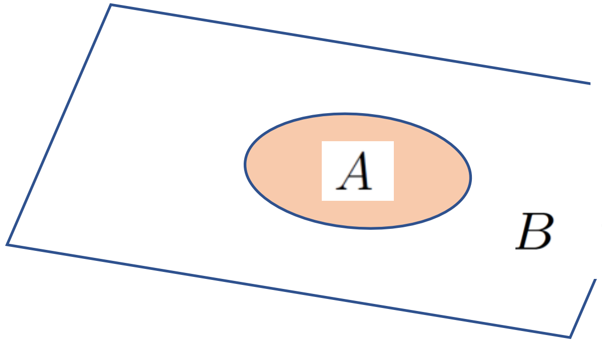
TARGET SPACE ENTANGLEMENT, FINITENESS AND HOLOGRAPHY

Sumit R. Das

- (S.R.Das, A. Kaushal, G. Mandal and S.P. Trivedi : J.Phys. A53 (2020) 44, 444002)
- (S.R.Das, A. Kaushal, S. Liu, G. Mandal and S.P. Trivedi: JHEP 04 (2021) 225)
- (S.R. Das, A. Jevicki & J. Zheng – JHEP 12 (2022) 052.)
- (S.R.Das, A. Kaushal, M.H. Radwan, G. Mandal, K.K. Nanda & S. Trivedi – arXiv:2212.11640)

Target Space Entanglement

- A familiar kind of entanglement in quantum field theories is entanglement in base space – this is the notion of entanglement of the degrees of freedom localized in some subregion of the space on which the theory is defined, with the rest.



$$\rho(\phi_A(x), \phi'_A(x)) = \int \mathcal{D}\phi_B(x)\Psi[\phi_A(x), \phi_B(x)]\Psi^\star[\phi'_A(x), \phi_B(x)]$$

- This can be described in terms of a subalgebra of operators – those which can be used to perform measurements in A .
- In holographic theories this is evaluated by the Ryu-Takayanagi formula and its generalizations.
- In usual *relativistic* field theory, the entanglement is divergent.

- In this talk I will discuss notions of entanglement of internal degrees of freedom of a quantum theory.
- In particular, gauge theories of matrices e.g. $N \times N$ Matrices

$$X^I_{ij}(\vec{x}, t)$$

$$X^I \to U(\vec{x}, t) X^I(\vec{x}, t) U^\dagger(\vec{x}, t)$$

- We are interested in defining notions of entanglement among the matrix degrees of freedom.
- To isolate this kind of entanglement from entanglement among regions of the base space, concentrate on quantum mechanics of matrices

$$X^I(t)$$

- One motivation for studying this problem is holography.
- Usual base space entanglement plays a key role in the emergence of a smooth gravitational dual – best understood when the bulk is AdS.
- There are examples of holography where the dual theory is 0+1 dimensional – e.g. D0 brane holography, two dimensional non-critical strings.
- We would like to understand if there is a connection of smooth bulks with entanglement – such entanglement is necessarily that between the internal degrees of freedom.
- In fact, in most known examples of holography, the bulk space-time has an internal compact space, e.g. $AdS_m \times Y^n$. The internal space geometrizes a R symmetry. We would like to learn if a smooth Y^n is related to some kind of entanglement.
- Apart from holography, gauge theories of matrices also appear in other interesting physical situations as well – as we will mention later.

Single Matrix Quantum Mechanics

- Consider single matrix quantum mechanics. The action is

$$S = \beta \int dt \; \mathrm{Tr}\left[(\partial_t M - i[A_t, M])^2 - V(M) \right] \qquad \beta \sim N$$

- In the $A_t = 0$ gauge, the Gauss Law constraint requires the states to be singlets under the global $U(N)$
- The Hamiltonian is now

$$H = \mathrm{Tr}\left[-\frac{1}{\beta} \frac{\partial}{\partial M} \frac{\partial}{\partial M} + \beta V(M) \right]$$

- This by itself is the earliest model of holography.

- An obvious guess is to consider the entanglement of a block in the matrix with the rest, e.g.

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} \end{pmatrix}$$

- A priori this does not make much sense, since a gauge transformation mixes up the matrix elements.
- One of our main aims will be to define something like this in a gauge invariant fashion.
- *(S.R.D., A. Kaushal, G. Mandal & S. Trivedi, 2020;*
- *(S.R.D., A. Kaushal, S. Liu, G. Mandal & S. Trivedi, 2020)*

- The gauge invariant operators are of the form

$$\hat{C} = \mathrm{Tr}\left(\hat{M}^m \hat{\Pi}_M^n\right)_{\mathrm{order}}$$

- Now consider a projector for an interval A on the real line

$$(\hat{P}_A) = \int_A dx\ \delta(x\mathbf{I} - \hat{M})$$

- The projected operators

$$\hat{C}_A = \mathrm{Tr}\left(\hat{P}_A M \hat{P}_A M \cdots \hat{P}_A M \hat{P}_A \hat{\Pi}_M \hat{P}_A \cdots \hat{\Pi}_M \hat{P}_A\right)$$

  form a subalgebra.

- Expectation values of these operators are evaluated by a reduced density matrix with an associated von Neumann entropy.

- To see the nature of this projection, consider the gauge in which M is diagonal.

$$M_{ij} \to \mathrm{diag}[\lambda_1, \lambda_2, \cdots \lambda_N]$$

- The operators are

$$\hat{\mathcal{C}}_{n,m} = \sum_{i=1}^{N} \hat{\lambda}_i^n \hat{\pi}_i^m$$

- The projected operators are then

$$(\hat{\mathcal{C}}_{n,m})_A^P = \sum_{i=1}^{N} (\hat{P}_A)_i \hat{\lambda}_i^n (\hat{P}_A)_i \hat{\pi}_i (\hat{P}_A)_i \hat{\pi}_i (\hat{P}_A)_i \cdots (\hat{P}_A)_i \hat{\pi}_i (\hat{P}_A)_i$$

$$(\hat{P}_A)_i = \int_A dx\ \delta(x - \hat{\lambda}_i)$$

- The theory becomes that of N free non-relativistic fermions in some external potential. The λ_i are the coordinates of these fermions.
- Consider now the expectation value of such a projected operator in a slater determinant state

- Consider the simplest type of operator

$$\sum_{i=1}^{N} (\hat{\lambda}_i)^n$$

- The projected version is then

$$\mathcal{O}_n^P = \sum_{i=1}^{N} \int_A \; dx \delta(x - \lambda_i) \; \lambda_i^n$$

- Consider now the expectation value of such an operator in a system of 2 particles

$$\langle \Psi | \mathcal{O}_3^P | \Psi \rangle = \int_A d\lambda_1 \int_A d\lambda_2 \Psi^\star(\lambda_1, \lambda_2)(\lambda_1^3 + \lambda_2^3)\Psi(\lambda_1, \lambda_2)$$
$$+ 2 \int_A d\lambda_1 \int_{\bar{A}} d\lambda_2 \Psi^\star(\lambda_1, \lambda_2)(\lambda_1^3)\Psi(\lambda_1, \lambda_2)$$

- This therefore measures the sum of powers of the eigenvalue only if it lies in the region of interest A.

- This means that the Hilbert space becomes a sum over sectors, labelled by the number of particles k which lie in the region of interest. (*S.R.D, G. Mandal & S. Trivedi*; *Mazenc & Ranard, 2019*)

$$\mathcal{H}_N = \oplus_k \mathcal{H}_{k,N-k}$$

- Acting on a state in this sector the operator has a non-trivial action only on those particles – so we can think in terms of an operator $\mathcal{O}_k$ which lives in this smaller Hilbert space pertaining to the region of interest.
- The entire answer may be written in terms of reduced density matrices

$$\langle\Psi|\mathcal{O}^P|\Psi\rangle = \sum_{k=1}^{N-1} \mathrm{Tr}_A\left[\tilde{\rho}_k \mathcal{O}_k\right]$$

- Where the trace is over the smaller Hilbert space

$$\langle\lambda_a|\hat{\rho}_k|\lambda'_a\rangle = \binom{N}{k} \int \prod_{\alpha=k+1}^{N} [d\lambda_\alpha]\ \Psi^\star\left(\lambda'_a, \lambda_\alpha\right)\ \Psi\left(\lambda_a, \lambda_\alpha\right)$$

- Note that each of the $\tilde{\rho}_k$ is not normalized. Its trace is the probability for k particles to lie in A

$$p_k = \mathrm{tr}_k \tilde{\rho}_k$$

- The full reduced density matrix is block diagonal – each block corresponds to a sector

$$\rho = \begin{pmatrix} \tilde{\rho}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \tilde{\rho}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\rho}_N \end{pmatrix} \qquad \mathrm{tr}\rho = 1$$

- The associated entanglement entropy is then given by

$$S = -\mathrm{tr}(\rho \log \rho) = -\sum_{k=0}^{N} \mathrm{tr}_k(\tilde{\rho}_k \log \tilde{\rho}_k)$$

- The $k = 0$ sector is special. The density matrix is a number, equal to p_0
- This generalizes to situations where the whole system is in a mixed state.

- The entropy can be re-expressed in terms of normalized density matrices

$$S = -\sum_k p_k \log p_k - \sum_k p_k \mathrm{tr}_k \hat{\rho}_k \log \hat{\rho}_k \qquad \hat{\rho}_k = \frac{1}{p_k}\tilde{\rho}_k$$

- The first term is a "classical" piece. The second term is the sum of entropies of each sector weighted by the probability of k particles to be in A.
- In fact, this notion exists even for a single particle. In this case the answer for the entanglement entropy has only the classical term

$$S = -p_A \log p_A - (1-p_A)\log(1-p_A)$$

- Where p_A is the probability of this particle to be in the region.

- The theory can be also expessed in a second quantized form

$$H = \int d\lambda \left[\frac{1}{2\beta} \frac{d\Psi^\dagger}{d\lambda} \frac{d\Psi}{d\lambda} + \beta V(\lambda) \Psi^\dagger \Psi + \beta \mu_F \Psi^\dagger \Psi \right]$$

- Here μ_F is the Fermi Level.
- In this fermion field theory, we could consider the entanglement of an interval in the space defined by λ in the filled fermi level ground state.
- This usual base space entanglement of this second quantized theory is exactly the target space entanglement we have been discussing.

Finiteness of the EE

- When the external potential is absent, the result for entanglement entropy associated with an interval (a,b) in eigenvalue space is – for large N

$$S = \begin{cases} \frac{1}{3}\left[\log(k_F(b-a)) + 1 + \gamma + \log\ 2\right], & \text{if } k_F(b-a) \gg 1. \\ \frac{1}{3}\left[\pi k_F(b-a) + k_F^2(b-a)^2 + \cdots\right], & \text{if } k_F(b-a) \ll 1. \end{cases}$$

- The large interval answer looks very much like the entanglement entropy of a single massless scalar in 1+1 dimensions – with k_F playing the role of the cutoff (*S.R.D.1995*)
- For generic potentials, the local fermi momentum which plays this role. This can be explicitly seen in a WKB approximation

- The appearance of the local fermi momentum as a cutoff is perfectly natural from the point of view of fermions.
- However, from the point of view of the collective field theory this is interesting, since the local fermi momentum is a position dependent coupling.

$$H = \int dq \left[\Pi_q^2 + \frac{\pi^2}{2}(\partial_q \eta)^2 + \frac{1}{\beta \rho_0^2}\left((\partial_q \eta)^3 + \Pi_\eta (\partial_q \eta) \Pi_\eta\right) \right] \qquad q = \int^\lambda \frac{d\lambda'}{\rho_0(\lambda')}$$

- Treating the collective field theory perturbatively, the lowest order result is that of 1+1 dimensional CFT – with the usual UV divergent result.
- Clearly interactions should convert this to a finite result. We want to know how does this happen.

$$\rho(\lambda, t) = \frac{1}{\beta} \mathrm{Tr}\ \delta(\lambda \cdot I_{N\times N} - M(t))$$

$$\rho(\lambda, t) = \rho_0(\lambda) + \frac{1}{\beta \rho_0(\lambda)} \partial_q \eta(q, t) \qquad \partial_\lambda \Pi_\rho = -\beta \frac{1}{\rho_0(\lambda)} \Pi_\eta$$

$$\rho_0(\lambda) = \frac{1}{\pi}\sqrt{2(\mu_F - V(\lambda))} \qquad \Pi_{\rho,0} = 0$$

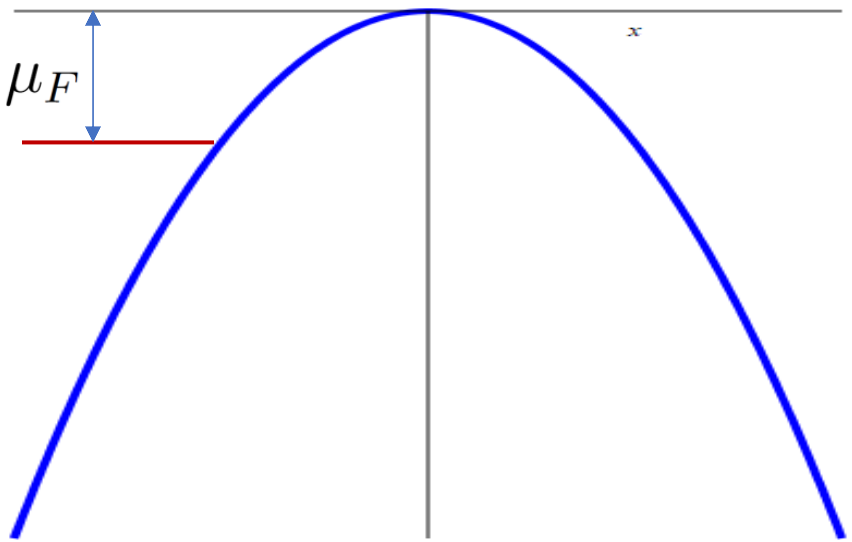
- For the inverted harmonic oscillator potential relevant for two dimensional strings, the result for large enough intervals far from the turning point is similar (*Hartnoll and Mazenc, 2015*)

$$S_{EE} = \frac{1}{3} \log \frac{q_2 - q_1}{\sqrt{g_{eff}(q_1) g_{eff}(q_2)}}$$

- As was guessed, the cutoff is indeed provided by the local depth of the fermi sea.

$$\lambda = \sqrt{2\mu_F} \;\cosh\; q$$

$$g_{eff}(q) = \frac{1}{\beta \rho_0(\lambda)^2} = \frac{g_s}{\sinh^2 q}$$



Multiple Matrices and D0 Branes

- When we have gauged quantum mechanics of many matrices, e.g BFSS D0 brane theory

$$S = \frac{N}{2(g_s N) l_s} \mathrm{Tr} \int dt \left[\sum_{I=1}^{9} (D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J=1}^{9} [X^I, X^J]^2 \right] + \text{fermions}$$

we cannot of course choose a gauge where all the matrices are diagonal.

- However, a similar construction involving a projection operator provides a gauge invariant notion of entanglement in target space.
- For low energy states with the eigenvalues of X^I well separated, these eigenvalues have an interpretation as coordinates of D0 branes.

- Consider some Hermitian matrix operator made out of the matrices $f(\hat{X}^I)$
- For example $f(X^I) = X^1$, or $f(\hat{X}^I) = \hat{R}^2 - r_0^2$ with $\hat{R}^2 = \sum_{I=1}^{9} (\hat{X}^I)^2$
- Now construct a projection operator

$$P_1 = \int_A dx\ \delta(x \cdot I - f(\hat{X}^I))$$

- Starting with the gauge invariant operators

$$\hat{\mathcal{O}} = Tr(\mathcal{O}) = Tr(\cdots \hat{X}^I \cdots \hat{\Pi}_J \cdots)$$

- We can now construct a subalgebra of projected operators

$$\hat{\mathcal{O}}^{P_1} = Tr(\cdots (\hat{X}^I)^{P_1} \cdots (\hat{\Pi}_J)^{P_1} \cdots)$$

- where

$$\hat{X}^I \to (\hat{X}^I)^{P_1} = \hat{P}_1 X^I \hat{P}_1 \qquad \hat{\Pi}_J \to (\hat{\Pi}_J)^{P_1} = \hat{P}_1 \hat{\Pi}_J \hat{P}_1$$

- We can then associate a reduced density matrix and a corresponding von Neumann entropy for this subalgebra.

- For example, consider the case

$$f(X^I) = X^1$$

- The projection then retains those eigenvalues of X^1 which lies in the interval A
- This is clear if we choose a gauge in which X^1 is diagonal.

$$\hat{X}^1 \to \mathrm{diag}\left(\hat{\lambda}_1, \cdots \hat{\lambda}_N\right) \qquad \hat{\Pi}_1 \to \mathrm{diag}\left(\hat{\pi}_1 \cdots \hat{\pi}_N\right)$$

- The remaining symmetries are Weyl transformations

$$(\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_N) \mapsto (\hat{\lambda}_{\sigma(1)}, \hat{\lambda}_{\sigma(2)}, \cdots, \hat{\lambda}_{\sigma(N)}), \; \sigma \in S(N)$$

$$\hat{X}^L \mapsto \sigma(\hat{X}^L), \; \sigma(\hat{X}^L_{ij}) = \hat{X}^L_{\sigma(i)\sigma(j)}, \; L = 2, \cdots, 9.$$

- And $U(1)^N$ transformations

$$\hat{X}^L_{ij} \mapsto \hat{X}^L_{ij} e^{i(\theta_i - \theta_j)}$$

- These need to be imposed on the states.

- Once again there are sectors labelled by the number of eigenvalues of X^1, k which lie in the region of interest.
- What does this projection do to the other matrices which are not diagonal ?
- In the sector labelled by k it is easy to see that the projector in the matrix space is given by

$$P_1 = \begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(N-k)} \\ \mathbf{0}_{(N-k)\times k} & \mathbf{0}_{(N-k)\times(N-k)} \end{pmatrix}$$

- Thus, we retain all operators in the $k \times k$ block.

- Once again there are sectors labelled by the number of eigenvalues of X^1, k which lie in the region of interest.
- What does this projection do to the other matrices which are not diagonal ?
- In the sector labelled by k it is easy to see that the projector in the matrix space is given by

$$P_1 = \begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(N-k)} \\ \mathbf{0}_{(N-k)\times k} & \mathbf{0}_{(N-k)\times(N-k)} \end{pmatrix}$$

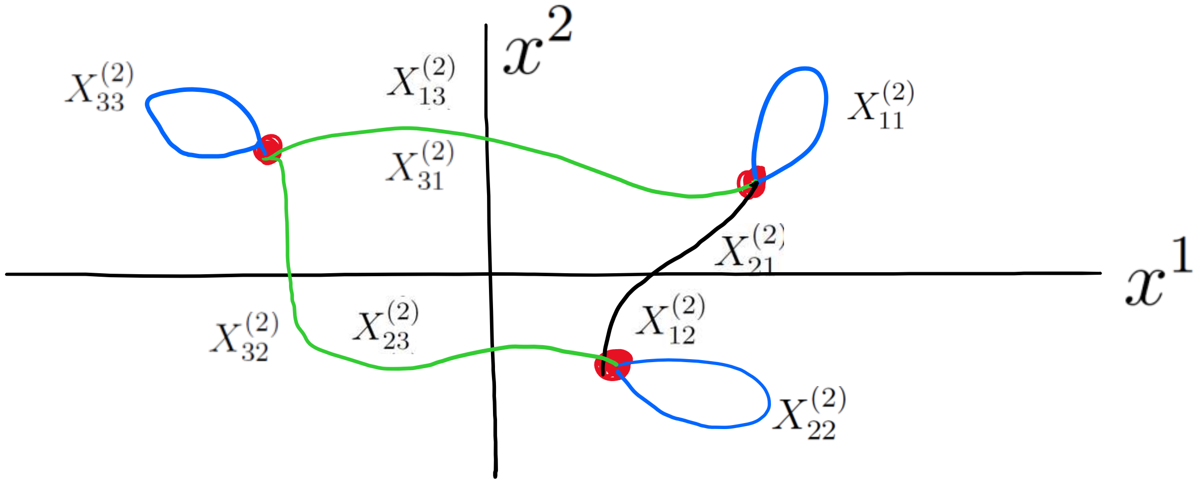
- Thus, we retain all operators in the $k \times k$ block.

$$(X^I)^{P_1}_{ij} = \int dx_1 \delta(x_1 - \lambda_i) X^I_{ij} \int dx_2 (x_2 - \lambda_j) = \begin{cases} X^I_{ij} & \text{if } i,j = 1 \cdots k \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

- Consider a typical snapshot of a configuration of the eigenvalues λ_α and the matrix elements $X^I_{\alpha\beta}$. Consider $N=3$, and the matrices

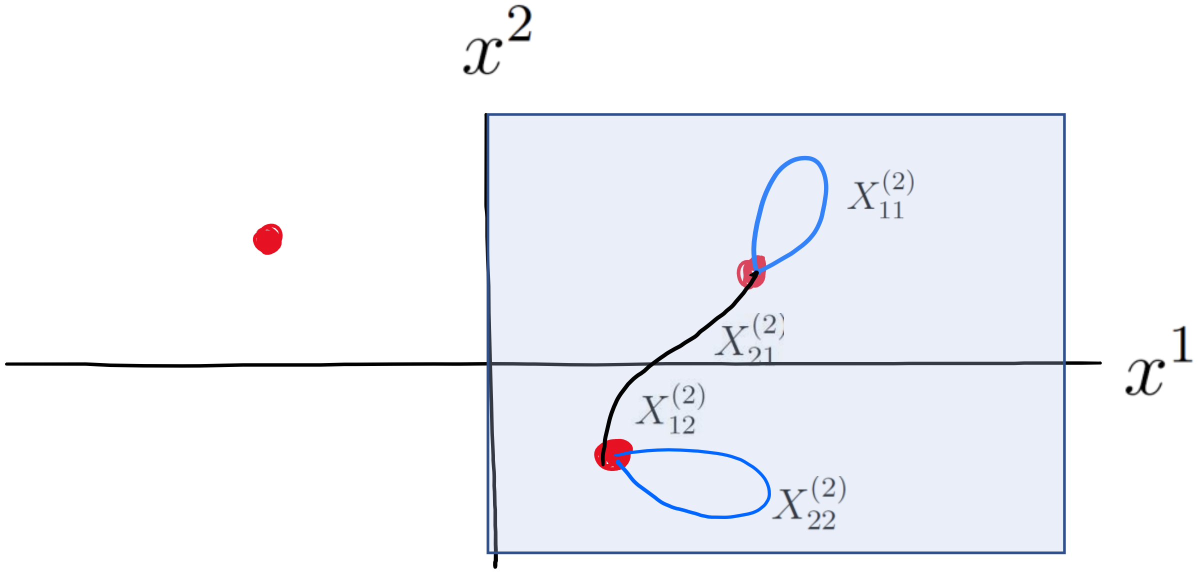
$$X^1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad X^2 = \begin{pmatrix} X^{(2)}_{11} & X^{(2)}_{12} & X^{(2)}_{13} \\ X^{(2)}_{21} & X^{(2)}_{22} & X^{(2)}_{23} \\ X^{(2)}_{31} & X^{(2)}_{32} & X^{(2)}_{33} \end{pmatrix}$$

- A configuration can be pictorially represented as



- For the constraint $x^1 > 0$ the projector $\hat{P}_1$ then keeps

$$X^1 = \begin{pmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ & & \end{pmatrix} \qquad X^2 = \begin{pmatrix} X^{(2)}_{11} & X^{(2)}_{12} & \\ X^{(2)}_{21} & X^{(2)}_{22} & \\ & & \end{pmatrix}$$

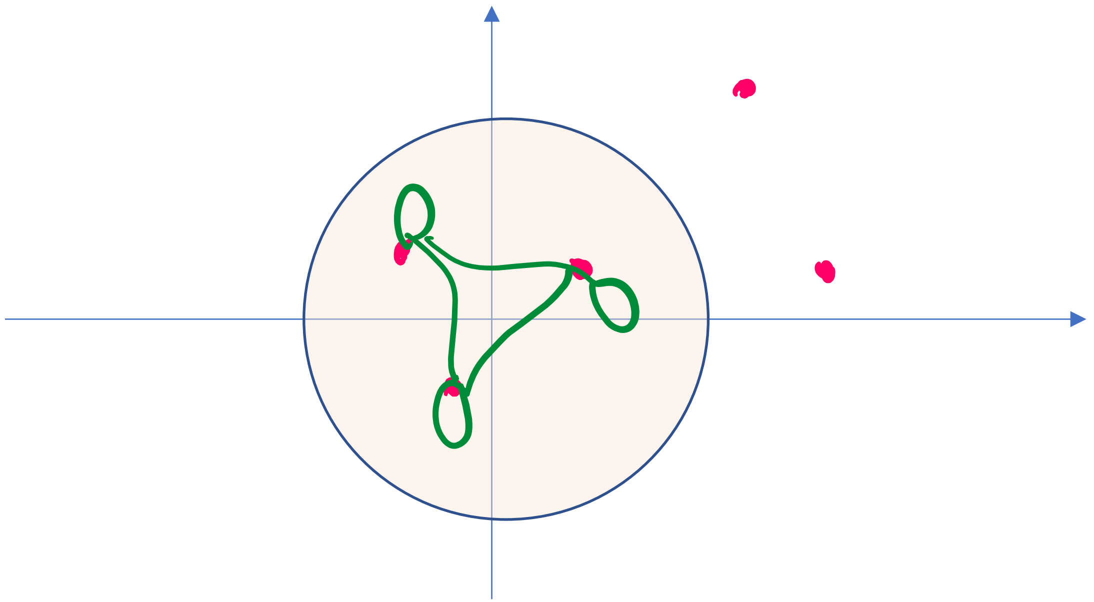


- In a similar spirit, a constraint

$$f(\hat{X}^I) = \hat{R}^2 - r_0^2 \qquad\qquad \hat{R}^2 = \sum_{I=1}^{9} (\hat{X}^I)^2$$

- Could be associated with a bulk region

$$r^2 = \sum_{I=1}^{9} (x^I)^2 > r_0^2$$



We will come to some caveats later

- For example, consider the case of two matrices. Construct the complex matrix

$$\hat{Z} = \hat{X}^1 + i\hat{X}^2$$

- This can be written in terms of a Hermitian matrix $\hat{R}$ and a unitary matrix $\hat{Q}$

$$\boxed{\hat{Z} = (\mathfrak{L}_{\hat{Q}}\hat{R})\hat{Q}}$$

$$\mathfrak{L}_{\hat{V}}\hat{M} \equiv \sqrt{2}\left[\sum_{n=0}^{\infty}(-1)^n(\hat{V}^\dagger)^n\hat{M}^2\hat{V}^n\right]^{1/2}$$

- This construction is based on the work of *Masuku & Rodrigues*, which we modified to ensure that

$$\hat{R}^2 = (\hat{X}^1)^2 + (\hat{X}^2)^2$$

- The construction can be generalized to arbitrary number of matrices.

- There is also a projection where the open strings joining branes in the region of interest with those in the complement are kept.

- Even though the open strings which join branes in complementary regions are not observables in the region of interest, the $U(1)^N$ charges carried by these strings are observables.
- Gauss Law constraints then restrict the charges on the open strings which lie entirely in the region of interest.
- For a given number of D0 branes in the region of interest, the Hilbert space has a further decomposition into charge super-selection sectors.

(*Hampapura, Harper and Lawrence*, 2021; *Frenkel and Hartnoll*,2021)

Dp Branes

- Our considerations extend to Dp brane theories with the same target space constraint at all points on the Dp brane base space.
- For example, in a "unitary" gauge where one the scalar fields X^1 is diagonal the following form a basis of states

$$\sum_\sigma (\text{sgn } \sigma) |\lambda_{\sigma(i)}(\xi), (A_\mu)_{\sigma(i)\sigma(j)}(\xi), X^L_{\sigma(i)\sigma(j)}(\xi)\rangle$$

- We can now define a sub-algebra of operators which have nontrivial matrix elements between states in this basis which have some number of eigenvalues of X^1 lie in the region of interest, at all points on the D-brane.
- However, now we have a richer possibility: we could impose target space constraints which apply to part of the base space.
- This would be a combination of base space and target space entanglement which we are exploring.

What would the answer be ?

- Multi matrix quantum mechanics is notoriously difficult.
- There are a few results for this kind of entanglement

(1) *Hampapura, Harper and Lawrence* perform a Born-Oppenheimer calculation. For a given sector they get an answer proportional to

(2) *Frenkel and Hartnoll* – deal with a 2 matrix problem where one of the matrices is canonically conjugate to the other. This kind of model is of interest in Quantum Hall physics. They find a very interesting interplay of entanglement produced by the off-diagonal matrix elements.

- It is natural to expect that the answer is proportional to N^2 . This would make sense since this is the inverse of the Newton's constant.
- We have set up explicit path integral expressions for the Renyi entropies resulting from a thermal density matrix – with the hope that one can evaluate them numerically.
- Recent years have witnessed remarkable progress in calculating finite temperature partition functions of BFSS/BMN and finding precise agreement with holographic expectations.

(*Caterall & Wiseman*; *Hanada, Hyakutake, Ishiki & Nishimura*; *Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki & Vranas*)

- Hopefully these calculations can be extended to these Renyi entropies.
- Finally, regardless of holography, this formulation should be useful for many body systems where the wave function is better understood in the first quantized description.

Finiteness of EE in Collective Field Theory

(S.R.D, A. Jevicki and J. Zheng)

- As we saw the entanglement entropy in single matrix quantum mechanics is finite – the role of the UV cutoff is played by the local fermi momentum.
- Significantly this remains finite even when $N \to \infty$ provided the fermi momenta remain finite.
- E.g. For free fermions in a box this is $N \to \infty$ and $L \to \infty$ keeping N/L finite.
- For inverted oscillator this is the double scaling limit.
- We now turn to the question: how does collective field theory manage to make the final answer finite ?

- In non-relativistic fermion field theory the entanglement entropy has a cumulant expansion (*Song, Flindt, Rachel, Klich & Le Hur*)

$$S_A = \lim_{M\to\infty} \sum_{m=1}^{M} \alpha_{2m}(M) C_{2m}, \quad C_m = (-i\partial_\lambda) \log\langle[\exp(i\lambda N_A)]\rangle|_{\lambda=0},$$

$$N_A = \int_A d^d\vec{x}\ \psi^\dagger(\vec{x})\psi(\vec{x}).$$

- For large N systems usually the lowest order term – which is the dispersion of the fermion number in the interval - dominates.
- This can be evaluated in a WKB/ Thomas-Fermi approximation for intervals which are far from the turning point and where the potential varies slowly.

- For an interval of size a centered at a location x_0 the second cumulant leads to

$$S = \frac{\pi}{3\hbar}\, 2aP_F(x_0) - \frac{1}{3}\left[\frac{4P_F(x_0)}{\hbar}\mathrm{Si}(4P_F(x_0)a/\hbar) + \mathrm{Ci}(4P_F(x_0)a/\hbar) + \cos(4P_F(x_0)a/\hbar)\right] + \frac{1}{3}[1 + \gamma_E + \log(4P_F(x_0)a/\hbar)] \tag{2.21}$$

- Where $P_F(x_0)$ is the local fermi momentum.
- In the regime

$$4P_F(x_0)a/\hbar \gg 1$$

- The result is

$$S_{EE} = \frac{1}{3}\left[1 + \gamma_E + \log\left(4P_F(x_0)a/\hbar\right)\right]$$

- As promised, the answer is pretty much like that of a single massless scalar field in 1+1 dimensions with the local momentum playing the role of the UV cutoff.

- Cumulants of the fermion number in the region of interest are equal time correlation functions of the collective field - the fermion number density.

$$S_A^{(2)} = \frac{\pi^2}{3} \int_a^b dx \int_a^b dx' \left[\langle F|\rho(x)\rho(x')|F\rangle - \langle F|\rho(x)|F\rangle\langle F|\rho(x')|F\rangle \right]$$

- We will now evaluate this using the collective field theory Hamiltonian.
- Reminder: in the collective field theory Hamiltonian the local fermi momentum appears as a coupling.
- In lowest order of perturbation theory this will give the conformal field theory result which is UV divergent.
- Therefore, the finiteness is an effect of interactions.
- The way the coupling enters suggests that the effect must be non-perturbative.

Exact evaluation for vanishing potential

- The question of UV finiteness has little to do with the nature of the external potential. We therefore consider the simplest case: vanishing potential

$$H = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right)$$

- Introduce a $U(N)$ matrix

$$U = \exp\left(\frac{2\pi i}{L}M\right)$$

- This is like putting the space of eigenvalues in a box.
- The Hamiltonian now becomes a Laplacian on $U(N)$

$$H = \left(\frac{2\pi}{L}\right)^2 \sum_\alpha C_\alpha C_\alpha, \qquad C_\alpha = \mathrm{Tr}\left(t^\alpha U \frac{\partial}{\partial U}\right)$$

- The exact eigenfunctions and eigenstates of this Hamiltonian are known (Nomura, 1986; Jevicki, 1991).
- Introduce the variables

$$\phi_n = \mathrm{Tr}\, U^n$$

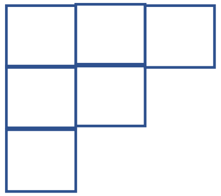
- This is the fourier transform (in eigenvalue space) of the collective field. The fluctuation about the saddle point is

$$\delta\phi_n = \int dx\, e^{-\frac{2\pi i n}{L}} \eta(x) = \sqrt{n}(a_n + a_n^\dagger).$$

- The Hamiltonian for fluctuations can be then written in terms of these oscillators.

$$\begin{aligned} H &= H_2 + H_3 \\ H_2 &= \frac{2\pi}{L} k_F \sum_{n\neq 0} |n| a_n^\dagger a_n \\ H_3 &= \frac{2\pi^2}{L^2} \sum_{n,m>0;n,m<0} \sqrt{nm|n+m|}(a_n^\dagger a_m^\dagger a_{n+m} + a_{n+m}^\dagger a_n a_m), \end{aligned}$$

- The exact eigenstates are labelled by a Young diagram



$$\lambda \equiv \{\lambda_1, \lambda_2, \cdots\}, \quad \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots$$

$$\sum_j \lambda_j = n$$

- They are given by the action of Schur Polynomials of the $a_m^\dagger$

$$|\lambda\rangle = s_\lambda(\sqrt{j}a_j^\dagger)\,|0\rangle$$

- The eigenvalue for this state is

$$E_\lambda = \left(\frac{\sqrt{2}\pi}{L}\right)^2 \left[Nn + \sum_j \lambda_j(\lambda_j - 2j + 1)\right]$$

- The connection of Schur polynomials with slater determinants show that the exact eigenstates are states of the N fermion system – as they should be

- We need to calculate the connected two-point function of the collective field.
- The fourier transform of this is

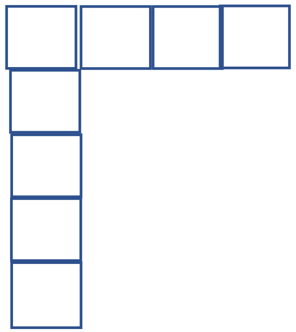
$$\langle 0|\delta\phi_n \delta\phi_{-n}|0\rangle_c = \sum_\lambda e^{-iE_\lambda t}|\langle\lambda|\delta\phi_n|0\rangle|^2$$

- The only states which contribute to this sum are states labelled by two integers

$$n = 0, \pm 1, \pm 2 \cdots \qquad\qquad -(N-1)/2 \leq m \leq (N-1)/2$$

- Which have

$$\begin{aligned} \lambda_1 &= m + n - (N-1)/2 \\ \lambda_2 &= \lambda_3 \cdots \lambda_{(N-1)/2-m} = 1 \end{aligned}$$



- The matrix element which appears above is unity since for these states

$$s_\lambda(a^\dagger) \sim \frac{1}{\sqrt{n}} a_n^\dagger + \cdots$$

- These states are in fact the bosonic description of fermion-hole excitations where we remove a single fermion at a level m inside the fermi sea to a state labelled by m+n
- The energy is

$$E_\lambda(n,m) = \frac{1}{2}\left(\frac{2\pi}{L}\right)^2 (n^2 + 2nm).$$

- In the limit of large L this is simply

$$E_{\lambda(p,k)} = \frac{1}{2}(k^2 + 2pk) = \frac{1}{2}[(p+k)^2 - p^2].$$

$$k = \frac{2\pi(m+n)}{L}$$

$$p = \frac{2\pi m}{L}$$

- In the correlation function there is an integral over $-k_F < p < k_F$
- The final result is

$$\tilde{G}(\omega,k) = \frac{1}{2\pi k}\left(\log\frac{i\omega - k_F k + k^2/2}{i\omega - k_F k - k^2/2} - \log\frac{i\omega + k_F k + k^2/2}{i\omega + k_F k - k^2/2}\right)$$

- In exact agreement with the fermion answer.

- Extracting the equal time correlator one can now compute the second cumulant contribution to the entanglement entropy in an interval

$$S_A^{(2)} = \frac{1}{3}\{-\mathrm{Ci}[2k_F(b-a)] - 2k_F(b-a)\,\mathrm{Si}[2k_F(b-a)] + \log[k_F(b-a)] + \pi k_F(b-a) + 2\sin^2[k_F(b-a)] + \gamma + \log 2\}\,,$$

- Which agrees with a direct fermion calculation.
- In two extreme limits

$$S = \begin{cases} \frac{1}{3}\left[\log(k_F(b-a)) + 1 + \gamma + \log\ 2\right], & \text{if } k_F(b-a) \gg 1. \\ \frac{1}{3}\left[\pi k_F(b-a) + k_F^2(b-a)^2 + \cdots\right], & \text{if } k_F(b-a) \ll 1. \end{cases}$$

EE in Perturbation Theory

- For nontrivial potentials we do not have the luxury of exact solutions – so it is important to ask if this finite result can be obtained in a perturbation calculation.
- Introduce chiral fields

$$\alpha_L = \frac{1}{\sqrt{2\pi}}(\partial_x \pi + \pi\eta), \quad \alpha_R = \frac{1}{\sqrt{2\pi}}(\partial_x \pi - \pi\eta),$$

$$[\alpha_{L,R}(x), \alpha_{L,R}(x')] = \mp \partial_x \delta(x - x'), \qquad [\alpha_L(x), \alpha_R(x')] = 0.$$

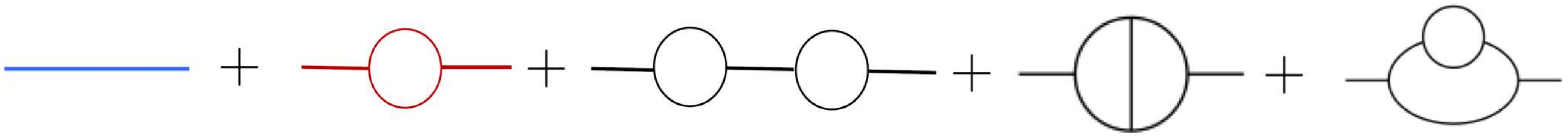
- The Hamiltonian becomes

$$H = \frac{k_F}{2}\int dx \left\{ \alpha_L^2 + \frac{\sqrt{2\pi}}{3k_F}\alpha_L^3 \right\} + \frac{k_F}{2}\int dx \left\{ \alpha_R^2 - \frac{\sqrt{2\pi}}{3k_F}\alpha_R^3 \right\}$$

- The perturbation expansion is a low momentum expansion in powers of k/k_F
- To see what to expect, consider the momentum space equal time Green's function of collective fields calculated exactly.

$$G_0(k) = \begin{cases} |k|/\pi \text{ for } |k| < 2k_F \\ 2k_F/\pi \text{ for } |k| > 2k_F \end{cases}$$

- This kind of result cannot be obtained in any tranucation of the expansion in k/k_F
- The series can be summed. The result is in exact agreement with the exact answer.



- Similar expansion appears in XXZ chain (*Pereira, Sirker, Cux, Hagemans, Maillet, White and Affleck*, 2007)

Other Notions

- In matrix theories there are other notions of entanglement which deal with entanglement of color degrees of freedom. One such notion is “matrix entanglement” is natural in theories with partial deconfinement. (*Gautam, Hanada, Jevicki and Peng).*
- Entwinement : is a notion similar to k-body density matrix used by chemists. In the context of holography this notion is relevant to duals of symmetric product orbifolds.

Lessons ?

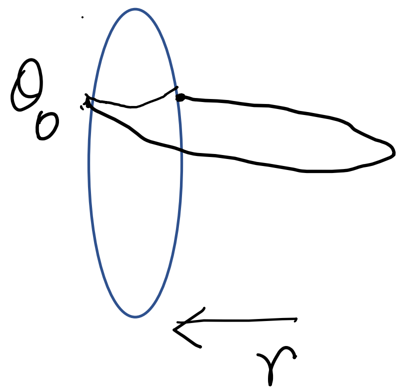
- These results suggest that this kind of entanglement entropy in gravitational theories is finite because the Newton's constant is finite.
- This finiteness is possibly invisible in a perturbation theory unless one can sum the entire perturbation expansion.
- Note that finiteness of N brings in a "stringy exclusion principle". In our exact calculation this is built in – the variables ϕ_n and $n < N$. In the perturbation calculation this is hard to trace. However, it is natural to believe that this plays a role.

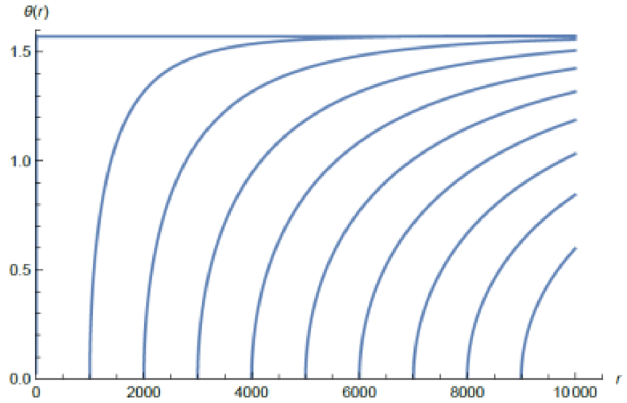
RT Surfaces ?

(S.R.D, A. Kaushal, G. Mandal, K. Nanda, M. Radwan & S. Trivedi)

- Are there notions of entanglement in internal space which relate to RT surfaces ?
- Consider for example the familiar example of $AdS_m \times S^n$
- The usual RT surfaces which measure base space entanglement are anchored on a region of the boundary of AdS_m and smeared on the S^n
- One can ask: do extremal surfaces which are anchored on a subregion of the and completely smeared along the AdS_m directions measure any kind of entanglement ?
- This question has been addressed in the past, with no clear answer (*Mollabashi, Shiba and Takayanagi*; *Karch and Uhlemann*; *Anous, Karczmarek, Mintun, Van Raamsdonk and B. Way)*

- This question is somewhat confusing because of a result of Graham and Karch: If such an RT surface goes into the bulk and end on the boundary of a subregion of the internal compact space – that boundary itself has to be extremal.
- When the internal space is a S^n, this RT surface has to end on an equator of the located at the boundary of the
- If we allow the AdS_m space to end on a cutoff boundary, it is possible for the RT surface to end on e.g. a cap of arbitrary size.



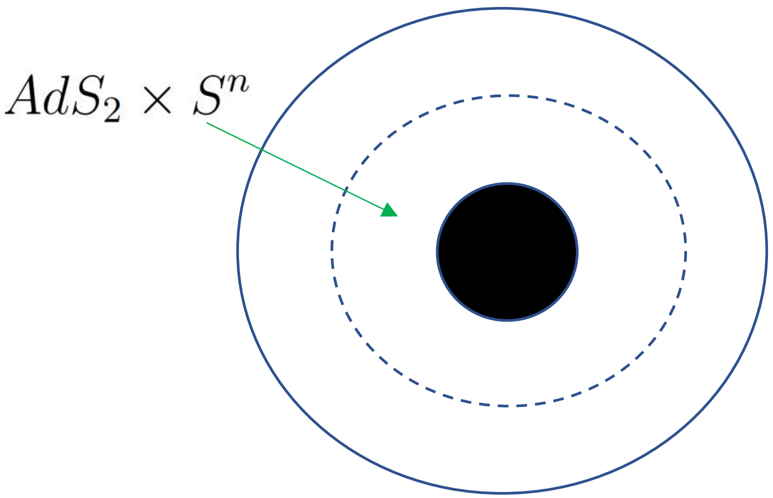


- For $AdS_2 \times S^n$ the only extremal surface is the one which hugs the boundary along the internal space.

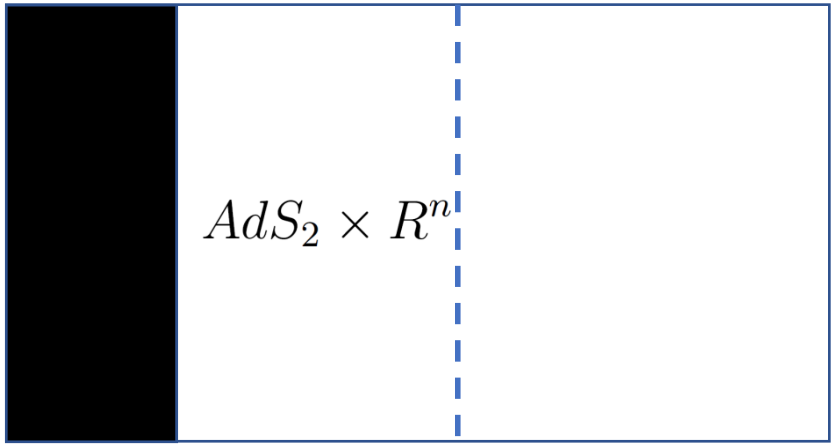
- When the internal space is non-compact, the Graham-Karch result does not directly apply.
- However, a similar analysis shows that the only extremal surfaces which go all the way to the boundary of AdS_m are those which end on regions of infinite size.
- These facts, which follow from asymptotic properties of AdS_m, makes it confusing to associate any entanglement entropy with such surfaces.
- Significantly the Graham-Karch result can be avoided if we consider warped products where the size of the internal space depends on the AdS radial direction.

- The meaning of these RT surfaces become clear in cases when such product spaces appear as IR geometries of a higher dimensional asymptocially AdS spaces.
- The most well-known example is the near horizon geometry of extremal AdS black holes or black branes. The scale of the flow is the chemical potential μ

$$AdS_{n+2} \to AdS_2 \times S^n$$

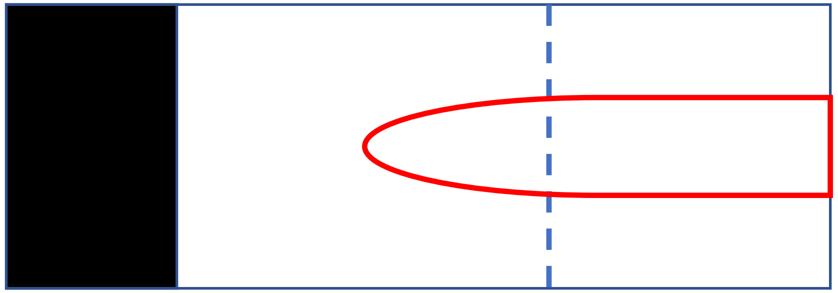


$$AdS_{n+2} \to AdS_2 \times R^n$$



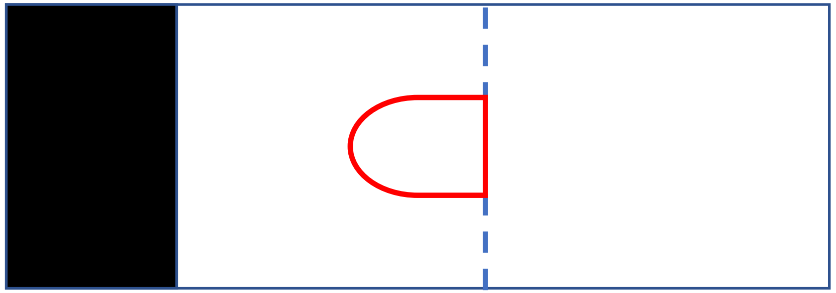
- Other examples include the dual of 3+1 dimensional N=4 theory in the presence of a constant magnetic field, $AdS_5 \to AdS_3 \times R^2$. The magnetic field provides the scale of the flow. (*D'Hoker and Kraus*)

- Consider e.g. $AdS_4 \to AdS_2 \times R^2$ and a RT surface anchored on a strip of some width l on the boundary.
- The area of this RT surface is the usual base space EE of the UV theory.
- When $l \gg 1/\mu$ the surface does not traverse much of the strip direction till it enters the $AdS_2 \times R^2$ region.
- Such a RT surface can be then thought of a RT surface which lives in the IR geometry which is anchored on the internal space R^2
- More precisely, the l derivative of the area is determined by the IR geometry which is a warped product of AdS_2 and R^2 .



- In this calculation warping plays a key role. In higher dimensions not so important.

- This quantity can be interpreted as a quantity in the 0+1 dimensional dual of 1+1 dimensional JT gravity living in the IR geometry – pretty much like many other quantities (e.g. thermodynamics).
- This now appears as an entanglement entropy of internal degrees of freedom – since this dual theory has no space – this part of the entropy is extensive.



$$\frac{A}{2G_N} \sim N^2 L l \mu^2$$

- There is a similar story for the higher dimensional examples. In these cases, the RT surface can be thought of being anchored on a region of internal space, and smeared over the base space directions.
- The situation for $AdS_{n+2} \to AdS_2 \times S^n$ is similar for caps on the S^n which are large enough – but much more involved.

- To determine the kind of entanglement in the IR theory this is evaluating consider the case of $AdS_4 \to AdS_2 \times S^2$
- From the point of view of the UV, the dual field theory is characterized by operators

$$\mathcal{O}(t, \theta, \phi)$$

- The entanglement we are evaluating is that of a region of (θ, ϕ) space.
- Equivalently these operators may be characterized by angular momentum quantum numbers,

$$\mathcal{O}(t, l, m)$$

- In the IR, these angular momentum quantum numbers become internal symmetry quantum numbers – $\mathcal{O}_{l,m}(t)$
- The entanglement becomes that in the internal space.

- Consider the familiar example of $AdS_m \times S^n$ and the standard set of gauge invariant operators in the boundary theory

$$\mathcal{O}_{l,\vec{m}}(\xi) = \mathcal{O}^{IJKL\cdots}(\xi) = \mathrm{Tr}\left(X^{(I}X^J X^K \cdots - \mathrm{trace}\right)$$

- Folding with spherical harmonics one can construct operators which are

$$\mathcal{O}(\xi,\theta,\phi_i) = \sum_{l,\vec{m}} Y_{l,\vec{m}}(\theta,\phi_i)\mathcal{O}_{l,\vec{m}}(\xi)$$

- Where θ, ϕ_i are a set of coordinates on the S^n
- The subalgebra of operators obtained by taking products and sums of these can be used to define a reduced density matrix which quantifies a notion of entanglement of internal degrees of freedom.
- Note that this notion of entanglement does not deal with entanglement of the color degrees of freedom- the projections are applied *after* a color trace is performed.
- This entanglement is closely related to supergravity modes rather than D branes.

Epilouge

- We have explored possible notions of entanglement of internal degrees of freedom.
- One such notion is natural from the point of view of "bulk entanglement" as perceived by D branes, e.g. in the Matrix Model description of two-dimensional string theory.
- Other notions are more natural from the view of supergravity modes.
- There are other possible notions of entanglement in string theory, e.g. considering String Theory on a non-compact orbifold to mimic replica calculation of entanglement entropy in usual field theories (*Dabholkar*; *He, Numasawa, Takayanagi & Watanabe*; *Witten*).
- Usual base space entanglement plays a key role in obtaining a smooth bulk space-time. Target space entanglement – or generally entanglement of internal degrees of freedom should also play a role in ensuring that the internal spaces are smooth.

THANK YOU

- It is also possible to construct another projector which retains the $(N-k)\times k$ and the $k\times(N-k)$ blocks as well.
- To do this we first define the projector to the complement

$$\tilde{P}_1 = \int_{x<0} dx \delta(x\mathbf{I} - f(\hat{X}^I))$$

- Then the corresponding operator subalgebra is obtained by the replacements

$$\hat{X}^I \to (\hat{X}^I)^{P_2} = \hat{X}^I - \tilde{P}_1 \hat{X}^I \tilde{P}_1$$

- In the matrix space

$$P_2 = \begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{I}_{k\times(N-k)} \\ \mathbf{I}_{(N-k)\times k} & \mathbf{0}_{(N-k)\times(N-k)} \end{pmatrix}$$

- While the second projector keeps

$$X^1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \end{pmatrix} \qquad X^2 = \begin{pmatrix} X^{(2)}_{11} & X^{(2)}_{12} & X^{(2)}_{13} \\ X^{(2)}_{21} & X^{(2)}_{22} & X^{(2)}_{23} \\ X^{(2)}_{31} & X^{(2)}_{32} & \end{pmatrix}$$