangle \qquad (\sigma^{z}_{(1)} - \mathbf{1})|\psi\rangle \quad \propto \quad |\downarrow\rangle\, |\uparrow\rangle$$

We can generate a basis for Hilbert space by acting only on the first qubit in the entangled pair.

# Reeh-Schleider Theorem (QFT)

$|\Omega\rangle \equiv$ vacuum state. $\mathcal{H}_0 \equiv$ vacuum sector of the full Hilbert space $\mathcal{H}$.

Assume algebra of local fields is generated by a hermitian scalar field $\phi(x^\mu)$.

Introduce the smeared field $\phi_f \equiv \int d\vec{x} f(x^\mu)\phi(x^\mu)$ and states

$$|\Psi_{\{f_1,\cdots,f_n\}}\rangle = \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle$$

Let $\Sigma$ be a Cauchy hypersurface, let $\mathcal{V} \subset \Sigma$ be an arbitrarily small open set and let $\mathcal{U}_\mathcal{V}$ be a small neighbourhood of $\mathcal{V}$ in spacetime.

Reeh-Schleider theorem states restricting $f_i$ to support in $\mathcal{U}_\mathcal{V}$, the states $|\Psi_{\{f_1,\cdots,f_n\}}\rangle$ generate $\mathcal{H}_0$.

Reflects enormous entanglement in quantum field theory vacuum.

# Reeh-Schleider Theorem

$$|\Psi_{\{f_1,\cdots,f_n\}}\rangle = \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle \qquad \phi_f \equiv \int d\vec{x} f(x^\mu)\phi(x^\mu)$$

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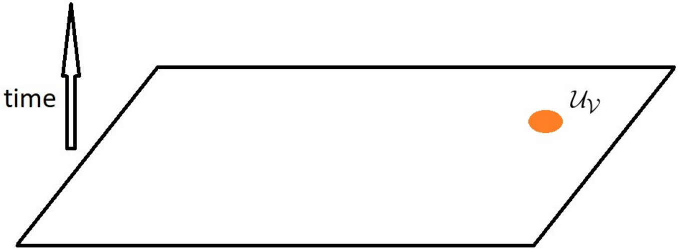


# Gauss Law for Gravity

The Gauss law $\Rightarrow$ energy of a state in gravity can be measured from near the boundary

ADM Hamiltonian is a surface term.

$\Rightarrow$ the projector onto the state of lowest energy

$$P_\Omega = |\Omega\rangle\langle\Omega|$$

is an element of the boundary algebra of operators.

# Holography of Information

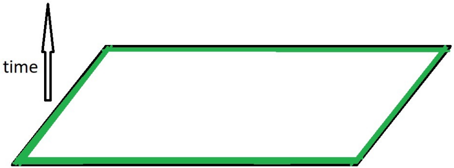
Any observable $\mathcal{O}$ in $\mathcal{H}_0$ is a linear combination of operators of the form

$$|a\rangle\langle b|$$

with $|a\rangle$ and $|b\rangle$ any states in $\mathcal{H}_0$: $\mathcal{O} = \sum_{a,b} c_{ab}|a\rangle\langle b|$.

By Reeh-Schleider theorem, the complete set of these operators can be written as

$$|a\rangle\langle b| = \phi_{f_1^{(a)}}\phi_{f_2^{(a)}}\cdots\phi_{f_{n(a)}^{(a)}}|\Omega\rangle\langle\Omega|\phi_{f_1^{(b)}}\phi_{f_2^{(b)}}\cdots\phi_{f_{n(b)}^{(b)}}$$



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Gauss law implies that $P_\Omega = |\Omega\rangle\langle\Omega| \in$ is an element of the boundary algebra of operators, and a product of operators in boundary algebra $\in$ boundary algebra $\Rightarrow$ the complete set of operators $|a\rangle\langle b|$ belong to the boundary algebra.

Consequently, any observable (including operators one naively thinks are localized deep in the bulk of spacetime) acting on $\mathcal{H}_0$ is an element of the boundary algebra of operators.

# Goal and Comments

The principle of the holography of information is pointing out an unusual localization of quantum information in quantum gravity.

AdS/CFT gives a non-perturbative definition of quantum gravity on negatively curved spacetime, as a CFT. Can we see the principle of the holography of information in the CFT?

In AdS/CFT is the holography of information trivially true? The principle is a statement about quantum gravity itself. The proof does not invoke AdS/CFT and it holds (for example) for flat spacetime.

Use AdS/CFT to map the principle of the holography of information into a statement about CFT and explore it using only CFT.

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# Vector model / Higher spin duality

Klebanov and Polyakov, hep-th/0210114

Free $O(N)$ vector model in $2+1$ dimensions

$$S = \int d^3x \sum_{a=1}^{N} \left( \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a \right)$$

Single trace primaries: $O_{\Delta=1}(t,\vec{x}) = \sum_{a=1}^{N} \phi^a(t,\vec{x})\phi^a(t,\vec{x})$

$$J_{\mu_1\mu_2\cdots\mu_{2s}}(t,\vec{x})\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_{2s}} = \sum_{a=1}^{N}\sum_{k=0}^{2s} \frac{(-1)^k \; :(\alpha\cdot\partial)^{2s-k}\phi^a \; (\alpha\cdot\partial)^k\phi^a :}{k!(2s-k)!\Gamma(k+\frac{1}{2})\Gamma(2s-k+\frac{1}{2})}$$

The gravity dual is higher spin gravity in $AdS_4$, with a bulk scalar and a gauge field $A_{\mu_1\cdots\mu_{2s}}$ for each conserved current.

# Large $N$ Higher spin equations of motion

Metsaev, arXiv:hep-th/9906217

Work in Poincare patch of $AdS_4$

$$X^+ \equiv X^2 + X^0, \quad X^- \equiv X^2 - X^0, \quad X \equiv X^1, \quad Z$$

Work in lightcone gauge

$$A_{+\mu_2\cdots\mu_{2s}} = 0$$

All components $A_{-\mu_2\cdots\mu_{2s}}$ are determined by constraints. Dynamical fields are $X, Z$ polarizations: $A_{XZXZ\cdots ZZ}$.

Free equation of motion is (obtained after gauge fixing and solving the constraint)

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-} + \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)\frac{A_{XXZ\cdots ZX}}{Z} = 0$$

# Conformal symmetry

Metsaev, arXiv:hep-th/9906217

Work out the conformal generators after reducing to physical degrees of freedom:

• Fix a gauge.

• Conformal transformation must be supplemented with a compensating gauge transformation that restores the gauge.

• Solve the symmetric and traceless constraints.

Result is a set of transformation defined on $A_{xxz\cdots zx}$ fields.

# Repackaging Higher Spin Gravity

An infinite number of spinning fields in $\mathrm{AdS}_4$ = a single field on $\mathrm{AdS}_4 \times \mathrm{S}^1$

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z$

$$ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$$

Fields: $A_{\mu_1 \cdots \mu_{2s}}(X^+, X^-, X, Z)$, $\Phi(X^+, X^-, X, Z)$

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z \qquad \theta$

$$ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$$

Fields: $\Phi(X^+, X^-, X, Z, \theta)$

$$\Phi(X^+, X^-, X, Z, \theta) = \sum_{s=0}^{\infty} \left( \cos(2s\theta) \frac{A^{XX \cdots XX}}{Z} + \sin(2s\theta) \frac{A^{XX \cdots XZ}}{Z} \right)$$

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Basic claim: Holography is accomplished by a change to gauge invariant (bilocal) field variables in the CFT. A change of spacetime coordinates is needed to give the bulk interpretation of the bilocal collective field theory.

Das and Jevicki, hep-th/0304093

# Bilocal Holography

Basic claim: Holography is accomplished by a change to gauge invariant (bilocal) field variables in the CFT. A change of spacetime coordinates is needed to give the bulk interpretation of the bilocal collective field theory.

Das and Jevicki, hep-th/0304093

The loop expansion parameter of the original CFT is $\hbar$. After changing to invariant (bilocal) variables the loop expansion parameter is $1/N$ matching the loop expansion parameter of the dual gravity.

Jevicki and Sakita, Nucl. Phys. B **165** (1980), 511

The bilocal transforms in a tensor product. The complete collection of higher spin fields transform in a direct sum. The natural change of basis

$$V_{[\frac{1}{2},0]} \otimes V_{[\frac{1}{2},0]} \longrightarrow V_{[1,0]} \ \oplus \bigoplus_{s=2,4,\cdots} V_{[s+1,s]}$$

determines a map between CFT and bulk coordinates.

(Think of addition of angular momentum: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.)

# Change of Variables

To solve QFT "all" we need to do is evaluate a complicated integral.

$$\int d\phi^a e^{-S(\phi^a)} \qquad a = 1, \cdots, N$$

Hard to do when $N \to \infty$, but things simplify when the theory has an $O(N)$ symmetry, so the action is an $O(N)$ invariant.

Suppose the $\phi^a$ are in vector rep of $O(N)$. Then $S$ is a function of $\sigma = \phi^a\phi^a$, the unique invariant. One integration variable and not $N$ - much simpler!

$$\int d\sigma e^{-N\tilde{S}(\sigma)}$$

$N$ appears because we had a total of $N$ variables. Saddle point approximation produces an expansion with $1/N$ the loop counting parameter.

2+1 Minkowski in light cone coordinates $x^+ \equiv x^0 + x^2$, $x^- \equiv x^2 - x^0$, $x \equiv x^1$. Invariant variables

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sum_{a=1}^{N} \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2)$$

# Change of Variables

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Invariant variables

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The field $\sigma(x^+, x_1^-, x_1, x_2^-, x_2)$ develops a large $N$ expectation value. It is the fluctuation $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ about this large $N$ background that maps to bulk AdS fields.

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \frac{1}{\sqrt{N}}\eta(x^+, x_1^-, x_1, x_2^-, x_2)$$

$\sigma_0(x^+, x_1^-, x_1, x_2^-, x_2)$ is the large $N$ two point function.

RdMK, Jevicki, Jin and Rodrigues, arXiv:1008.0633

# Change of Co-ordinates

The bilocal transforms in $V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0}$ $\qquad (L^A \in \mathrm{so}(2,3))$

$$L^A_{\otimes}\sigma = \Big( L^A \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) + \phi^a(x^+, x_1^-, x_1) L^A \phi^a(x^+, x_2^-, x_2) \Big)$$

$$V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0} = V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s}$$

The complete collection of higher spin fields fill out the reducible representation $V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s}$

$$L^A_{\oplus}\Phi(X^+, X^-, X, Z, \theta) = \sum_{s=0}^{\infty} \left( \cos(2s\theta) L^A_{2s} \frac{A^{XX\cdots XX}}{Z} + \sin(2s\theta) L^A_{2s} \frac{A^{XX\cdots XZ}}{Z} \right)$$

We want to change from the natural representation ($L^A_{\otimes}$) of the CFT to the representation that is natural for the bulk gravity ($L^A_{\oplus}$).

# Change of Coordinates

Symmetry: $X^- \to X^- + a$ in gravity and $x^- \to x^- + b$ in CFT motivates Fourier transform: $X^- \to P^+$ and $x^- \to p^+$

$$x_1 = X + Z\tan\left(\frac{\theta}{2}\right) \qquad x_2 = X - Z\cot\left(\frac{\theta}{2}\right) \qquad x^+ = X^+$$
$$p_1^+ = P^+\cos^2\left(\frac{\theta}{2}\right) \qquad p_2^+ = P^+\sin^2\left(\frac{\theta}{2}\right)$$

$$X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}$$
$$P^+ = p_1^+ + p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$L^A_\oplus \Phi = 2\pi P^+ \sin\theta \; L^A_\otimes \eta$$

RdMK, Jevicki, Jin and Rodrigues, arXiv:1008.0633

# Summary: Bilocal Holography

$$\begin{aligned}
\sigma(x^+, x_1^-, x_1, x_2^-, x_2) &= \sum_{a=1}^{N} \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) \\
&= \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \frac{1}{\sqrt{N}}\eta(x^+, x_1^-, x_1, x_2^-, x_2)
\end{aligned}$$

$$X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+} \qquad X^+ = x^+$$

$$P^+ = p_1^+ + p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$\begin{aligned}
\Phi &= \sum_{s=0}^{\infty}\left(\cos(2s\theta)\frac{A^{XX\cdots XX}(X^+, P^+, X, Z)}{Z} + \sin(2s\theta)\frac{A^{XX\cdots XZ}(X^+, P^+, X, Z)}{Z}\right) \\
&= 2\pi P^+ \sin\theta \; \eta(X^+, P^+\cos^2\frac{\theta}{2}, X + Z\tan\frac{\theta}{2}, P^+\sin^2\frac{\theta}{2}, X - Z\cot\frac{\theta}{2})
\end{aligned}$$

We are studying the free O(N) vector model. Its a UV fixed point.

By perturbing and flowing we can reach an IR fixed point. The bilocal holography for this fixed point has also been worked out. Mulokwe and Rodrigues, arXiv:1808.00042, Johnson, Mulokwe and Rodrigues, arXiv:2201.10214

# Bulk Reconstruction

Does the proposed bulk field $\Phi(X^+, P^+, X, Z, \theta)$ obey the correct bulk equation of motion with the correct boundary condition? CFT equation of motion:

$$\left(\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + \frac{\partial^2}{\partial x^2}\right)\phi^a(x^+, x^-, x) = 0$$

implies

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-} + \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)\Phi\left(X^+, X^-, X, Z, \theta\right) = 0$$

$$(p_1^+ + p_2^+)^s \cos\left(2s \arctan\sqrt{\frac{p_2^+}{p_1^+}}\right) = \mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k (p_1^+)^{s-k}(p_2^+)^k}{\Gamma\left(s-k+\frac{1}{2}\right)\left(\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!\right)}$$

implies that ($\mathcal{N} = \Gamma\left(\frac{1}{2}\right)s!\Gamma\left(s+\frac{1}{2}\right)$)

$$\frac{\partial^s}{\partial X^{-s}}\Phi_s(X^+; X^-, X, 0) = 16\pi\mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k \partial_-^{s-k}\phi^a(X^+, X^-, X)\partial_-^k\phi^a(X^+, X^-, X)}{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!}$$

# Consistency with GKPW dictionary

GKPW is usually formulated in de Donder gauge: $D^A H_{AA_2\cdots A_{2s}} = 0$. The residual gauge symmetry can be used to make $H_{A_1A_2\cdots A_{2s}}$ traceless.

From e.o.m. near $Z=0$ we have ($M$ is a boundary index - does not take $Z$ values)

$$H_{M_1\cdots M_{2s}} \sim Z^{2-2s} B_{M_1\cdots M_{2s}}(X^+,X^-,X) + Z^{1+2s} A_{M_1\cdots M_{2s}}(X^+,X^-,X)$$

$$H_{M_1\cdots M_{2s-k}Z\cdots Z} \sim Z^{2-2s-k} B_{M_1\cdots M_{2s}}(X^+,X^-,X) + Z^{1+2s+k} A_{M_1\cdots M_{2s}}(X^+,X^-,X)$$

GKPW says:

$$j_{M_1,M_2\cdots M_{2s}} \propto A_{M_1,M_2\cdots M_{2s}}$$

CFT operator is related to component of boundary field falling off as $Z^{1+2s}$.

# Consistency with GKPW dictionary

Transform to lightcone gauge: $H'_{A_1 A_2 \cdots A_{2s}} = H_{A_1 A_2 \cdots A_{2s}} - D_{(A_1} \Lambda_{A_2 \cdots A_{2s})}$

Requiring $H'_{+A_2 \cdots A_{2s}} = 0$ fixes the gauge parameter $\Lambda_{A_1 \cdots A_{2s-1}}$.

**Example: Spin** $2s$ After the gauge transformation, GKPW says

$$A'_{XXX \cdots X} = -A'_{ZZX \cdots X} \propto Z \partial_-^{-2s} j_{-- \cdots --}$$

$$\Rightarrow \partial_-^{2s} \frac{A'_{XXX \cdots X}}{Z} = -\partial_-^{2s} \frac{A'_{ZZX \cdots X}}{Z} \propto j_{-- \cdots --}$$

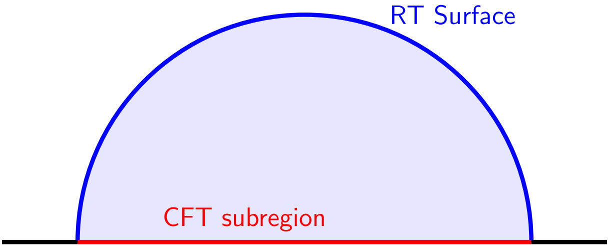
so that bilocal holography does indeed constitute a correct bulk reconstruction.

Mintun and Polchinski, arXiv:1411.3151

# Subregion Duality

Which subregion of the CFT (if any) is dual to a given subregion of the bulk?

Entanglement wedge reconstruction claims that everyting from the boundary up to the RT surface can be reconstructed.



# Subregion Duality

Which subregion of the CFT (if any) is dual to a given subregion of the bulk?

Consider localized CFT excitations, at time $x^+$, with the first at $(x_1, p_1^+)$ and the at $(x_2, p_2^+)$, described as wavepackets, tightly peaked at $x_1$ and $x_2$ along direction $x$ transverse to the light cone, and smeared along $x^-$.

$$\left(X - \frac{x_1 + x_2}{2}\right)^2 + Z^2 = \left(\frac{x_1 - x_2}{2}\right)^2$$

The bulk excitation is on a semicircle in the $X, Z$ plane, with radius $(x_1 - x_2)/2$ and center $X = (x_1 + x_2)/2$ and $Z = 0$. To locate the excitation specify angle $\theta$

$$\tan\theta = \frac{Z}{X - \frac{x_1+x_2}{2}} = \frac{2\sqrt{p_1^+ p_2^+}}{p_1^+ - p_2^+}$$

This angle $\theta$ is the angle $\theta$ appearing in the map.

# Subregion Duality

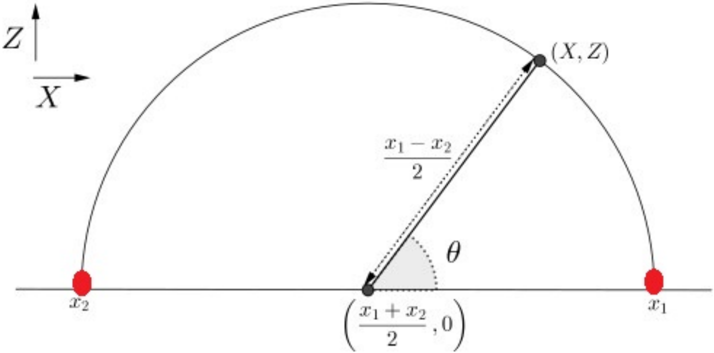



Figure: The bilocal describing excitations localized at $(x_1, p_1^+)$ and $(x_2, p_2^+)$ corresponds to a bulk excitation localized at $(X, Z)$ as shown. This figure lives on a constant $x^+ = X^+$ slice. The angle $\theta$ is $\theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$.

# Bulk Reconstruction

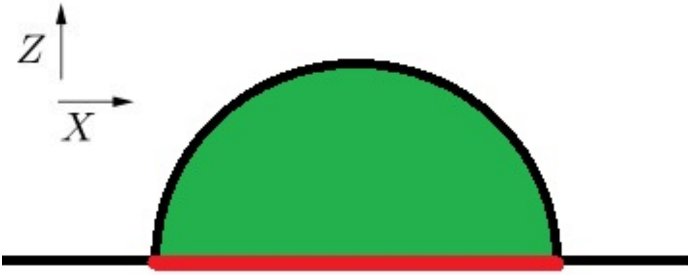



Figure: Using bilocals restricted to the red region of the CFT we are able to reconstruct bulk field living in the area shaded in green. This figure is for fixed time $x^+ = X^+$.

What is the interpretation of the boundary of the green region?

# Subregion Duality

CFT subregion: $x^+ = 0$, $-\infty \le x^- \le \infty$, $-L \le x \le L$.
Bulk excitations: $X^+ = 0$ and

$$X^2 + Z^2 \le L^2 \qquad -\infty \le X^- \le \infty \qquad 0 \le \theta \le \pi$$

The metric of $\mathrm{AdS}_4$ on a constant $X^+$ slice is

$$ds^2 = \frac{dX^2 + dZ^2}{Z^2}$$

Consider region $\mathcal{E}_{\mathcal{R}}$ of the $(X, Z)$ plane bounded by $Z = 0$ and the curve $Z = Z(X)$. The length of the boundary of this region is given by

$$\ell = \int dX \frac{\sqrt{Z'^2 + 1}}{Z} \qquad Z' = \frac{dZ}{dX}$$

Minimizing we find

$$\frac{dZ}{dX} = \frac{\sqrt{R^2 - Z^2}}{Z} \qquad \Rightarrow \qquad X^2 + Z^2 = R^2$$

$\mathcal{E}_{\mathcal{R}}$ is reminiscent of the entanglement wedge.

# Bulk Reconstruction

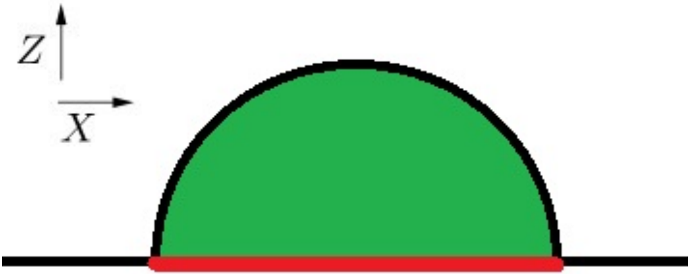



Figure: Using bilocals restricted to the red region of the CFT we are able to reconstruct bulk field living in the area shaded in green. This figure is for fixed time $x^+ = X^+$.

Using bilocal holography we can derive the entanglement wedge reconstruction result for higher spin gravity.

RdMK, E. Gandote, N. H. Tahiridimbisoa and H. J. R. Van Zyl, arXiv:2106.00349

# Bulk Reconstruction: Quantum Error Correction

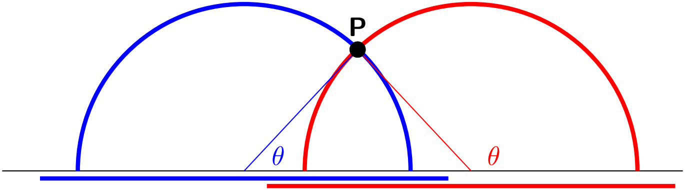



Figure: The bulk field is located at bulk point $P$ specified by $(X^+, P^+, X, Z)$. A field at this point can be constructed using the bilocal $\sigma(x^+, p_1^+, x_1, p_2^+, x_2)$ or using the bilocal $\sigma(x^+, p_1'^+, x_a, p_2'^+, x_b)$. $\theta$ differs in the two reconstructions so that we reconstruct a different linear combination of fields at the given bulk point.

$$\Phi = \sum_{s=0}^{\infty} \left( \cos(2s\theta) \frac{A^{XX\cdots XX}(X^+, P^+, X, Z)}{Z} + \sin(2s\theta) \frac{A^{XX\cdots XZ}(X^+, P^+, X, Z)}{Z} \right)$$

RdMK, E. Gandote, N. H. Tahiridimbisoa and H. J. R. Van Zyl, arXiv:2106.00349

# Bulk Reconstruction: Quantum Error Correction

We now want to study the physics of a low energy observer. In this case we restrict to a code subspace obtained by truncating to a subspace of the full Hilbert space.

We cut off both

- the number of fields that can be reconstructed and
- the occupation number of each field.

For illustration consider the most drastic reduction to just a single field.

$$\begin{aligned}\Phi &= \sum_{s=0}^{\infty}\left(\cos(2s\theta)\frac{A^{XX\cdots XX}(X^+,P^+,X,Z)}{Z}+\sin(2s\theta)\frac{A^{XX\cdots XZ}(X^+,P^+,X,Z)}{Z}\right)\\ &\to \frac{A(X^+,P^+,X,Z)}{Z}\end{aligned}$$

independent of $\theta$. The code subspace cut off always restricts dependence on $\theta$ and hence some fluidity in the bulk reconstruction.

# Bulk Reconstruction: Quantum Error Correction

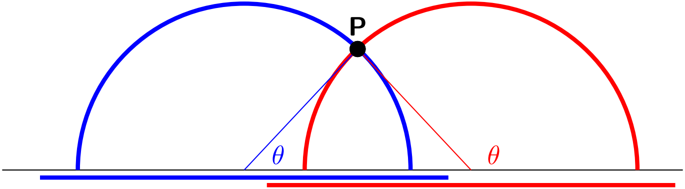



Figure: After restricting to the code subspace the bulk field located at bulk point $P$ specified by $(X^+, P^+, X, Z)$ can be constructed using the bilocal $\sigma(x^+, p_1^+, x_1, p_2^+, x_2)$ or using the bilocal $\sigma(x^+, p_1'^+, x_a, p_2'^+, x_b)$ due to fluidity in the reconstruction of bulk fields from bilocals in the conformal field theory. There are an infinite number of different possible reconstructions corresponding to the fact that an infinite number of semi-circles with distinct endpoints, all passing through $P$, can be drawn.

RdMK, E. Gandote, N. H. Tahiridimbisoa and H. J. R. Van Zyl, arXiv:2106.00349

# Location of single trace primaries

Where do single trace primaries map to in the $AdS_4$ bulk? ($Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}$)

Scalar primary $= \phi^a(x^+, x^-, x)\phi^a(x^+, x^-, x)$ i.e. located on boundary $Z = 0$.

Conserved currents are

$$\begin{aligned} J_s(x^+, x^-, x, \alpha) &= J_{\mu_1\mu_2\cdots\mu_s}(x^+, x^-, x)\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_s} \\ &= \sum_{k=0}^{s} \frac{(-1)^k \; :(\alpha\cdot\partial)^{s-k}\phi^a(x^+, x^-, x)\;(\alpha\cdot\partial)^k\phi^a(x^+, x^-, x):}{k!(s-k)!\Gamma(k+\frac{1}{2})\Gamma(s-k+\frac{1}{2})} \\ &= \sum_{k=0}^{s} \frac{(-1)^k\,(\alpha\cdot\partial_1)^{s-k}\;(\alpha\cdot\partial_2)^k}{k!(s-k)!\Gamma(k+\frac{1}{2})\Gamma(s-k+\frac{1}{2})}\eta(x^+, x_1^-, x_1, x_2^-, x_2)\Big|_{x_1=x_2=x, x_1^-=x_2^-=x^-} \end{aligned}$$

To construct spinning currents it is enough to separate $x_1$ and $x_2$ by a small amount $\epsilon$, evaluate the derivatives and then send $x_2 \to x_1$. We can $|x_1 - x_2| < \epsilon$ where $\epsilon$ can be arbitrarily small.

# Location of single trace primaries

Where do single trace primaries map to in the $AdS_4$ bulk?

$$Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}$$

$$|x_1 - x_2| < \epsilon \qquad \Rightarrow \qquad Z < \epsilon$$

where $\epsilon$ can be arbitrarily small.

The complete set of single trace primary operators, after mapping to the dual gravity, are supported in an arbitrarily small neighbourhood of the boundary.

# Operators deep in the bulk

Which CFT bilocals map to operators localized deep in the bulk of $AdS_4$?

$$Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}$$

$p_1^+$ and $p_2^+$ are both positive $\Rightarrow$ the ratio $0 < \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} < 1$.

To obtain a large value for $Z$ we must consider a large separation between the two fields in the bilocal, i.e. $x_1 - x_2$ must be large.

# Holography of information and OPE

HOI predicts that bulk operators can be expressed as elements of the boundary algebra.

All single trace primary operators are supported in an arbitrarily small neighbourhood of the boundary.

By separating $x_1$ and $x_2$ to be arbitrarily distant, the bilocal field $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ corresponds to a bulk operator located arbitrarily deep in the bulk.

HOI is true if we can replace the bilocal field by a sum of single trace primaries.

This is exactly what the OPE does!
$(\eta(x^+, x_1^-, x_1, x_2^-, x_2) =: \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2 :)$

Burden of the proof is to show the OPE converges....

# Outline

The principle of the holography of information.

Vector model/higher spin duality.

Bilocal holography for the vector model.

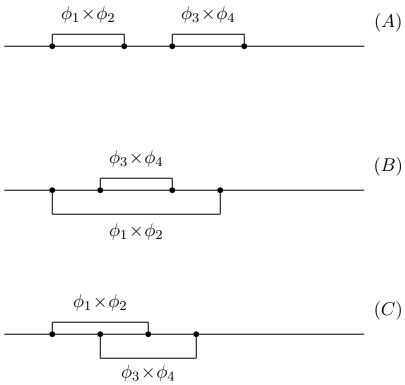
**Convergence of the (Lorentzian) OPE.**

Conclusions.

# Convergence of the OPE

An important point: the radius of convergence of the OPE inside a correlator is not predetermined but depends on the next-closest operator insertion.

We might spoil the convergence of the OPE with straddling bilocals.



# Convergence of the OPE

Work in Euclidean space: using radial quantization we map OPE convergence into the question about convergence of vectors in Hilbert space. (Pappadopulo, Rychkov, Espin, Rattazzi, arXiv:1208.6449)

Analytically continue to Minkowski space. (Qiao, arXiv:2005.09105)

Convergence results are stated using the conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} \qquad\qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

If neither $z$ nor $\bar{z}$ belong to $(1,\infty)$ then the $s$-channel OPE is convergent.

# OPE for $O(N)$ model

The OPE of $\phi^a$ with itself includes the currents $J_{2s}$ and $J_0$ as well as their descendants so that

$$\sum_{a=1}^{N} :\phi^a(x^\mu + y^\mu)\phi^a(x^\mu - y^\mu): \quad = \quad \sum_{s=0}^{\infty}\sum_{d=0}^{\infty} c_{sd}\left(y^\mu \frac{\partial}{\partial x^\mu}\right)^{2d} J_{2s}(y,x)$$

where

$$c_{0d} = \frac{1}{2^{2d}(d!)^2} \qquad \text{and} \qquad c_{sd} = \frac{(2s)!(4s-1)!!}{d!2^{2d+4s-1}(d+2s)!} \qquad\qquad s>0$$

# Convergence Results

As an explicit check of OPE convergence, calculate the s-channel OPE for the case $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4)$. This should converge.

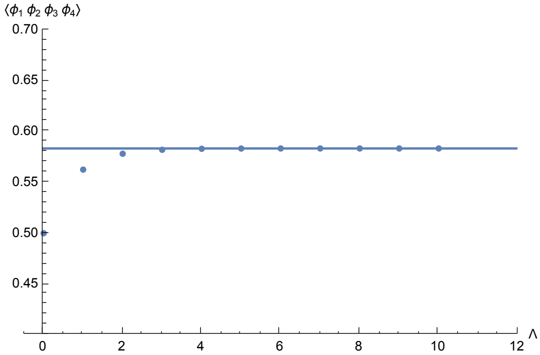
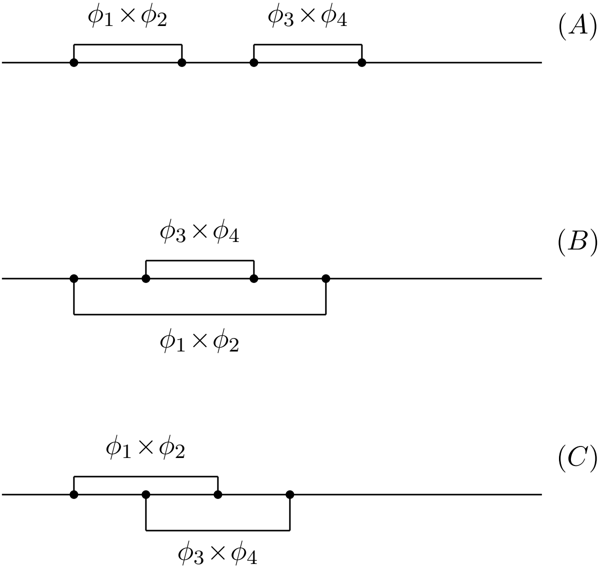



Figure: We compare the exact four-point function $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ value with the series expansions obtained from the OPE. The exact value of the four point function is the horizontal line at $7/12$. The series expansion is truncated with cut off $\Lambda$. The horizontal axis shows the value of $\Lambda$. Convergence is extremely rapid.

# OPE Configurations



# Convergence Results

Consider $(x_1, x_2, x_3, x_4) = (1, 3, 2, 4)$. The $s$-channel OPE should not converge.

The exact value of the four point function is $4/3$.

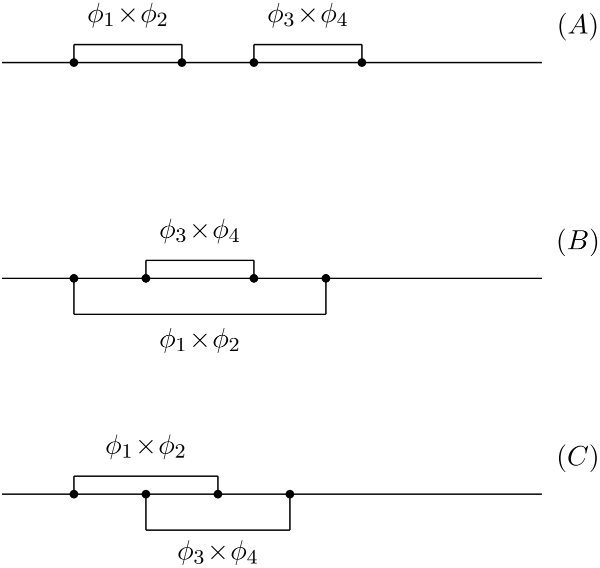
Truncating the series obtained from the OPE, with cut off values $\Lambda = 0, 1, 2, \cdots, 7$ we obtain

$$\{2, 6, 22, 86, 342, 1366, 5462, 21846\}$$

for the value of the sum.

The sum is clearly diverging.

# OPE Configurations



# Convergence Results

Finally, study $(x_1, x_2, x_3, x_4) = (6, -1, 3, 4)$. The $s$-channel OPE converges. For the convergent OPE expansion transform to a different conformal frame with conformal inversion operation

$$I \; : \; x^\mu \to x'^\mu = \frac{x^\mu}{x \cdot x} \tag{1}$$


We have the equality

$$\langle \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^- x_2)\phi^b(x^+, x_3^-, x_3)\phi^b(x^+, x_4^-, x_4)\rangle$$

$$= \quad \langle I \cdot I\phi^a(x^+, x_1^-, x_1)I \cdot I\phi^a(x^+, x_2^- x_2)I \cdot I\phi^b(x^+, x_3^-, x_3)I \cdot I\phi^b(x^+, x_4^-, x_4)I \cdot I\rangle$$

The free scalar field has $\Delta = \frac{1}{2}$ so that

$$I\phi^a(x_i^\mu)I = (x_i' \cdot x_i')^\Delta \phi'^a(x_i'^\mu) = \sqrt{x_i' \cdot x_i'}\phi'^a(x_i'^\mu) \qquad i = 1, 2 \tag{2}$$

# Holography Of Information

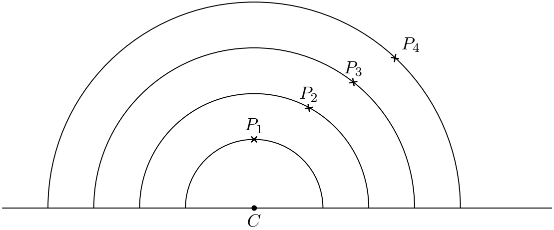



Figure: It is always possible to choose a centre ($C$) for $K$ non-intersecting semicircles such that a bulk point $P_i$ lies on the $i$th semicircle. At each semi-circle endpoint there is an operator, which is contracted with the operator at the other endpoint.

# Discussion and Future Directions

Have demonstrated the principle of the holography of information in bilocal holography.

The principle of holography of information is implied by a familiar but remarkable statement in conformal field theory: the operator product expansion.

Another positive indication that bilocal holography is indeed constructing the quantum gravity dual to the original conformal field theory.

Is this lesson relevant for the original $\mathcal{N} = 4$ super Yang-Mills/$\mathrm{AdS}_5 \times \mathrm{S}^5$ example of holography?

For matrix theories, many more gauge invariant variables. Bilocal, trilocal and in general multi-local operators appear. Scale-radius duality of AdS/CFT $\Rightarrow$ multi-local operators with well separated locations explore deep in the bulk, while the local limit gives a bulk operator located at the boundary. OPE can be used to take the product of separated operators and express them in terms of local operators.

Thanks for your attention!