The bulk Hilbert space from chords

Henry Lin

based on arXiv:2208.07032

+ upcoming w/ Douglas Stanford and Zhenbin Yang

Fifth Mandelstam Workshop

Incomplete References

JT gravity review:

[Maldacena, Stanford, Yang 1606.01857]

[Sarosi 1711.08482]

[Mertens & Turiaci 2210.10846]

JT gravity in the gauge-invariant formalism:

[Harlow & Jafferis, 1804.01081],

[Harlow & Wu, 2108.04841],

[HL, Maldacena, Rozenberg, Shan, 2207.00408]

SYK, etc.

SYK [Maldacena & Stanford, 1604.07818]

traversable wormholes [Maldacena, Stanford, Yang 1704.05333] [HL, Maldacena, Zhao, 1904.12820]

double scaled SYK [Berkooz *et al.* 1811.02584] [HL 2208.07032]

Double scaled SYK

SYK: N Majorana fermions with q-body Hamiltonian

$$H = i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \psi_{i_1} \cdots \psi_{i_q}, \quad \left\langle J_{i_1 \ldots i_q}^2 \right\rangle = \mathcal{J}^2 / \binom{N}{q},$$

Double scaling limit

$$q \to \infty, \ \ N \to \infty, \ \ \lambda = 2q^2/N = \text{fixed}.$$

2 dimensionless parameters: $\lambda \sim G_N/(\phi_r \epsilon)$, and $\beta \mathcal{J} \sim \beta/\epsilon$.

Double scaled SYK

SYK: N Majorana fermions with q-body Hamiltonian

$$H = i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \psi_{i_1} \cdots \psi_{i_q}, \quad \left\langle J^2_{i_1 \ldots i_q} \right\rangle = \mathcal{J}^2 / \binom{N}{q},$$

Double scaling limit

$$q \to \infty, \;\; N \to \infty, \;\; \lambda = 2q^2/N = \text{fixed}.$$

2 dimensionless parameters: $\lambda \sim G_N/(\phi_r \epsilon)$, and $\beta\mathcal{J} \sim \beta/\epsilon$.

- Standard large q limit: $\lambda \to 0$, $\beta\mathcal{J} = \text{fixed}$.

Double scaled SYK

SYK: N Majorana fermions with q-body Hamiltonian

$$H = i^{q/2} \sum_{1\le i_1<\cdots<i_q\le N} J_{i_1\ldots i_q} \psi_{i_1} \cdots \psi_{i_q}, \quad \left\langle J^2_{i_1\ldots i_q} \right\rangle = \mathcal{J}^2 / \binom{N}{q},$$

Double scaling limit

$$q \to \infty, \;\; N \to \infty, \;\; \lambda = 2q^2/N = \text{fixed}.$$

2 dimensionless parameters: $\lambda \sim G_N/(\phi_r \epsilon)$, and $\beta\mathcal{J} \sim \beta/\epsilon$.

- Standard large q limit: $\lambda \to 0$, $\beta\mathcal{J} = \text{fixed}$.
- Triple scaling limit: $\lambda \to 0, \beta\mathcal{J} \to \infty$, $\lambda\beta\mathcal{J} = \text{fixed}$.

Analogy with $\mathcal{N} = 4$ SYM

$1/\lambda \sim N_{\mathrm{YM}}^2$. When $1/\lambda$ is large, the saddle point approximation is valid.

$(\beta\mathcal{J})^{-1} \sim \lambda_{\mathrm{YM}} = g_{\mathrm{YM}}^2 N_{\mathrm{YM}}$. This controls effects that make the chaos exponent sub-maximal. In string theory, this is finite α' corrections. At finite temperatures, don't get JT gravity but some more complicated action $\sim$ higher derivative corrections to Einstein gravity.

Part of the motivation for this work is to understand chaos at finite chaos exponent. $\beta\mathcal{J} \to 0$ the Lyapunov exponent $\mathcal{L} = \mathcal{J}$, whereas at $\beta\mathcal{J} \to \infty$, $\mathcal{L} = 2\pi/\beta$.

Review of $Z(\beta)$ Computation

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.
2. Do the disorder average (Wick contract the J's in each H).

$$\operatorname{tr} HHHH \supset \operatorname{tr} \overbrace{HHHH}^{\text{contractions (1,3),(2,4)}} = \operatorname{tr}\left(\Psi_I^q \Psi_J^q \Psi_I^q \Psi_J^q\right)$$

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.
2. Do the disorder average (Wick contract the J's in each H).

$$\operatorname{tr} HHHH \supset \operatorname{tr} \contraction{}{H}{H}{H}\contraction[2ex]{H}{H}{HH}{H} HHHH = \operatorname{tr}\big(\Psi_I^q \Psi_J^q \Psi_I^q \Psi_J^q\big)$$

3. To evaluate the remaining trace, anti-commute like fermions next to each other and then annihilate them using $\psi_i^2 = 1$.

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.
2. Do the disorder average (Wick contract the J's in each H).

$$\operatorname{tr} HHHH \supset \operatorname{tr} \overbrace{HHHH} = \operatorname{tr}\big(\Psi_I^q \Psi_J^q \Psi_I^q \Psi_J^q\big)$$

3. To evaluate the remaining trace, anti-commute like fermions next to each other and then annihilate them using $\psi_i^2 = 1$. Anti-commuting two sets of fermions $\Psi_{I_1}^q$ and $\Psi_{I_2}^q$ past each other gives a sign $(-1)^f$, $f = |I_1 \cap I_2|$.

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.
2. Do the disorder average (Wick contract the J's in each H).

$$\operatorname{tr} HHHH \supset \operatorname{tr} \contraction{}{H}{H}{H}\contraction[2ex]{H}{H}{H}{H}HHHH = \operatorname{tr}\big(\Psi_I^q \Psi_J^q \Psi_I^q \Psi_J^q\big)$$

3. To evaluate the remaining trace, anti-commute like fermions next to each other and then annihilate them using $\psi_i^2 = 1$. Anti-commuting two sets of fermions $\Psi_{I_1}^q$ and $\Psi_{I_2}^q$ past each other gives a sign $(-1)^f$, $f = |I_1 \cap I_2|$. In the double-scaling limit, $\mathfrak{q} = \big\langle (-1)^f \big\rangle = e^{-\lambda}$.

Review of $Z(\beta)$ Computation

1. Expand $e^{-\beta H} = \sum_m \beta^m H^m / m!$ and focus on evaluating the sum term by term.
2. Do the disorder average (Wick contract the J's in each H).

$$\operatorname{tr} HHHH \supset \operatorname{tr} \overbrace{HHHH}^{\text{contracted: 1–3, 2–4}} = \operatorname{tr}\left(\Psi_I^q \Psi_J^q \Psi_I^q \Psi_J^q\right)$$

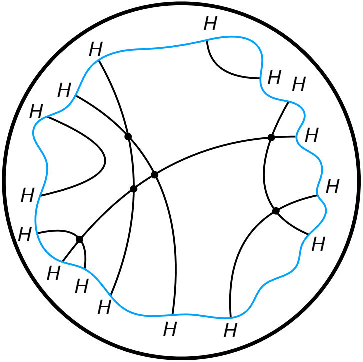
3. To evaluate the remaining trace, anti-commute like fermions next to each other and then annihilate them using $\psi_i^2 = 1$. Anti-commuting two sets of fermions $\Psi_{I_1}^q$ and $\Psi_{I_2}^q$ past each other gives a sign $(-1)^f$, $f = |I_1 \cap I_2|$. In the double-scaling limit, $\mathfrak{q} = \left\langle (-1)^f \right\rangle = e^{-\lambda}$.
4. Sum over Wick contractions and m.

Chord diagrams

The last two steps are achieved by writing down all "chord diagrams" and associating a factor $\mathfrak{q} = e^{-\lambda}$ to each vertex.

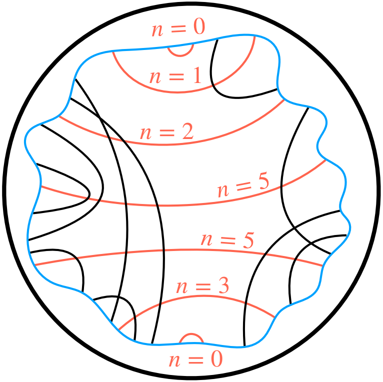
Chord diagrams

The last two steps are achieved by writing down all "chord diagrams" and associating a factor $\mathfrak{q} = e^{-\lambda}$ to each vertex.



A contribution to $\operatorname{tr} H^{16} \supset \mathfrak{q}^6$.

How do we enumerate all such diagrams? The idea is that we slice open all the diagrams and write



$$Z(\beta) = \langle 0| \, e^{-\beta T} \, |0\rangle$$

T accounts for the insertion of H on the boundary using chords.

Chord Hilbert Space = bulk Hilbert Space

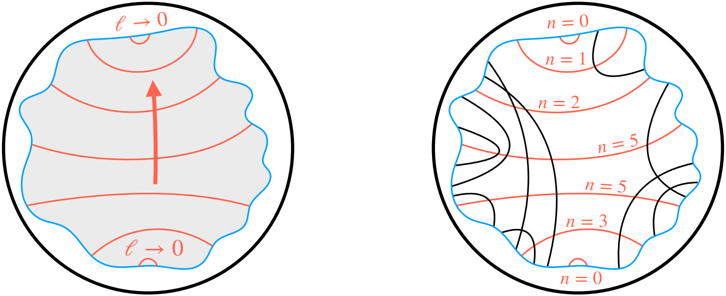


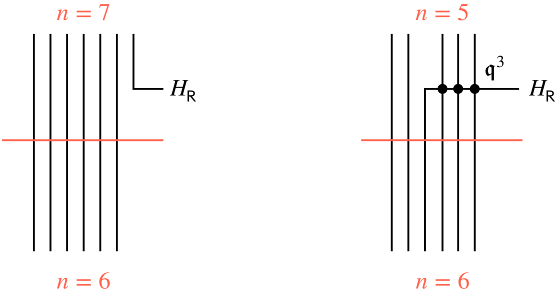
Figure: pure JT gravity vs chords.

The states $|n\rangle$ are "bulk" states with n open chords. We immediately suspect that $n \in \mathbb{Z}$ is the discrete analog of ℓ.

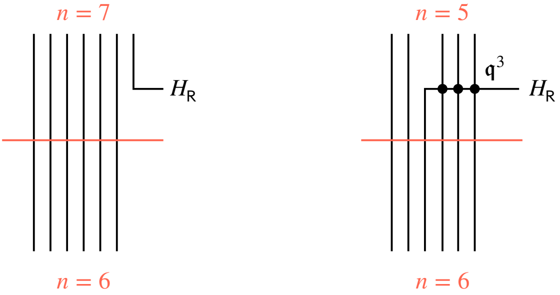
To relate n and ℓ more explicitly, let's work out the action of T on a state with n open chords.

In principle, we have two operators T_{L} and T_{R}, which correspond to inserting H on the L or R. But without matter these operators are identical.

Two processes that happen when we act with H_{R}:

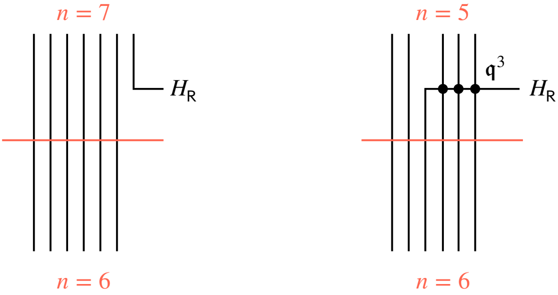


Two processes that happen when we act with H_{R}:



On the left, a chord is created. By convention, it does not cross any chords.

Two processes that happen when we act with H_R:



On the left, a chord is created. By convention, it does not cross any chords. On the right, a chord is annihilated. It crosses 3 chords giving a factor of $\mathfrak{q}^3$.

$$
\begin{aligned}
T &= \alpha^\dagger + \alpha W, \quad \alpha |n\rangle = |n-1\rangle, \quad n > 0 \\
W_n &= \mathfrak{q}^0 + \mathfrak{q}^1 + \cdots + \mathfrak{q}^{n-1} = \frac{1-\mathfrak{q}^n}{1-\mathfrak{q}}
\end{aligned} \tag{1}
$$

Inner product

So far, we have defined a bulk *vector* space, spanned by chord states $|n\rangle$. We have also defined the action of the Hamiltonian on these bulk states.

Inner product

So far, we have defined a bulk *vector* space, spanned by chord states $|n\rangle$. We have also defined the action of the Hamiltonian on these bulk states.

To complete our definition of a Hilbert space, we need to specify an inner product. This will also facilitate contact with JT gravity.

Inner product

So far, we have defined a bulk *vector* space, spanned by chord states $|n\rangle$. We have also defined the action of the Hamiltonian on these bulk states.

To complete our definition of a Hilbert space, we need to specify an inner product. This will also facilitate contact with JT gravity.

The inner product is defined by $\langle 0| T^{a+b} |0\rangle = \Big\langle T^a |0\rangle , T^b |0\rangle \Big\rangle$.

$$(T^a)^0{}_n (T^b)^n{}_0 = g_{mn} (T^a)^m{}_0 (T^b)^n{}_0 .$$

where $g_{mn} = \Big\langle |m\rangle , |n\rangle \Big\rangle$ and $T |0\rangle = T^m_{\ 0} |m\rangle$.

Inner product

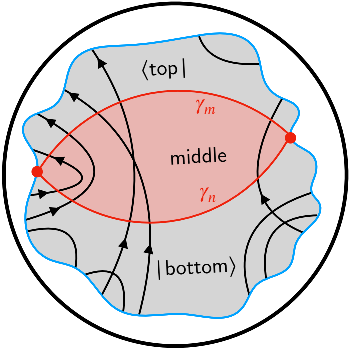


Figure: Interpretation of the chord diagram as ⟨top|bottom⟩. The middle region (pink) defines an inner product. All chords entering through γ_n must exit through γ_m.

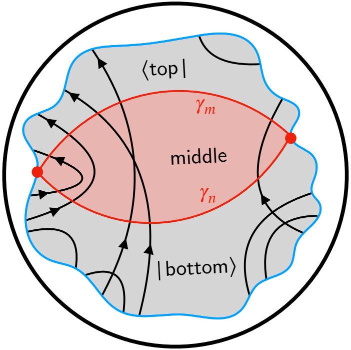
$$(T^a)_{0,n}(T^b)_{n,0} = g^{mn}(T^a)_{m,0}(T^b)_{n,0}.$$

Since all chords that cross γ_n must cross γ_m, $\langle m|n\rangle \propto \delta_{mn}$.

Since all chords that cross γ_n must cross γ_m, $\langle m|n\rangle \propto \delta_{mn}$.

Delete-a-chord recursion relation:

$$\langle n|n\rangle = W_n \langle n-1|n-1\rangle \,. \tag{2}$$



Evaluating T in an orthonormal basis gives

$$\begin{aligned} H &= -g^{1/2} T g^{-1/2} = \alpha\sqrt{W} + \sqrt{W}\alpha^{\dagger} \\ &= -\frac{1}{\sqrt{1-\mathfrak{q}}}\left[e^{i\lambda k}\sqrt{1-e^{-\ell}} + \sqrt{1-e^{-\ell}}e^{-i\lambda k}\right] \end{aligned}$$

Evaluating T in an orthonormal basis gives

$$\begin{aligned} H &= -g^{1/2} T g^{-1/2} = \alpha\sqrt{W} + \sqrt{W}\alpha^\dagger \\ &= -\frac{1}{\sqrt{1-\mathfrak{q}}}\left[e^{i\lambda k}\sqrt{1-e^{-\ell}} + \sqrt{1-e^{-\ell}}e^{-i\lambda k}\right] \end{aligned}$$

Here we have identified $\boxed{\ell = \lambda n}$. Taking the *triple-scaling limit*:

$$\lambda \to 0, \quad \ell \to \infty, \quad e^{-\ell}/\lambda^2 = e^{-\tilde{\ell}} = \text{fixed},$$

we recover Liouville quantum mechanics

$$H - E_0 \propto k^2 + e^{-\tilde{\ell}} \tag{3}$$

Chord Hilbert Space = bulk Hilbert Space

	JT gravity	Double Scaled SYK
$\mathcal{H}_{\text{grav}}$	length $\|\ell\rangle$	chords $\|n\rangle$
H	$k^2 + e^{-\ell}$	$e^{ik\ell}\sqrt{1-e^{-\ell}} + cc$
$Z(\beta)$	$\langle \ell = 0\| \, e^{-\beta H} \, \|\ell = 0\rangle$	$\langle 0\| \, e^{-\beta T} \, \|0\rangle$
TFD	$e^{-\beta H/2} \, \|\ell = 0\rangle$	$e^{-\beta H/2} \, \|0\rangle$

Operator Size & Chord Number

Expand any 2-sided state $|\chi\rangle$ in the "size basis":

$$|\chi\rangle = \sum_{s,I} c_{s,I} \Psi_I^s |\Omega\rangle ,$$

The size of this state is measured by the 2-sided operator:

$$\text{size} = \frac{1}{2} \sum_{\alpha=1}^{N} \left(1 + i\psi_\alpha^{\mathrm{L}} \psi_\alpha^{\mathrm{R}} \right) .$$

Let $\bar{n}$ be the total chord number, weighted by dimension:

$$\boxed{\bar{n} = \text{size}/q} \tag{4}$$

Let $\bar{n}$ be the total chord number, weighted by dimension:

$$\bar{n} = \text{size}/q \tag{5}$$

Let's focus on a particular term in the computation of the average size:

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \operatorname{tr}\left(HHHH\psi_\alpha HHHH\psi_\alpha\right) \\
&\propto \sum_{I,\alpha} \operatorname{tr}\left(\Psi^q_{I_1}\Psi^q_{I_1}\Psi^q_{I_2}\Psi^q_{I_3}\psi_\alpha\Psi^q_{I_4}\Psi^q_{I_2}\Psi^q_{I_4}\Psi^q_{I_3}\psi_\alpha\right) \\
&\propto \sum_{I,\alpha} \operatorname{tr}\left(\Psi^q_{I_2}\Psi^q_{I_3}\psi_\alpha\Psi^q_{I_2}\Psi^q_{I_3}\psi_\alpha\right)
\end{aligned} \tag{6}$$

Bulk-to-boundary map: a warmup

Gram-Schmidt the set of vectors:

$$|\Omega\rangle, H|\Omega\rangle, H^2|\Omega\rangle, \cdots, \tag{7}$$

Since $T \sim \alpha^\dagger + \alpha$, this generates the chord basis:

$$|0\rangle, |1\rangle, |2\rangle, \cdots, \tag{8}$$

Explicitly,

$$|\Omega\rangle,\ H|\Omega\rangle,\ H^2|\Omega\rangle - |\Omega\rangle,\ H^3|\Omega\rangle - (2+\mathfrak{q})H|\Omega\rangle \tag{9}$$

Similar to Krylov complexity.

Correlators and matter chords

Matter chords

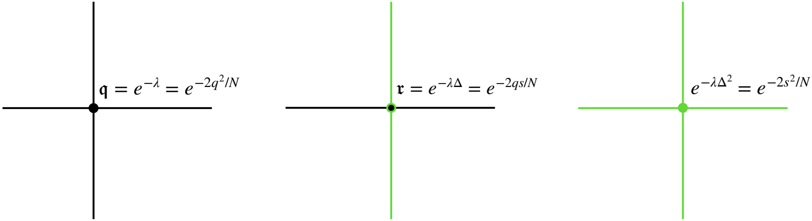
Following [Berkooz *et al.*], we will consider thermal correlators of "matter operators" with $\Delta = s/q$ fixed:

$$M_s = i^{s/2} \sum_I K_I \psi_I^s \tag{10}$$

Multi-index notation: $\Psi_I^s = \psi_{i_1}\psi_{i_2}\cdots\psi_{i_s}$.

Case with matter

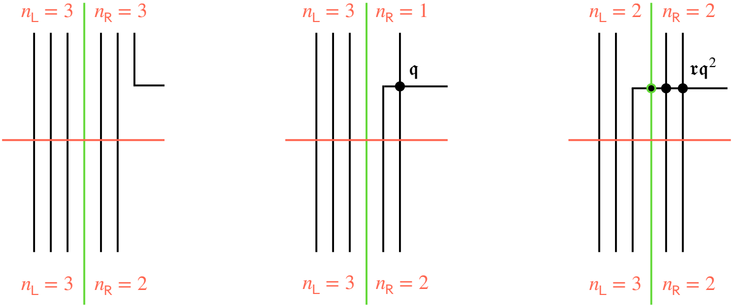
Feynman rules for matter operators M_s:



A microscopic origin of Newton's Law: $GM_1M_2 \sim (2q^2/N)\Delta_1\Delta_2$

Case with matter

The next simplest case is a wormhole with one green particle. Then states are labeled by $|n_\mathsf{L}, n_\mathsf{R}\rangle$.



Repeating the logic from before,

$$\begin{aligned} T_\mathsf{R} &= \alpha_\mathsf{R}^\dagger + \alpha_\mathsf{R} W_\mathsf{R} + \alpha_\mathsf{L} \mathfrak{r}\mathfrak{q}^{n_\mathsf{R}} W_\mathsf{L}, \\ T_\mathsf{L} &= \alpha_\mathsf{L}^\dagger + \alpha_\mathsf{L} W_\mathsf{L} + \alpha_\mathsf{R} \mathfrak{r}\mathfrak{q}^{n_\mathsf{L}} W_\mathsf{R}. \end{aligned} \tag{11}$$

Case with matter

For states with multi-particles, states are labeled by

$$\left|n_0, n_1, \cdots, n_m\right\rangle_{s_1, s_2, \cdots, s_m} \tag{12}$$

General expressions for T_{L}, T_{R}:

$$\begin{aligned} T_{\mathrm{L}} &= \alpha_{\mathrm{L}}^{\dagger} + \sum_{i=0}^{m} \alpha_i \left[\frac{1-e^{-\ell_i}}{1-\mathfrak{q}}\right] \prod_{j<i} \mathfrak{r}_j e^{-\ell_j}, \\ T_{\mathrm{R}} &= \alpha_{\mathrm{R}}^{\dagger} + \sum_{i=0}^{m} \alpha_i \left[\frac{1-e^{-\ell_i}}{1-\mathfrak{q}}\right] \prod_{j>i} \mathfrak{r}_j e^{-\ell_j}. \end{aligned} \tag{13}$$

Note that there is only one creation $\alpha^{\dagger}$ operator.

The chord algebra

We obtained the general expressions for T_L, T_R acting on arbitrary states in the double-scaled Hilbert space.

We would like to use these to understand the bulk dual. 0th order question: what are the symmetries of the bulk (if any)?

In NAdS_2, the gauge-invariant bulk isometries of AdS_2 are subtle. They do not commute with the $H_{\mathrm{L}}, H_{\mathrm{R}}$, but form an algebra which includes the Hamiltonian and the length.

The chord algebra

The total chord number $\bar{n}$ and $T_{\mathsf{L/R}}$ form an algebra, independent of the matter content:

$$
\begin{aligned}
[T_{\mathsf{L}}, T_{\mathsf{R}}] &= 0 \\
[T_{\mathsf{L/R}}, \bar{n}] &= T_{\mathsf{L/R}} - 2\alpha^{\dagger}_{\mathsf{L/R}} \\
[\alpha^{\dagger}_{\mathsf{L}}, \alpha^{\dagger}_{\mathsf{R}}] &= 0 \\
[\bar{n}, \alpha^{\dagger}_{\mathsf{L/R}}] &= \alpha^{\dagger}_{\mathsf{L/R}} \\
[T_{\mathsf{L/R}}, \alpha^{\dagger}_{\mathsf{L/R}}]_{\mathfrak{q}} &= 1 + (1-\mathfrak{q}) \left(\alpha^{\dagger}_{\mathsf{L/R}}\right)^2 \\
[T_{\mathsf{L/R}}, \alpha^{\dagger}_{\mathsf{R/L}}] &= \mathfrak{q}^{\bar{n}}
\end{aligned}
$$

Here $[A, B]_{\mathfrak{q}} = AB - \mathfrak{q}BA$. This algebra has implications for the bulk dual of double scaled SYK. [HL, Stanford, Yang, upcoming].

Can find a subalgebra that commutes with $\bar{n}$, generated by 4 elements:

$$
\begin{aligned}
F_{\mathsf{LL}} &= \alpha_{\mathsf{L}}^{\dagger}(T_{\mathsf{L}} - \alpha_{\mathsf{L}}^{\dagger}) \\
F_{\mathsf{RR}} &= \alpha_{\mathsf{R}}^{\dagger}(T_{\mathsf{R}} - \alpha_{\mathsf{R}}^{\dagger}) \\
F_{\mathsf{LR}} &= \alpha_{\mathsf{L}}^{\dagger}(T_{\mathsf{R}} - \alpha_{\mathsf{R}}^{\dagger}) \\
F_{\mathsf{RL}} &= \alpha_{\mathsf{R}}^{\dagger}(T_{\mathsf{L}} - \alpha_{\mathsf{L}}^{\dagger})
\end{aligned}
\tag{14}
$$

To see that these commute with $\bar{n}$, recall that $T_i - \alpha_i^{\dagger}$ only annihilates. One can work out the commutation relations of F using the chord algebra.

These 4 elements form a subalgebra of $U_{\mathfrak{q}^{1/2}}(\mathfrak{sl}(2,\mathbb{R}))$:

$$[K^2, \mathcal{E}]_{\mathfrak{q}} = [\mathcal{F}, K^2]_{\mathfrak{q}} = 0, \quad \mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \frac{K^2 - K^{-2}}{\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2}}$$

For each $n \in \mathbb{Z}$, we get unitary finite-dim reps of this algebra!

Gravitational algebra

In the triple scaling limit, the chord algebra becomes the JT gravitational algebra:

$$
\begin{aligned}
[H_{\mathrm{L}}, H_{\mathrm{R}}] &= 0 \\
i[H_{\mathrm{L/R}}, \tilde{\ell}] &= 2k_{\mathrm{L/R}} \\
[\tilde{\ell}, k_{\mathrm{L/R}}] &= i \\
[k_{\mathrm{L}}, k_{\mathrm{R}}] &= 0 \\
-i[k_{\mathrm{L/R}}, H_{\mathrm{L/R}}] &= H_{\mathrm{L/R}} - k_{\mathrm{L/R}}^2 \\
-i[k_{\mathrm{L/R}}, H_{\mathrm{R/L}}] &= e^{-\tilde{\ell}}
\end{aligned}
$$

In [Harlow & Wu '21] this algebra was derived classically using Poisson brackets; here we obtained them quantum mechanically.

JT with matter

Taking the triple scaling limit of our expressions for $T_{\mathrm{L}}, T_{\mathrm{R}} \Rightarrow$ gives concrete reps of the JT algebra.

JT with matter

Taking the triple scaling limit of our expressions for $T_\mathsf{L}, T_\mathsf{R} \Rightarrow$ gives concrete reps of the JT algebra. The simplest case is for $m = 1$ matter particles in the wormhole:

$$\begin{aligned} \tilde{\ell}_\mathsf{L} &= \ell_\mathsf{L} + \log\lambda, \quad \tilde{\ell}_\mathsf{R} = \ell_\mathsf{R} + \log\lambda \\ H_\mathsf{L} &\approx -\tilde{\partial}_\mathsf{L}^2 + \Delta e^{-\tilde{\ell}_\mathsf{L}} + e^{-\tilde{\ell}_\mathsf{L}}(\partial_\mathsf{R} - \partial_\mathsf{L}) + e^{-\tilde{\ell}_\mathsf{L} - \tilde{\ell}_\mathsf{R}} \\ H_\mathsf{R} &\approx -\tilde{\partial}_\mathsf{R}^2 + \Delta e^{-\tilde{\ell}_\mathsf{R}} - e^{-\tilde{\ell}_\mathsf{R}}(\partial_\mathsf{R} - \partial_\mathsf{L}) + e^{-\tilde{\ell}_\mathsf{L} - \tilde{\ell}_\mathsf{R}}. \end{aligned} \tag{15}$$

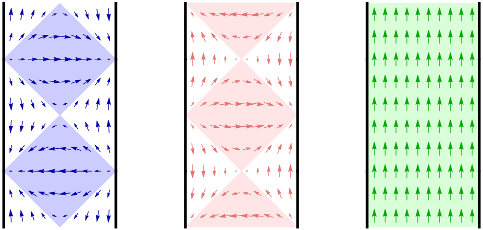
This is some generalization of Liouville that involves two coordinates. For m-particle states, there is a generalization involving $m + 1$ coordinates.

Symmetries near the horizon

An interesting sub-algebra of the JT gravitational algebra is generated by elements which commute with $\tilde{\ell}$. This forms an $\mathfrak{sl}(2,\mathbb{R})$ algebra [HL, Maldacena, Zhao] that is the near horizon symmetries of the wormhole.

Symmetries near the horizon

An interesting sub-algebra of the JT gravitational algebra is generated by elements which commute with $\tilde{\ell}$. This forms an $\mathfrak{sl}(2,\mathbb{R})$ algebra [HL, Maldacena, Zhao] that is the near horizon symmetries of the wormhole. These generators satisfy $[L_m, L_n] = i(m-n)L_{m+n}$ and move matter around in the black hole interior!



Symmetries

Plugging in our expressions for $H_{\mathrm{L}}, H_{\mathrm{R}}$ for 1 matter particle gives:

$$L_0 = -i\partial_x, \quad L_\pm = (\Delta \pm \partial_x)\, e^{\mp x} \tag{16}$$

x is essentially the distance from the horizon $x = \lambda(n_{\mathrm{L}} - n_{\mathrm{R}})/2$.

Symmetries

Plugging in our expressions for $H_{\mathsf{L}}, H_{\mathsf{R}}$ for 1 matter particle gives:

$$L_0 = -i\partial_x, \quad L_\pm = (\Delta \pm \partial_x)\, e^{\mp x} \tag{16}$$

x is essentially the distance from the horizon $x = \lambda(n_{\mathsf{L}} - n_{\mathsf{R}})/2$. This $\mathfrak{sl}(2,\mathbb{R})$ algebra is a contraction of the $U_{\mathfrak{q}^{1/2}}(\mathfrak{sl}(2,\mathbb{R}))$ subalgebra we discussed before.

Finite temperature $\mathfrak{sl}(2, \mathbb{R})$ symmetries

We can back off the low temperature limit and consider $\lambda \to 0$ holding temperature fixed. (Equivalently, $\bar{n} \to \infty$, holding $e^{-\lambda \bar{n}}$ fixed.)

Finite temperature $\mathfrak{sl}(2,\mathbb{R})$ symmetries

We can back off the low temperature limit and consider $\lambda \to 0$ holding temperature fixed. (Equivalently, $\bar{n} \to \infty$, holding $e^{-\lambda \bar{n}}$ fixed.) In this limit, we still find an $\mathfrak{sl}(2,\mathbb{R})$ symmetry! Immediate questions:

1. Why is chaos sub-maximal?
2. Is there a hyperbolic space on which this symmetry acts as the isometries?

Finite temperature $\mathfrak{sl}(2,\mathbb{R})$ symmetries

We can back off the low temperature limit and consider $\lambda \to 0$ holding temperature fixed. (Equivalently, $\bar{n} \to \infty$, holding $e^{-\lambda\bar{n}}$ fixed.) In this limit, we still find an $\mathfrak{sl}(2,\mathbb{R})$ symmetry! Immediate questions:

1. Why is chaos sub-maximal?
2. Is there a hyperbolic space on which this symmetry acts as the isometries?

Stay tuned...

Bulk-to-boundary map

In the case without matter, we performed Gram-Schmidt to obtain $|0\rangle \to |1\rangle \to |2\rangle \to \cdots$. The next simplest case is to consider 1 particle states.

$$\begin{bmatrix} |0,0\rangle \to & |0,1\rangle \to & |0,2\rangle \to & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |1,0\rangle \to & |1,1\rangle \to & |1,2\rangle \to & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |2,0\rangle \to & |2,1\rangle \to & |2,2\rangle \to & \cdots \\ \downarrow & \downarrow & \downarrow & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Bulk-to-boundary map

To make the first column of this matrix, we can consider the states with no particle $|n\rangle$ and then act with $(M_s)_{\mathrm{R}}\,|n\rangle = |n,0\rangle$.

$$\begin{bmatrix} |0,0\rangle \rightarrow & |0,1\rangle \rightarrow & |0,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |1,0\rangle \rightarrow & |1,1\rangle \rightarrow & |1,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |2,0\rangle \rightarrow & |2,1\rangle \rightarrow & |2,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Bulk-to-boundary map

To make the first column of this matrix, we can consider the states with no particle $|n\rangle$ and then act with $(M_s)_{\mathrm{R}}\,|n\rangle = |n, 0\rangle$.

$$\begin{bmatrix} |0,0\rangle \rightarrow & |0,1\rangle \rightarrow & |0,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |1,0\rangle \rightarrow & |1,1\rangle \rightarrow & |1,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ |2,0\rangle \rightarrow & |2,1\rangle \rightarrow & |2,2\rangle \rightarrow & \cdots \\ \downarrow & \downarrow & \downarrow & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To carry out this orthogonalization procedure, need to know $\mathrm{tr}\, M_s H^a M_s H^b$. Known explicitly [Berkooz *et al.*].

Subtlety: $\langle n, m | n', m' \rangle \neq 0$ unless $n + m \neq n' + m'$.

Bulk-to-boundary map

Have an algorithm to construct multi-particle states by organizing the states into a higher dimensional array.Need the *n*-pt functions of the theory, known in terms of Γ_q [Berkooz *et al.*].

Works for all values of $\mathfrak{q}$ and temperatures; even when quantum corrections are large! More complete bulk reconstruction than HKLL.

Roughly analogous to summing α' corrections and $1/N$ corrections, but *not* e^{-N} corrections.

Double scaled algebra

The one-sided algebra of observables generated by $\{H, M_s, M_{s'}, M_{s''}, \cdots\}$ form a Type II_1 algebra.

Double scaled algebra

The one-sided algebra of observables generated by $\{H, M_s, M_{s'}, M_{s''}, \cdots\}$ form a Type II_1 algebra.

Unlike the Type $\text{III}/\text{II}_\infty$ in [Leutheusser & Liu, Witten] the Hamiltonian H is part of the algebra. Normally the Hamiltonian (and its moments) diverge in the large N limit. But in SYK, $H \sim N/q^2$ so H is finite in the double scaling limit. Therefore, we can include it.

Double scaled algebra

The one-sided algebra of observables generated by $\{H, M_s, M_{s'}, M_{s''}, \cdots\}$ form a Type II_1 algebra.

Unlike the Type III/II_∞ in [Leutheusser & Liu, Witten] the Hamiltonian H is part of the algebra. Normally the Hamiltonian (and its moments) diverge in the large N limit. But in SYK, $H \sim N/q^2$ so H is finite in the double scaling limit. Therefore, we can include it.

In [Leutheusser & Liu], the near horizon symmetries were constructed using an algebraic approach. Here I also constructed the near horizon symmetries. Any relation?

Future directions

1. Tensor networks
2. Can one derive a QES formula using chords [Lewkowycz & Maldacena]?
3. Wormholes [Jafferis *et al.*]
4. $\mathcal{N} = 2$ supersymmetry [HL, Maldacena, Rozenberg, Shan; Berkooz *et al.*]
5. Bulk implications of the q-deformation [HL, Stanford, Yang, in prep]